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# Some properties of epimorphisms of Hilbert algebras

**Research Article**

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**Abstract:** This paper represents a start in the study of epimorphisms in some categories of Hilbert algebras. Even if we give a complete characterization for such epimorphisms only for implication algebras, the following results will make possible the construction of some examples of epimorphisms which are not surjective functions. Also, we will show that the study of epimorphisms of Hilbert algebras is equivalent with the study of epimorphisms of Hertz algebras.

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## 1. Introduction

The study of epimorphisms in categories of algebraic logic was started by Balbes and Dwinger in [1], with the characterization of epimorphisms of distributive lattices and Boole algebras. Generally speaking, in category theory an *epimorphism* is a morphism $f : X \to Y$ such that for all morphisms $h_1, h_2 : Y \to Z$ with the property $h_1 \circ f = h_2 \circ f$, we have $h_1 = h_2$. A similar notion is the notion of epic subalgebra: if $\mathcal{C}$ is an algebraic category and $A, A_1 \in \mathcal{C}$ such that $A$ is a subalgebra of $A_1$, then $A$ is said to be an *epic subalgebra* of $A_1$, if the inclusion morphism $1_{AA_1} : A \to A_1$ is an epimorphism.

In the case of Boole algebras it is shown that every epimorphism is a surjective function. For the case of distributive lattices, the authors are using the notion of free boolean extension of a distributive lattice. The existence of such extension is assured by constructing a reflector from the category of distributive lattices to the category of Boole algebras (for more details, see [1], pag.97). Denote that reflector by $R$ and recall the result from [1]:

### *Theorem 1.1.*

*Let $L, L_1$ be two bounded distributive lattices with $L$ a subalgebra of $L_1$. The following are equivalent:*

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(i) *L is an epic subalgebra of* $L_1$*;*

(ii) *The morphism* $R(1_{L,L_1})$ *is an isomorphism of Boole algebras;*

(iii) *For every prime ideals* $I_1, I_2$ *in* $L_1$*,* $I_1 \cap L = I_2 \cap L \Rightarrow I_1 = I_2$*.*

We will try to give an analogous result for some subcategories of Hilbert algebras as implication algebras or Hertz algebras. The condition (iii) from the theorem above will refer to deductive systems of a Hilbert algebra and for the condition (ii), we refer to some results from [10]. A reflector from the category of Hilbert algebras to the category of Hertz algebras is given in [10] (see also [2]).
Throughout this paper, sometimes it will be easier to work with epic subalgebras, sometimes we will work just with epimorphisms. In fact, the two notions are equivalent:

***Remark 1.1.***
Let $\mathcal{C}$ be an algebraic category, $A, B$ objects in $\mathcal{C}$, $f : A \to B$ a morphism. Then $f$ is an epimorphism iff $f(A) \subseteq B$ is an epic subalgebra.

In this paper the symbols $\Rightarrow$ and $\Leftrightarrow$ will be used for logical implication and logical equivalence, respectively.

## 2. Preliminaries

We include some elementary aspects of Hilbert algebras that are necessary for this paper and for more details we refer to [2], [5] and [11].

***Definition 2.1 ([5, 11]).***
A Hilbert algebra is an algebra $(A, \to, 1)$ of type $(2, 0)$ such that the following axioms are verified for every $x, y, z \in A$:

($a_1$) $x \to (y \to x) = 1$;

($a_2$) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$;

($a_3$) If $x \to y = y \to x = 1$, then $x = y$.

For a Hilbert algebra $(A, \to, 1)$, the relation $x \leq y$ defined by $x \leq y \Leftrightarrow x \to y = 1$ is a partial order on $A$ (called the *natural order* on $A$); with respect to this order, 1 is the largest element of $A$. $A$ will be called *bounded* if it has a smallest element 0; in this case, for $x \in A$ we denote $x^* = x \to 0$.
Relative to the natural order, sometimes $A$ may become a semilattice and we will refer to $A$ as a *Hilbert algebra with infimum* if it is a meet-semilattice, or as a *Hilbert algebra with supremum* if it is a join-semilattice.
A subset $D$ of a Hilbert algebra $A$ is called a *deductive system* if $1 \in D$ and $x, x \to y \in D \Rightarrow y \in D$. Some trivial examples of deductive systems are $\{1\}$ and $A$. A deductive system $D$ is said to be *proper* if $D \neq A$. The set of all deductive systems of the Hilbert algebra $A$ will be denoted by $D_s(A)$.
For two distinct elements $a, b \in A$ and a non-empty set $\mathcal{D}$ of deductive systems of $A$, we shall say that they are *separable by deductive systems from* $\mathcal{D}$, if there exists a deductive system $D \in \mathcal{D}$ such that $a \in D, b \notin D$ or $a \notin D$, $b \in D$ (our study is based on some results of how two distinct elements of a Hilbert algebra can be separated by a deductive system).
Let $A$ and $B$ two Hilbert algebras. A function $f : A \to B$ is called a *morphism of Hilbert algebras* if for every $x, y \in A$, $f(x \to y) = f(x) \to f(y)$. Clearly, if $f$ is a morphism, then $f(1) = 1$.
Let $f : A \to B$ be a morphism of Hilbert algebras. If $A$ and $B$ are bounded Hilbert algebras and $f(0) = 0$, we say that $f$ is a *morphism of bounded Hilbert algebras*. If $A$ and $B$ have the structure of meet-semilattices relative to the natural order and "$\wedge$" is the operation of infimum, $f$ is called a *morphism of Hilbert algebras with infimum* if for every $x, y \in A$, $f(x \wedge y) = f(x) \wedge f(y)$. Similarly, if $A$ and $B$ are Hilbert algebras with supremum and "$\vee$" is the operation of supremum, $f$ is called a *morphism of Hilbert algebras with supremum* if for every $x, y \in A$, $f(x \vee y) = f(x) \vee f(y)$.

