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Approximation and asymptotics of eigenvalues of unbounded self-adjoint Jacobi matrices acting in l^2 by the use of finite submatrices

Research Article

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Abstract: We consider the problem of approximation of eigenvalues of a self-adjoint operator J defined by a Jacobi matrix in the Hilbert space $l^2(\mathbb{N})$ by eigenvalues of principal finite submatrices of an infinite Jacobi matrix that defines this operator. We assume the operator J is bounded from below with compact resolvent. In our research we estimate the asymptotics (with $n \to \infty$) of the joint error of approximation for the eigenvalues, numbered from 1 to N, of J by the eigenvalues of the finite submatrix J_n of order $n \times n$, where $N = \max\{k \in \mathbb{N} : k \leq rn\}$ and $r \in (0, 1)$ is arbitrary chosen. We apply this result to obtain an asymptotics for the eigenvalues of J. The method applied in this research is based on Volkmer's results included in [23].

MSC: 47B25, 47B36, 15A18

Keywords: Self-adjoint unbounded Jacobi matrix • Asymptotics • Point spectrum • Tridiagonal matrix • Eigenvalue

1. Introduction

Tridiagonal matrices appear in various problems in mathematics. Infinite symmetric tridiagonal matrices called Jacobi matrices have essential meaning, consequently spectral properties of linear operators associated with Jacobi matrices are investigated (see, e.g.,[6]-[17],[21]-[23] and others). It can happen that a linear operator defined by a Jacobi matrix is compact or has a compact resolvent and its spectrum consists of eigenvalues of finite multiplicity (see, e.g., [7, 11, 14] and [13]). Sometimes it is possible to calculate exact formulas for eigenvalues of such Jacobi matrices (see, e.g., [9, 17, 20] and [12]), but it is not possible in general. So, asymptotic and approximate approaches to localize the point spectrum are applied (see, e.g., [3–6, 14, 16, 18, 20, 23] and [25]). This work continues the research started in [15]. We consider the problem of approximation for eigenvalues of some self-adjoint bounded from below discrete operator in the Hilbert space $l^2 = l^2(\mathbb{N})$ by eigenvalues of properly chosen principal finite submatrices of an infinite Jacobi matrix that defines the operator. Projective methods, that use finite submatrices to investigate spectral properties of operators given by

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infinite Jacobi matrices, were applied by Arveson ([1, 2]), Ifantis, Kokologiannaki and Petropoulou ([11]), Volkmer ([23]) and others.
The eigenvalues of a self-adjoint, bounded from below operator with compact resolvent may be arranged non-decreasingly. In [23], Volkmer estimated the error of approximation for the eigenvalue and eigenvector, whose number is fixed. In our research we estimate the asymptotics (with $n \to \infty$) of the joint error of approximation for the first $[rn] = \max\{k \in \mathbb{N} : k \leq rn\}$ eigenvalues of the infinite Jacobi matrix by the eigenvalues of the principal finite submatrix of order $n \times n$, where $r \in (0,1)$ is arbitrary chosen. To obtain the result we use some weaker assumptions then in [15] and slightly different from that in [23]. The method applied is based on Volkmer's results included in [23]. Finally we use the main result of this paper to obtain an asymptotic behaviour of the point spectrum of J. This approach to the problem of asymptotics is different then the methods based on diagonalization (applied, e.g., in [4–6, 14, 16] or [18]) and the methods that use an analytic model of the spectral equation (see [12] and [17]).
Let us consider an operator J in the space l^2 with the canonical basis given by the tridiagonal symmetric Jacobi matrix

$$J = \begin{pmatrix} d_1 & c_1 & 0 & \cdots & \cdots \\ c_1 & d_2 & c_2 & 0 & \ddots \\ 0 & c_2 & d_3 & c_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{1}$$

and consider the finite submatrices of order $n \times n$

$$J_n = \begin{pmatrix} d_1 & c_1 & 0 & \cdots & \cdots & 0 \\ c_1 & d_2 & c_2 & 0 & \cdots & \\ 0 & c_2 & d_3 & c_3 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & c_{n-2} & d_{n-1} & c_{n-1} \\ 0 & \cdots & \cdots & 0 & c_{n-1} & d_n \end{pmatrix}, n \geq 1. \tag{2}$$

We assume that J is an operator in the Hilbert space l^2 and acts on the maximum domain

$$D(J) = \{\{f_n\}_{n=1}^{\infty} \in l^2 : \ \{c_{n-1}f_{n-1} + d_nf_n + c_nf_{n+1}\}_{n=1}^{\infty} \in l^2\}$$