We focus now on the importance of deductive systems of a Hilbert algebra: the congruences of Hilbert algebras can be given in terms of deductive systems (see [5] and [11]). More exactly, if $A$ is a Hilbert algebra and $D \in D_s(A)$, then $\theta_D = \{(x, y) \in A \times A : x \to y, y \to x \in D\}$ is a congruence of $A$. $A/\theta_D = A/D$ becomes a Hilbert algebra where $\mathbf{1} = D$. It is easy to prove that $D_s(A)$ is closed under arbitrary intersections. So, given a non-empty set $X \subseteq A$, the set $< X >= \bigcap\{D \in D_s(A) : X \subseteq D\}$, is called *the deductive system generated by* $X$. If for $x, x_1, x_2, \ldots, x_n \in A$, $n \geq 1$, we define $(x_1, \ldots, x_n; x) = x_1 \to (x_2 \to \ldots (x_n \to x) \ldots)$, then the deductive system generated by a subset $X \subseteq A$ can be characterized as the set $< X >= \{x \in A :$ there exist $x_1, x_2, \ldots, x_n \in A$, such that $(x_1, x_2, \ldots, x_n; x) = 1\}$. If $a \in A$, then $D =< \{a\} >= [a) = \{x \in A : a \leq x\}$ is called the *principal deductive system generated by* $a$.
Let $A$ be a Hilbert algebra and $D$ a proper deductive system of $A$. We say that $D$ is *irreducible* if for any $D_1, D_2 \in D_s(A)$ such that $D = D_1 \cap D_2$, it follows that $D = D_1$ or $D = D_2$. We say that $D$ is *completely irreducible* if for any family $\{D_i : i \in I\} \subseteq D_s(A)$ such that $D = \bigcap\{D_i : i \in I\}$, then $D = D_i$ for some $i \in I$. The set of all irreducible (completely irreducible) deductive systems of a Hilbert algebra $A$ is denoted by $D_{si}(A)$ ($D_{ci}(A)$).
We include some useful results relative to this kind of deductive systems, and for more details we refer to [5].

### Theorem 2.1 ([5]).
*Let $A$ be a Hilbert algebra, $a \in A$ and $D \in D_s(A)$ such that $a \notin D$. Then there exists $M \in D_{ci}(A)$ such that $D \subseteq M$ and $a \notin M$.*

### Corollary 2.1 ([5]).
*Let $A$ be a Hilbert algebra and $a, b \in A$ such that $b \nleq a$. Then there exists $M \in D_{ci}(A)$ such that $a \notin M$ and $b \in M$.*

### Remark 2.1.
Using Corollary 2.1, if $A$ is a Hilbert algebra and $a, b \in A, a \neq b$, then $a$ and $b$ can be separated by a completely irreducible deductive system (because if $a$ and $b$ are distinct then at least one of the relations $a \leq b$ and $b \leq a$ does not hold).

Another particular kind of deductive systems are the maximal deductive systems. For a Hilbert algebra $A$ and a deductive system $M$, $M$ is said to be *maximal* if it is proper and for any other deductive system $D$, if $M \subseteq D$, then $M = D$ or $D = A$. The set of all maximal deductive systems of a Hilbert algebra $A$ will be denoted by $Max(A)$. This kind of deductive systems will help us give a characterization of epimorphisms for *implication algebras*:

### Definition 2.2 ([11]).
An implication algebra is a Hilbert algebra $(A, \to, 1)$ such that the following equation holds

$$(x \to y) \to x = x, \text{ for all } x, y \in A.$$

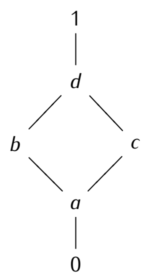
### Remark 2.2.
In [4, 9, 12] for implication algebras other names are used, such as Tarski algebras ([4]), commutative Hilbert algebras ([9]) or Abbott algebras ([12]).

We saw that for a Hilbert algebra $A$, any two distinct elements can be separated by a completely irreducible deductive system. Unfortunately such separation is not possible in general using a maximal deductive system.

### Example 2.1.
Consider the Hilbert algebra $A = \{0, a, b, c, d, 1\}$ from [7], example 1.1. The implication is given by the order (for $x, y \in A$, if $x \leq y$, then $x \to y = 1$, otherwise $x \to y = y$) and the natural order is offered in the Hasse diagram from Figure 1. It is easy to see that the only maximal deductive system of $A$ is $M = \{a, b, c, d, 1\}$. Then the elements $b, c$ cannot be separated by $M$, since both of them are contained by $M$.

Although in the case of Hilbert algebras we do not have a result of separation with maximal deductive systems (see Example 2.1), in the case of implication algebras such a separation is possible and it is assured by the following theorem.

**Figure 1.**




### *Theorem 2.2 ([11]).*

*The following conditions are equivalent for every deductive system $M$ of an implication algebra $(A, \rightarrow, 1)$:*

$(i)$ *$M$ is a maximal deductive system;*

$(ii)$ *$M$ is a proper deductive system such that for all $x, y \in M$, $(x \rightarrow y) \rightarrow y \in M$ implies either $x \in M$ or $y \in M$;*

$(iii)$ *$M$ is an irreducible deductive system;*

$(iv)$ *$A/M$ is a two-element implication algebra.*

### *Remark 2.3.*

Let $A$ be an implication algebra. Since $A$ is also a Hilbert algebra, we can separate every two distinct elements $a, b$ by a completely irreducible deductive system $M$ (see Remark 2.1). Following the above theorem, $M$ will be maximal. This means that in an implication algebra every two distinct elements can be separated by a maximal deductive system.

Some useful characterizations for maximal deductive systems can be found in [13]. The author introduces the following notation in a Hilbert algebra: for any two elements $x, y$, let be $x \sqcup y = (x \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x)$.

### *Theorem 2.3 ([13]).*

*For a proper deductive system $M$ of a Hilbert algebra $A$, the following are equivalent:*

$(i)$ *$M \in Max(A)$;*

$(ii)$ *If $x, y \in A$ and $x \sqcup y \in M$, then $x \in M$ or $y \in M$;*

$(iii)$ *If $x \notin M$, then $x \rightarrow y \in M$ for every $y \in A$.*

The following results refer to morphisms of Hilbert algebras. We are interested in seeing how deductive systems are transported through morphisms. It is easy to see that if $f : A \rightarrow B$ is a morphism of Hilbert algebras then for every $D \in D_s(B)$, $f^{-1}(D) \in D_s(A)$. Such results are not always true for any kind of deductive systems (such as irreducibles or maximals). The next example clarifies this problem.

### *Example 2.2.*

We consider again the Hilbert algebra $A = \{0, a, b, c, d, 1\}$ from Example 2.1. Let $B = \{0, b, c, 1\} \subset A$ be a Hilbert subalgebra of $A$ and $D = \{1\} \in D_s(A)$. It is clear that $D$ is an irreducible deductive system in $A$, but $D$ is not an irreducible deductive system in $B$. This means that if we denote by $i : B \rightarrow A$ the inclusion morphism, then $i^{-1}(D) \notin D_{si}(B)$, although $D \in D_{si}(A)$.

A similar result for maximal deductive systems in the case of bounded Hilbert algebras can be found in [2]. Using some other results from [13], we obtain the following theorem which generalizes a result established in [2] for the bounded case of Hilbert algebras.

### Theorem 2.4.

*If $f : A \to B$ is a morphism of Hilbert algebras and $M \in Max(B)$, then $f^{-1}(M) \in Max(A) \cup \{A\}$. So if $f^{-1}(M) \neq A$, then $f^{-1}(M)$ is a maximal deductive system of $A$.*

**Proof.** Let $M \in Max(B)$. Then $f^{-1}(M) \in Ds(A)$. If $f^{-1}(M) \neq A$, we use Theorem 2.3. Let $x, y \in A$ such that $x \sqcup y \in f^{-1}(M)$. Then $f(x \sqcup y) = f(x) \sqcup f(y) \in M$ and since $M$ is maximal, either $f(x) \in M$ or $f(y) \in M$. This means $x \in f^{-1}(M)$ or $y \in f^{-1}(M)$, so $f^{-1}(M) \in Max(A)$. □

In what follows we highlight a class of Hilbert algebras known as *Hertz algebras*:

### Definition 2.3 ([7, 10]).

A Hertz algebra is an algebra $(A, \to, \wedge, 1)$ of type $(2, 2, 0)$ which satisfies the following axioms:

$(a_4)$ $x \to x = 1$;

$(a_5)$ $(x \to y) \wedge y = y$;

$(a_6)$ $x \wedge (x \to y) = x \wedge y$;

$(a_7)$ $x \to (y \wedge z) = (x \to z) \wedge (x \to y)$.

In [6] it is proved:

### Theorem 2.5.

*For a Hilbert algebra $A$, the following assertions are equivalent:*

*$(i)$ $A$ is a Hertz algebra;*

*$(ii)$ $A$ is a Hilbert algebra with infimum which verifies the following property: for every $x, y \in A, x \to (y \to (x \wedge y)) = 1$ (that is, for every $x, y \in A$, $x \wedge y \in <\{x, y\}>$).*

In Section 3, we will see that the study of epimorphisms of Hilbert algebras is equivalent with the same study in the case of Hertz algebras.

In [7], Figallo et al. show that congruences of Hilbert algebras with infimum can be characterized like in the case of Hilbert algebras in terms of deductive systems, but in this case, the congruences are generated by *absorbent* deductive systems (a deductive system $D$ of a Hilbert algebra with infimum $A$, is called *absorbent*, if $x \in D$, then $z \to (z \wedge x) \in D$, for every $z \in A$). In the case of Hertz algebras, it is easy to see that deductive systems coincide with filters, so every deductive system will be also absorbent, that means that the congruences of Hertz algebras can be given like in the case of Hilbert algebras in terms of deductive systems.

So, in each of the three cases, Hilbert, implication algebras or Hertz algebras, congruences can be given in terms of deductive systems. Now we can give a first result relative to the epimorphisms in the categories of Hilbert, Hertz, or implication algebras:

### Proposition 2.1.

*Let $f : A \to B$ be an epimorphism of Hilbert (Hertz or implication) algebras. Then $< f(A) >= B$.*

**Proof.** Suppose $D =< f(A) >\neq B$. If we consider $p_D, 0_B : B \to B/\theta_D$, $p_D$ the canonical surjection and $0_B$ the morphism $0_B(x) = [1]/\theta_D = \mathbf{1}$, for every $x \in B$, then for every $a \in A$, $(p_D \circ f)(a) = p_D(f(a)) = [f(a)]/\theta_D = [1]/\theta_D = \mathbf{1}$(since $f(a) \in f(A) \subseteq < f(A) >$)$= 0_B(f(a)) = (0_B \circ f)(a)$, hence we have $p_D \circ f = 0_B \circ f$. But since $p_D(x) = [1]/\theta_D \Leftrightarrow x \in D$ and $D \neq B$, we have $p_D \neq 0_B$. This means that $f$ is not an epimorphism. So we must have $D = B$. □

# 3. Epimorphisms of Hilbert algebras

We start our study with the following result which offers a condition similar to (*iii*) from Theorem 1.1, for a subalgebra $A$ of a Hilbert algebra $A_1$ to become an epic subalgebra.