(we here assume $c_0 = f_0 = 0$). We assume that the sequences $\{c_n\}$ and $\{d_n\}$ satisfy the following conditions:

$(A1)$ $d_n, c_n \in \mathbb{R}$ for all $n \geq 1$;
$(A2)$ there exist $\alpha > 0$, $\beta \in \mathbb{R}$, $\alpha > \beta$ and $\delta, S > 0$ such that $d_n = \delta n^{\alpha}(1 + \delta_n)$, where $\lim_{n\to\infty} \delta_n = 0$, and $|c_n| \leq Sn^{\beta}$ for $n \geq 1$.

If $(A1)$ and $(A2)$ are satisfied then the operator J is self-adjoint, has a compact resolvent and its spectrum is discrete (see, for instance, Janas and Naboko [13] or Cojuhari and Janas [7]). Moreover, the operator J is bounded from below and

$$D(J) = \{\{f_n\}_{n=1}^{\infty} \in l^2 : \ \{n^{\alpha}f_n\}_{n=1}^{\infty} \in l^2\}. \tag{3}$$

The main result of this work is the following theorem about the estimation of the joint error of approximation for a part of eigenvalues of the operator J by the eigenvalues of the finite submatrix J_n of order $n \times n$ (with $n \to \infty$).

Theorem 1.1.

Let J be an operator in the Hilbert space l^2 with the canonical basis defined by infinite matrix (1) satisfying $(A1)$ and $(A2)$. Let the spectrum of J consists of the non-decreasingly ordered eigenvalues: $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$. Denote by $\mu_{i,n}$, $1 \leq i \leq n$, the eigenvalues of the matrix J_n and assume that $\mu_{1,n} \leq \mu_{2,n} \leq \dots \leq \mu_{n,n}$. Then for every $\gamma > 0$ and $r \in (0,1)$ there exists $C > 0$ such that

$$\sup_{1 \leq k \leq rn} |\mu_{k,n} - \lambda_k| \leq Cn^{-\gamma}, \ where \ n > 1.$$

The complete proof of this theorem is included in section 3.
In section 4 we give some consequences of Theorem 1.1. Under an additional condition, which is formulated at the beginning of section 4, it is possible to estimate appropriate eigenvalues of the finite matrices (2) and use this result to obtain an asymptotics for the point spectrum of J.

2. Preliminaries

This section sets the notations and remarks as well as the background for the Rayleigh-Ritz method for approximation of eigenvalues and Volkmer's results (see [23]) we are going to apply in our investigations.
Let H be a Hilbert space and $J : D(J) \subset H \to H$ be a linear self-adjoint operator in H. Assume that J has a compact resolvent and is bounded from below. Then the spectrum of J consists of the eigenvalues that can be ordered non-decreasingly: $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$. By the minimum-maximum principle, for all $k \in \mathbb{N}$, there is

$$\lambda_k = \min_{E_k} \max\{(Jx, x) : x \in E_k, \|x\| = 1\}, \tag{4}$$

where the minimum is taken over all linear subspaces $E_k \subseteq D(J)$ of dimension k ($(.,.)$ stands for an inner product in H). Denote by x_k the eigenvector of J associated with the eigenvalue λ_k. We can assume that the system of chosen eigenvectors $\{x_1, x_2, x_3, \dots\}$ is an orthonormal basis in H. So, if $k \geq 1$ then

$$x_k = \{x_{k,j}\}_{j=1}^{\infty} \quad \text{and} \quad \|x_k\|^2 = \sum_{j=1}^{\infty} |x_{k,j}|^2 = 1.$$

Suppose that $H = l^2$ and $\{e_1, e_2, e_3, \dots\}$ is the canonical basis of l^2. Now, let J be a Jacobi operator given in the canonical basis by matrix (1). Let $E = Lin\{e_1, e_2, \dots, e_n\}$, P_n denote the orthogonal projection onto E and $Q_n = I - P_n$. Then the linear operator $P_nJ : E \to E$ has the matrix representation in the canonical basis given by the matrix J_n due to (2).
Denote by $\mu_{i,n}$, $1 \leq i \leq n$, the eigenvalues of the matrix J_n and assume that $\mu_{1,n} \leq \mu_{2,n} \leq \dots \leq \mu_{n,n}$. For a fixed $i \in \{1, 2, \dots, n\}$ let $y_{i,n} \in \mathbb{R}^n$ be an eigenvector of J_n associated with $\mu_{i,n}$ and assume that $\{y_{1,n}, y_{2,n}, \dots, y_{n,n}\}$ is an orthonormal basis in $\mathbb{R}^n$ such that