***Theorem 3.1.***
*Let $A, A_1$ be Hilbert algebras such that $A$ is a subalgebra of $A_1$. If for any deductive systems $D_1, D_2 \in D_s(A_1)$, $D_1 \cap A = D_2 \cap A \Rightarrow D_1 = D_2$, then $A$ is an epic subalgebra of $A_1$.*

***Proof.*** Suppose that $A$ is not an epic subalgebra of $A_1$. Then $1_{A,A_1}$ is not an epimorphism of Hilbert algebras, so there exist a Hilbert algebra $A_2$ and two distinct morphisms of Hilbert algebras $h_1, h_2 : A_1 \to A_2$ such that $h_1 \circ 1_{A,A_1} = h_2 \circ 1_{A,A_1}$. So $h_1(a) = h_2(a)$ for every $a \in A$ and there exists $x_0 \in A_1 \setminus A$ with $h_1(x_0) \neq h_2(x_0)$. Now we make use of Remark 2.1, so we can choose a deductive system $M \in D_{ci}(A_2)$ such that, for example, $h_1(x_0) \in M$ and $h_2(x_0) \notin M$. Then the deductive systems $h_1^{-1}(M), h_2^{-1}(M) \in D_s(A_1)$ will be distinct, since $x_0 \in h_1^{-1}(M) \setminus h_2^{-1}(M)$. We also have $h_1^{-1}(M) \cap A = h_2^{-1}(M) \cap A$, which leads to a contradiction. Truly, if $y \in h_1^{-1}(M) \cap A$, then $h_1(y) \in M, y \in A$ and since $h_1$ and $h_2$ are equal on $A$, we obtain $h_2(y) = h_1(y) \in M$, so $y \in h_2^{-1}(M) \cap A$. So $h_1^{-1}(M) \cap A \subseteq h_2^{-1}(M) \cap A$, analogously we obtain the other inclusion. $\square$

***Corollary 3.1.***
*Let $A, B$ be two Hilbert algebras and $f : A \to B$ a morphism of Hilbert algebras. If for any deductive systems $D_1, D_2 \in D_s(B)$, $D_1 \cap f(A) = D_2 \cap f(A) \Rightarrow D_1 = D_2$, then $f$ is an epimorphism of Hilbert algebras.*

The above result could be improved if we could suppose that morphisms of Hilbert algebras turn irreducible deductive systems into irreducible deductive systems. Although this is generally false (see Example 2), in the case of Hilbert algebras with supremum we have such a result (see [3]):

***Theorem 3.2 ([3]).***
*Let $A, B$ be two Hilbert algebras with supremum and let $f : A \to B$ be a morphism of Hilbert algebras. Then $f$ is a morphism of Hilbert algebras with supremum iff for every $M \in D_{si}(B)$, $f^{-1}(M) \in D_{si}(B)$ or $f^{-1}(M) = A$.*

Adding a supplementary condition, we obtain the following lemma:

***Lemma 3.1.***
*Let $A, B$ be two bounded Hilbert algebras with supremum and let $f : A \to B$ be a morphism of bounded Hilbert algebras. Then $f$ is a morphism of Hilbert algebras with supremum iff for every $M \in D_{si}(B)$, $f^{-1}(M) \in D_{si}(B)$.*

***Proof.*** Everything follows from Theorem 3.2 with the remark that for $M \in D_{si}(B)$ we cannot have $f^{-1}(M) = A$ (if $f^{-1}(M) = A$, then $0 \in A = f^{-1}(M)$, this means that $f(0) = 0 \in M \Leftrightarrow M = B$ which leads to a contradiction with the fact that $M$ is proper). $\square$

***Corollary 3.2.***
*Let $f : A \to B$ be a morphism of bounded Hilbert algebras with supremum. If for any irreducible deductive systems $D_1, D_2 \in D_{si}(B)$, $D_1 \cap f(A) = D_2 \cap f(A) \Rightarrow D_1 = D_2$, then $f$ is an epimorphism.*

***Proof.*** The proof follows the steps from Theorem 3.1, but in this case, with Lemma 3.1, the two distinct deductive systems $h_1^{-1}(M), h_2^{-1}(M)$ will be irreducible. $\square$

In what follows, we give a converse for Theorem 3.1. We need to extend a result from [2] (see Lemma 5.3.40, pp. 197).

### Theorem 3.3.

*Let $A$ be a Hilbert algebra and $M \in Ds(A), M \neq A$ . Then the function $f_M : A \to L_2$ ($L_2 = \{0,1\}$ is the Boole algebra with two elements) defined by:*

$$f_M(x) = \begin{cases} 0, & \text{if } x \notin M \\ 1, & \text{if } x \in M \end{cases}$$

*is a morphism of Hilbert algebras iff $M \in Max(A)$.*

**Proof.** The case of bounded Hilbert algebras is analyzed in [2]. To work with unbounded Hilbert algebras, we use Theorems 2.3 and 2.4.
($\Rightarrow$) Let $A$ be a Hilbert algebra and $M$ a proper deductive system such that the function $f_M$ is a morphism of Hilbert algebras. Since $\{1\}$ is a maximal deductive system in $L_2$ and $f^{-1}(\{1\}) = M \neq A$, then by Theorem 2.4, we have $f^{-1}(\{1\}) = M \in Max(A)$.
($\Leftarrow$) Let $A$ be a Hilbert algebra and $M \in Max(A)$. We have to prove that $f_M(x \to y) = f_M(x) \to f_M(y)$ for all $x, y \in A$.
If $x \to y \notin M$, then $y \notin M$ (because if by contrary $y \in M$, since $y \leq x \to y$, then $x \to y \in M$). If $x \notin M$, then by Theorem 2.3, we would have $x \to y \in M$ which is not possible. Therefore $x \in M$. So, in this case, $f_M(x \to y) = 0, f_M(x) = 1, f_M(y) = 0$ and we have the equality $f_M(x \to y) = f_M(x) \to f_M(y)$, because $0 = 1 \to 0$.
Suppose now that $x \to y \in M$. If $x \in M$, then $y \in M$ and we have the equality $f_M(x \to y) = f_M(x) \to f_M(y)$, because $1 = 1 \to 1$. If $x \notin M$, then either $y$ is in $M$ or not, we have the equality $f_M(x \to y) = f_M(x) \to f_M(y)$, because $1 = 0 \to 0 = 0 \to 1$. □

### Theorem 3.4.

*Let $A$, $A_1$ be Hilbert algebras such that $A$ is an epic subalgebra of $A_1$. Then for any maximal deductive systems $M_1, M_2 \in Max(A_1)$, $M_1 \cap A = M_2 \cap A \Rightarrow M_1 = M_2$.*

**Proof.** Suppose there exist distinct $M_1, M_2 \in Max(A_1)$ such that $M_1 \cap A = M_2 \cap A$. Consider the morphisms $f_{M_1}, f_{M_2} : A_1 \to \{0,1\}$ as in Theorem 3.3 and take $a \in A$. If $a \in M_1$, then $a \in M_1 \cap A = M_2 \cap A$, so $a \in M_2$. This implies that $f_{M_1}(a) = f_{M_2}(a) = 1$. If $a \notin M_1$, then $a \notin M_2$ (otherwise $a \in M_2 \Rightarrow a \in M_1$), so $f_{M_1}(a) = f_{M_2}(a) = 0$. We conclude that $f_{M_1} \circ 1_{A,A_1} = f_{M_2} \circ 1_{A,A_1}$. But, since $M_1$ and $M_2$ are distinct, $f_{M_1}$ and $f_{M_2}$ will also be distinct, which means that $1_{A,A_1}$ will not be an epimorphism. This is a contradiction. □

### Remark 3.1.

Since every implication algebra is a Hilbert algebra, Theorems 3.3 and 3.4 can also be used for implication algebras. But these results can be improved using Remark 2.3 and Theorem 2.4. The next theorem gives a characterization for epimorphisms of implication algebras.

### Theorem 3.5.

*Let $A$, $A_1$ be implication algebras such that $A$ is a subalgebra of $A_1$. Then $A$ is an epic subalgebra of $A_1$ if and only if $< A >= A_1$ (the deductive system generated by $A$ in $A_1$ is $A_1$) and for any maximal deductive systems $D_1, D_2 \in Max(A_1)$, $D_1 \cap A = D_2 \cap A \Rightarrow D_1 = D_2$.*

**Proof.** For the first implication, if $A$ is an epic subalgebra of $A_1$, following the above remark, we can use Theorem 3.4 and we obtain that for every $D_1, D_2 \in Max(A_1)$, $D_1 \cap A = D_2 \cap A \Rightarrow D_1 = D_2$. The equality $< A >= A_1$ follows from Proposition 2.1.
Conversely, suppose $< A >= A_1$ and for any maximal deductive systems $D_1, D_2 \in Max(A_1)$, $D_1 \cap A = D_2 \cap A \Rightarrow D_1 = D_2$. The proof follows the steps from Theorem 3.1.
If $A$ is not an epic subalgebra of $A_1$, then $1_{A,A_1}$ is not an epimorphism, so there exists an implication algebra $A_2$ and two distinct morphisms of Hilbert algebras $h_1, h_2 : A_1 \to A_2$ such that $h_1 \circ 1_{A,A_1} = h_2 \circ 1_{A,A_1}$. This means that $h_1(a) = h_2(a)$ for every $a \in A$ and there exists $x_0 \in A_1 \setminus A$ with $h_1(x_0) \neq h_2(x_0)$. Now we make use of Remark 2.3, so we can choose a maximal deductive system $M \in Max(A_2)$ such that, for example, $h_1(x_0) \in M$ and $h_2(x_0) \notin M$. Then the deductive systems $h_1^{-1}(M), h_2^{-1}(M) \in D_s(A_1)$ will be distinct, since $x_0 \in h_1^{-1}(M) \setminus h_2^{-1}(M)$. We will also have $h_1^{-1}(M) \cap A = h_2^{-1}(M) \cap A$.

We observe that since $x_0 \notin h_2^{-1}(M)$, $h_2^{-1}(M) \neq A_1$, hence by Theorem 2.4, $h_2^{-1}(M) \in Max(A_1)$. If $h_1^{-1}(M) \notin Max(A_1)$, then by Theorem 2.4, we must have $h_1^{-1}(M) = A_1$. From $h_1^{-1}(M) \cap A = h_2^{-1}(M) \cap A$, we obtain that $A = h_2^{-1}(M) \cap A \subseteq h_2^{-1}(M)$. Then $< A > \subseteq h_2^{-1}(M) \neq A_1$, which is a contradiction with $< A >= A_1$. So we must also have $h_1^{-1}(M) \in Max(A_1)$. We obtain $h_1^{-1}(M) = h_2^{-1}(M)$, which is a contradiction with $x_0 \in h_1^{-1}(M) \setminus h_2^{-1}(M)$. Then $A$ must be an epic subalgebra of $A_1$. $\square$

## 4. The category of Hilbert algebras is a category with proper epic subalgebras

In this section we give examples of nonsurjective epimorphisms of Hilbert algebras and we conclude that the study of epimorphisms in the category of Hilbert algebras is equivalent with the study of epimorphisms in the category of Hertz algebras. Denote by $\mathcal{H}$ the category of Hilbert algebras and by $\mathcal{H}_z$ the category of Hertz algebras.
To obtain a result similar to Theorem 1.1, we will need the following concept introduced by Porta in [10].

### *Definition 4.1.*

Let $A$ be a Hilbert algebra. A free Hertz extension of $A$ is a system $(A, \phi_A, H_A)$ where:

- $(h_1)$ $H_A$ is a Hertz algebra;
- $(h_2)$ $\phi_A : A \to H_A$ is an injective morphism in $\mathcal{H}$;
- $(h_3)$ $H_A$ is the Hertz algebra generated by $\phi_A(A)$;
- $(h_4)$ For any other Hertz algebra $H$ and any morphism $h : A \to H$ in $\mathcal{H}$, there exists a morphism $\bar{h} : H_A \to H$ in $\mathcal{H}_z$ such that $\bar{h} \circ \phi_A = h$.