$$(x_k, y_{k,n}) \geq 0 \quad \text{for} \quad k = 1, \dots, n, \tag{5}$$

where $(.,.)$ stands for the inner product in l^2, but here we treat $y_{k,n}$ as an infinite sequence, whose elements are equal to 0 for the indices greater then n.
Let $k \in \{1, \dots, n\}$ and

$$L^{(k,n)} = (L_{i,j}^{(n)})_{i,j=1\dots k}, \quad \text{where} \quad L_{i,j}^{(n)} = (Q_n x_i, x_j),$$

and

$$M^{(k,n)} = (M_{i,j}^{(n)})_{i,j=1\dots k}, \quad \text{where} \quad M_{i,j}^{(n)} = ((P_nJP_n - J)x_i, x_j),$$

be $k \times k$-matrices. The notation $\|T\|$ stands for the operator norm of an operator $T : \mathbb{C}^k \to \mathbb{C}^k$ given by the matrix T.

Lemma 2.1 (Volkmer, see [23]).

If $\|L^{(k,n)}\| < 1$ *then*

$$0 \leq \mu_{k,n} - \lambda_k \leq \frac{\|M^{(k,n)} + \lambda_k L^{(k,n)}\|}{1 - \|L^{(k,n)}\|},$$

where $1 \leq k \leq n$.

Lemma 2.2.
If $n \in \mathbb{N}$ and $k \in \{1, 2, ..., n\}$ then

$$\|L^{(k,n)}\| \le \sum_{i=1}^{k} \|Q_n x_i\|^2;$$

$$\|M^{(k,n)} + \lambda_k L^{(k,n)}\| \le |c_n|(\sum_{i=1}^{k} |x_{i,n+1}|^2)^{1/2}(\sum_{j=1}^{k} |x_{j,n}|^2)^{1/2} + (\sum_{i=1}^{k} |\lambda_k - \lambda_i|^2 \|Q_n x_i\|^2)^{1/2}(\sum_{j=1}^{k} \|Q_n x_j\|^2)^{1/2}.$$

Proof. The lemma is an easy consequence of the result of Volkmer (see [23]). □

Lemma 2.3 (Volkmer, see [23]).
Let

$$K_{k,n} = \|P_n J P_n x_k - P_n J x_k\|, \tag{6}$$

and

$$\Delta_{k,n} = \max\{|\mu_{i,n} - \lambda_k|^{-1} : i \in \{1, 2, ..., n\} \setminus \{k\}\}. \tag{7}$$

Then

$$K_{k,n} \le |c_n||x_{k,n+1}|$$

and

$$\|P_n x_k - y_k\|^2 \le \Delta_{k,n}^2 K_{k,n}^2 + (\|Q_n x_k\|^2 + \Delta_{k,n}^2 K_{k,n}^2)^2,$$

for $1 \le k \le n$.

Define

$$p_n = \max\{|\delta_k| k^\alpha : k \le n\}, \quad q_n = \max\{S n^\beta, S\}, \quad n \ge 1. \tag{8}$$

Lemma 2.4.
Under assumptions $(A1)$ and $(A2)$, the sequence $\{p_n\}$ is non-decreasing and

$$p_n = o(n^\alpha), \quad as \quad n \to \infty,$$

(i.e., $\lim_{n\to\infty} \frac{p_n}{n^\alpha} = 0$).

Proof. By definition $p_n = |\delta_{k_n}| k_n^\alpha$, for some $k_n \le n$. Assume that $\{p_n\}$ is unbounded, then $\lim_{n\to\infty} k_n = +\infty$. So,

$$\left|\frac{p_n}{n^\alpha}\right| = \frac{|\delta_{k_n}| k_n^\alpha}{n^\alpha} \le |\delta_{k_n}| \to 0, \quad n \to \infty,$$

because $\lim_{n\to\infty} \delta_n = 0$. □

At the end of this section, observe the following simple estimates for the eigenvalues of J.