We briefly recall the construction of $H_A$ and $\phi_A$.
For $A \in \mathcal{H}$, we denote by $F(A)$ the family of finite and non-empty subsets of $A$, and $I = \{1\}$. If $X, Y \in F(A)$, $X = \{x_1, x_2, \dots, x_n\}$, $Y = \{y_1, y_2, \dots, y_m\}$, we define $X \to Y = \bigcup_{j=1}^{m} \{(x_1, x_2, \dots, x_n; y_j)\}$ and $X \wedge Y = X \cup Y$. On $F(A)$ a binary relation is defined: $\rho_A = \{(X, Y) \in F(A) \times F(A) : X \to Y = Y \to X = I\}$. So $(X, Y) \in \rho_A$ iff $< X >=< Y >$. Then $\rho_A$ becomes a congruence relation on $(F(A), \to, \wedge)$. For $X \in F(A)$ we denote by $X/\rho_A$ the equivalence class of $X$ relative to $\rho_A$ and $H_A = F(A)/\rho_A$. Then $H_A$ becomes a Hertz algebra, where for any $X, Y \in F(A)$, we have $X/\rho_A \to Y/\rho_A = (X \to Y)/\rho_A$, $X/\rho_A \wedge Y/\rho_A = (X \cup Y)/\rho_A$. Also $X/\rho_A \leq Y/\rho_A$ iff $X/\rho_A \wedge Y/\rho_A = X/\rho_A$ iff $(X \cup Y)/\rho_A = X/\rho_A$. In $H_A$, $1_{H_A} = \{1\}/\rho_A$.
If we consider $\phi_A : A \to H_A$, $\phi_A(a) = \{a\}/\rho_A$, for every $a \in A$, then $\phi_A$ becomes an injective morphism in $\mathcal{H}$. Since for $X = \{x_1, x_2, \dots, x_n\} \in F(A)$, $X/\rho_A = \{x_1\}/\rho_A \wedge \dots \wedge \{x_n\}/\rho_A = \phi_A(x_1) \wedge \dots \wedge \phi_A(x_n)$, we deduce that $H_A$ is the Hertz algebra generated by $\phi_A(A)$. So the construction of $\bar{h}$ from $(h_4)$ is given by $\bar{h}(X/\rho_A) = \bigwedge_{i=1}^{n} h(x_i)$.

### *Remark 4.1.*

The morphism $\bar{h}$ is unique with respect to the property $(h_4)$.
Indeed, if we consider another morphism $\bar{\bar{h}} : H_A \to H$ in $\mathcal{H}_z$ such that $\bar{\bar{h}} \circ \phi_A = h$, then for $X = \{x_1, x_2, \dots, x_n\} \in F(A)$, $\bar{\bar{h}}(X/\rho_A) = \bar{\bar{h}}(\bigwedge_{i=1}^{n} \phi_A(x_i)) = \bigwedge_{i=1}^{n} \bar{\bar{h}}(\phi_A(x_i)) = \bigwedge_{i=1}^{n} (\bar{\bar{h}} \circ \phi_A)(x_i) = \bigwedge_{i=1}^{n} h(x_i) = \bar{h}(X/\rho_A)$, hence $\bar{\bar{h}} = \bar{h}$.

### *Remark 4.2 ([10]).*

Every Hilbert algebra $A$ has a unique free Hertz extension, that is, for any two free Hertz extensions of $A$, $(A, \phi_{1,A}, H_{1,A})$ and $(A, \phi_{2,A}, H_{2,A})$, $H_{1,A}$ and $H_{2,A}$ are isomorphic in $\mathcal{H}_z$.
More exactly, by $(h_4)$ from Definition 4.1 there exist $\alpha : H_{1,A} \to H_{2,A}$ and $\beta : H_{2,A} \to H_{1,A}$ such that $\alpha \circ \phi_{1,A} = \phi_{2,A}$ and $\beta \circ \phi_{2,A} = \phi_{1,A}$. So $\alpha \circ (\beta \circ \phi_{2,A}) = \phi_{2,A}$ and $\beta \circ (\alpha \circ \phi_{1,A}) = \phi_{1,A}$, hence $(\alpha \circ \beta) \circ \phi_{2,A} = \phi_{2,A}$ and $(\beta \circ \alpha) \circ \phi_{1,A} = \phi_{1,A}$.

By Remark 4.1, for the free Hertz extension $(A, \phi_{1,A}, H_{1,A})$ of $A$, if we consider the morphism $h = \phi_{1,A} : A \to A_1$, there exists a unique morphism $\bar{h} : A_1 \to A_1$ such that $\bar{h} \circ \phi_{1,A} = h = \phi_{1,A}$. But also $1_{H_{1,A}} \circ \phi_{1,A} = \phi_{1,A}$, so due to the uniqueness we must have $\bar{h} = 1_{H_{1,A}}$. This means that the unique morphism of Hertz algebras $\bar{h}$ with the property that $\bar{h} \circ \phi_{1,A} = \phi_{1,A}$, is the identity morphism $\bar{h} = 1_{H_{1,A}}$. Using the fact that $(\beta \circ \alpha) \circ \phi_{1,A} = \phi_{1,A}$, we obtain that $\beta \circ \alpha = 1_{H_{1,A}}$. Analogously we have $\alpha \circ \beta = 1_{H_{2,A}}$, hence $\alpha$ and $\beta$ are isomorphisms in $\mathcal{H}_z$ and $\alpha^{-1} = \beta$.

### *Remark 4.3.*

There exists a reflector $R : \mathcal{H} \to \mathcal{H}_z$.
Indeed, if for $A \in \mathcal{H}$ we put $R(A) = H_A$, we obtain the definition of the reflector $R$ on objects. To define $R$ on morphisms, let $A_1, A_2$ be two algebras in $\mathcal{H}$ and $f : A_1 \to A_2$ a morphism in $\mathcal{H}$. Consider the diagram

$$\begin{tikzcd}
A_1 \arrow[r, "f"] \arrow[d, "\phi_{A_1}"] \arrow[dr, dashed, "\bar{f}"] & A_2 \arrow[d, "\phi_{A_2}"] \\
H_{A_1} \arrow[r, dashed, "\bar{\bar{f}}"] & H_{A_2}
\end{tikzcd}$$

and the morphism $\bar{f} : A_1 \to H_{A_2}$, $\bar{f} = \phi_{A_2} \circ f$. Following Remark 4.1, there is a unique morphism $\bar{\bar{f}} : H_{A_1} \to H_{A_2}$ in $\mathcal{H}_z$ such that $\bar{\bar{f}} \circ \phi_{A_1} = \bar{f} = \phi_{A_2} \circ f$. If we define $R(f) = \bar{\bar{f}}$, we obtain the definition of $R$ on morphism. For more details see [2] (p.203).

The following results offer a wide range of examples for epimorphisms of Hilbert algebras.

### *Theorem 4.1.*

*Let $A$ be a Hilbert algebra. Then the morphism $\phi_A : A \to H_A$ is an epimorphism of Hilbert algebras.*

***Proof.*** By Remark 1.1, $\phi_A$ is an epimorphism iff $\phi_A(A) \subseteq H_A$ is an epic Hilbert subalgebra, and we prove that $\phi_A(A)$ is an epic Hilbert subalgebra of $H_A$ by using Theorem 3.1. So let $D_1, D_2 \in D_s(H_A)$ such that $D_1 \cap \phi_A(A) = D_2 \cap \phi_A(A)$. We prove that $D_1 = D_2$.
Let $X/\rho_A \in D_1$, $X = \{x_1, x_2, \ldots, x_n\} \in F(A)$. Since $X/\rho_A \wedge \{x_i\}/\rho_A = X/\rho_A$, for every $i = 1 \ldots n$, we have $X/\rho_A \le \{x_i\}/\rho_A$. Since $X/\rho_A \in D_1$, then $\{x_i\}/\rho_A \in D_1$, but also $\{x_i\}/\rho_A = \phi_A(x_i) \in \phi_A(A)$. We obtain $\{x_i\}/\rho_A \in D_1 \cap \phi_A(A) = D_2 \cap \phi_A(A)$. This means that for every $i = 1 \ldots n$, $\{x_i\}/\rho_A \in D_2$, and since every deductive system of a Hertz algebra is also a filter, we deduce $X/\rho_A = \wedge_{i=1}^{n} \{x_i\}/\rho_A \in D_2$. We have proved that $D_1 \subseteq D_2$. Analogously $D_2 \subseteq D_1$, from where we obtain $D_1 = D_2$. ☐

### *Proposition 4.1.*

*For a Hilbert algebra $A$, $\phi_A$ is surjective iff $A$ is a Hertz algebra.*

***Proof.*** Half of the problem is solved by Porta in [10]: if $A$ is Hertz, then $\phi_A$ is surjective (Corollary 7.3, pag. 64). For the direct implication, it is easy to see that if $\phi_A$ is surjective, then since it is a monomorphism, $\phi_A$ becomes an isomorphism of Hilbert algebras. In this way, the natural order in $A$ will be identical with the natural order in $H_A$, so $A$ becomes a Hilbert algebra with infimum. The isomorphism $\phi_A$ will transport the identity $x \to (y \to (x \wedge y)) = 1$, for every $x, y \in H_A$ (see Theorem 2.5), to the Hilbert algebra $A$, so by the same Theorem 2.5, $A$ will become a Hertz algebra. ☐

Using Theorem 4.1 and Proposition 4.1, we can now give examples of epimorphisms of Hilbert algebra which are not surjective functions.

### *Example 4.1.*

The Hilbert algebra $A$ from Example 2.1, is an example of a Hilbert algebra which is not a Hertz algebra because the identity $x \to (y \to (x \wedge y)) = 1$, is not verified for every $x, y \in A$. Indeed, $b \to (c \to (b \wedge c)) = b \to (c \to a) = b \to a = a \neq 1$. Then the epimorphism $\phi_A : A \to H_A$ is not a surjective function.

Although in general an epimorphism of Hilbert algebras is not surjective, there are particular cases in which this happens:

### Proposition 4.2.

*Let $A$ be a bounded Hilbert algebra, $B$ a Boole algebra and $f : A \to B$ an epimorphism of bounded Hilbert algebras. Then $f$ is a surjective function.*

**Proof.** We need a result from [8]: a reflector $R$ from the category of bounded Hilbert algebras to the category of Boole algebras can be constructed. For every bounded Hilbert algebra $A$, let be $R(A) = \{x^{**} : x \in A\}$ and $r_A : A \to R(A)$, $r_A(x) = x^{**}$. Then $R(A)$ becomes a Boole algebra and $r_A$ a surjective morphism of bounded Hilbert algebras (for more details see [8], theorems 12,14). In this way the reflector is defined on objects.
If $A_1, A_2$ are two bounded Hilbert algebras and $f : A_1 \to A_2$ is a morphism of bounded Hilbert algebras, then $R(f) : R(A_1) \to R(A_2)$ can be defined by $R(f)(x) = f(x)$ for every $x \in R(A_1)$. Clearly ([2],[8]) $R(f)$ is a morphism of Boole algebras and the diagram

$$\begin{tikzcd} A_1 \arrow[r, "f"] \arrow[d, "r_{A_1}"] & A_2 \arrow[d, "r_{A_2}"] \\ R(A_1) \arrow[r, "R(f)"] & R(A_2) \end{tikzcd}$$

is commutative. So we obtain the definition of the reflector $R$ on morphisms.
To prove $R$ is a reflector (see [2], p.27) we need to prove that for a bounded Hilbert algebra $A$, for any Boole algebra $B$ and any morphism of bounded Hilbert algebra $f : A \to B$, there exists a unique morphism of Boole algebras $\bar{f} : R(A) \to B$ such that $\bar{f} \circ r_A = f$. In [8], the existence of $\bar{f}$ is proved: $\bar{f} : R(A) \to B$, $\bar{f} = f|_{R(A)}$.
We return to the proof of our proposition. We consider a bounded Hilbert algebra $A$, a Boole algebra $B$ and an epimorphism of bounded Hilbert algebras $f : A \to B$. By the above construction of $R$, $f$ will be transported to the category of Boole algebras in $R(f) : R(A) \to R(B) = B$, with $R(f)(x) = f(x)$, for every $x \in R(A)$. Then $R(f) : R(A) \to R(B) = B$ is an epimorphism of Boole algebras, hence $R(f)$ is surjective (see [1] p.109, Corollary 2). This means that for every $y \in B$, there exists an element $x \in R(A)$ such that $R(f)(x) = y$, that is, $f(x) = y$. Hence $f$ is surjective. $\square$

The following results will help us to reduce the study of epimorphisms from the category of Hilbert algebras to the category of Hertz algebras. The first result is more general than our case, but it will be useful in what follows.

### Theorem 4.2.

*Let $\mathcal{C}$ be a category, $\mathcal{C}'$ a reflexive subcategory of $\mathcal{C}$ and*
*$R : \mathcal{C} \to \mathcal{C}'$ a reflector. For every $A \in \mathcal{C}$ we denote with $\phi_A : A \to R(A)$ the morphism from $\mathcal{C}$ induced by the reflector $R$. If $\phi_A$ is a monomorphism in $\mathcal{C}$ for every object $A \in \mathcal{C}$, then for any $A, B \in \mathcal{C}$ we have that $f : A \to B$ is an epimorphism in $\mathcal{C} \Leftrightarrow R(f) : R(A) \to R(B)$ is an epimorphism in $\mathcal{C}'$.*

**Proof.** The implication "$\Rightarrow$" is a well-known property of reflectors (see [1]). To prove "$\Leftarrow$", we consider $f : A \to B$ in $\mathcal{C}$, such that $R(f) : R(A) \to R(B)$ is an epimorphism in $\mathcal{C}'$ and let $C \in \mathcal{C}$, $g, h : B \to C$ be morphisms in $\mathcal{C}$ such that $g \circ f = h \circ f$. We prove that $g = h$. To be more precise, we consider the following diagrams.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi_A"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] \arrow[d, "\phi_B"] & C \arrow[d, "\phi_C"] \\ R(A) \arrow[r, "R(f)"] & R(B) \arrow[r, shift left, "R(g)"] \arrow[r, shift right, "R(h)"'] & R(C) \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, "\phi_A"] \arrow[dr, "\gamma"] & \\ R(A) \arrow[r, dashed, "\bar{\gamma}"] & R(C) \end{tikzcd}$$

Since $R$ is a reflector, we have $\phi_C \circ (g \circ f) = R(g \circ f) \circ \phi_A$ and $\phi_C \circ (h \circ f) = R(h \circ f) \circ \phi_A$. If we denote $\gamma = \phi_C \circ g \circ f = \phi_C \circ h \circ f$, then $\gamma : A \to R(C)$ is a morphism in $\mathcal{C}$ and using the property of universality of $\phi_A$, we conclude that there exists a unique morphism $\bar{\gamma} : R(A) \to R(C)$ in $\mathcal{C}'$ such that $\bar{\gamma} \circ \phi_A = \gamma$. We observe that both $R(g \circ f)$ and $R(h \circ f)$ are morphisms in $\mathcal{C}'$ and they verify $R(g \circ f) \circ \phi_A = R(h \circ f) \circ \phi_A = \gamma$. Then we must have $R(g \circ f) = R(h \circ f)$, so $R(g) \circ R(f) = R(h) \circ R(f)$. Since $R(f)$ is an epimorphism in $\mathcal{C}'$, we obtain $R(g) = R(h)$. Then $\phi_C \circ g = R(g) \circ \phi_B = R(h) \circ \phi_B = \phi_C \circ h$. So $\phi_C \circ g = \phi_C \circ h$, but $\phi_C$ is a monomorphism in $\mathcal{C}$. We conclude that $g = h$. $\square$

Using the above theorem for our reflector $R : \mathcal{H} \to \mathcal{H}_z$, we obtain a characterization of epimorphisms of Hilbert algebras similar to $(ii)$ from Theorem 1.1:

### Corollary 4.1.

*Let $f : A \to B$ be a morphism of Hilbert algebras. Then $f$ is an epimorphism of Hilbert algebras iff $R(f) : H_A \to H_B$ is an epimorphism of Hertz algebras.*

It is clear that the results from Theorems 3.1 and 3.3 are also true for Hertz algebras. In what follows, we improve Theorem 3.1 for the case of Hertz algebras. For that we need another result from [10].

### Theorem 4.3 ([10], pp. 66).

*Let $A$ be a Hilbert subalgebra of the Hertz algebra $H$. The following are equivalent:*

$(i)$ *$(A, 1_{A,H}, H)$ is a free Hertz extension of $A$;*

$(ii)$ *The mapping $\Phi : D_s(H) \to D_s(A)$, $\Phi(D) = D \cap A$, for every $D \in D_s(H)$, is injective.*

### Theorem 4.4.

*Let $A, A_1$ be Hertz algebras such that $A$ is a subalgebra of $A_1$. If for any deductive systems $D_1, D_2 \in D_s(A_1)$, $D_1 \cap A = D_2 \cap A \Rightarrow D_1 = D_2$, then $A = A_1$.*

***Proof.*** The condition $D_1 \cap A = D_2 \cap A \Rightarrow D_1 = D_2$, says that $\Phi$ from Theorem 4.3 is injective. So $(A, 1_{A,A_1}, A_1)$ is a free Hertz extension of $A$, hence by Remark 4.2, $H_A \approx A_1$ in $\mathcal{H}_z$. More precisely, there are two morphisms $\alpha : A_1 \to H_A$ and $\beta : H_A \to A_1$ in $\mathcal{H}_z$ such that $\alpha \circ 1_{A,A_1} = \phi_A$ and $\beta \circ \phi_A = 1_{A,A_1}$ (in fact $\alpha$ and $\beta$ are isomorphisms in $\mathcal{H}_z$ and $\beta = \alpha^{-1}$). Let $a_1 \in A_1$. Then $\alpha(a_1) \in H_A$. Since $A$ is a Hertz algebra, by Proposition 4.1 we deduce that $\phi_A$ is surjective, hence $\alpha(a_1) = \phi_A(a)$ with $a \in A$. So $1_{A,A_1}$ is a morphism and

$$\beta(\alpha(a_1)) = \beta(\phi_A(a)) \Leftrightarrow$$
$$(\beta \circ \alpha)(a_1) = (\beta \circ \phi_A)(a) \Leftrightarrow$$
$$1_{A_1}(a_1) = 1_{A,A_1}(a) \Leftrightarrow a_1 = a,$$

hence $A = A_1$. □

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