Proposition 2.1.
Under assumptions $(A1)$ and $(A2)$

$$\lambda_n \le \delta(n^\alpha + p_n) + 2q_n, \quad for \quad n \ge 1.$$

Proof. Applying the minimum-maximum principle (4) and using $(A1)$ and $(A2)$, we derive the following estimate

$$\lambda_n \le \mu_{n,n} \le \|J_n\| \le \max_{1\le k\le n} |d_k| + 2 \max_{1\le k\le n-1} |c_k| \le \delta(n^\alpha + p_n) + 2q_n,$$

for $n \ge 1$, $(c_0 = 0)$. □

3. Proof of Theorem 1.1

Let δ, α, β be the parameters of J that appear in $(A1)$ and $(A2)$, $\{p_n\}$ and $\{q_n\}$ be defined by (8). Define

$$f_{i,n} = \frac{|c_{n-1}|}{d_n - (\delta i^\alpha + \delta p_i + 2q_i) - |c_n|}, \quad 1 \le i < n. \tag{9}$$

Choose r' such that $r < r' < 1$. Let $n \ge 1$ and $1 \le i \le r'n$; then, applying Lemma 2.4 and Proposition 2.1, we can write the following estimates:

$$d_n - (\delta i^\alpha + \delta p_i + 2q_i) - |c_n| \ge \delta n^\alpha - \delta p_n - (\delta i^\alpha + \delta p_i + 2q_i) - q_n$$

$$\ge \delta n^\alpha - \delta (nr')^\alpha - 2\delta p_n - 3q_n \ge \delta(1 - r'^\alpha) n^\alpha + o(n^\alpha) \ge C_1 n^\alpha,$$

for $C_1 = \frac{1}{2}\delta(1 - r'^\alpha)$ and n large enough. Therefore, we obtain

$$0 \le f_{i,n} \le \frac{S(n-1)^\beta}{C_1 n^\alpha} \le \frac{a}{n^{\alpha-\beta}} \le \frac{1}{2} < 1, \quad \text{for} \quad n \ge n_1,\ 1 \le i \le r'n, \tag{10}$$

where $a = \frac{S}{C_1}$ and n_1 is large enough.

We need two lemmata given below to continue the proof. As in the previous sections, $x_k = \{x_{k,j}\}_{j=1}^\infty \in l^2$ is the eigenvector associated with the eigenvalue λ_k. We still assume $\{x_1, x_2, ...\}$ is an orthonormal basis in l^2 and $\|x_k\|^2 = \sum_{j=1}^\infty |x_{k,j}|^2 = 1$. Then (9), (10) and the calculations, that are based on Volkmer's results (see [23]), lead to the following lemma.

Lemma 3.1.
If $m \ge n_1$ and $1 \le i \le r'm$, then $|x_{i,m}| \le f_{i,m}|x_{i,m-1}|$.

Proof. Let $m \ge n_1$ and $1 \le i \le r'm$. There exists $k \ge m$ such that $|x_{i,k+1}| \le |x_{i,k}|$, because $x_i \in l^2$. Then, under the spectral equality $Jx_i = \lambda_i x_i$, we have

$$c_{k-1}x_{i,k-1} + (d_k - \lambda_i)x_{i,k} + c_k x_{i,k+1} = 0$$

and

$$|c_{k-1}x_{i,k-1}| \ge |d_k - \lambda_i||x_{i,k}| - |c_k||x_{i,k+1}|.$$

From the above inequality and Proposition 2.1 we derive

$$|c_{k-1}||x_{i,k-1}| \ge (|d_k - \lambda_i| - |c_k|)|x_{i,k}| \ge (d_k - \lambda_i - |c_k|)|x_{i,k}| \ge (d_k - (\delta i^\alpha + \delta p_i + 2q_i) - |c_k|)|x_{i,k}| > 0.$$

Thus

$$|x_{i,k}| \le \frac{|c_{k-1}|}{d_k - (\delta i^\alpha + \delta p_i + 2q_i) - |c_k|}|x_{i,k-1}| \le f_{i,k}|x_{i,k-1}| \le |x_{i,k-1}|,$$

because of (9) and (10). If $k - 1 > m$ we repeat the above procedure as long as we finally obtain $|x_{i,m}| \le f_{i,m}|x_{i,m-1}|$. $\square$

Notice that if $\alpha - \beta > 0$ then there exists an integer $s \ge 1$ such that

$$2(\alpha - \beta)s - 1 - 2|\beta| \ge \gamma. \tag{11}$$

Put

$$N_1 = 1 + \max\{n_1 + s, \frac{sr'}{r' - r}\}. \tag{12}$$

Lemma 3.2.

$$|x_{i,n}| \le f_{i,n} \cdot \ldots \cdot f_{i,n-(s-1)} \le A \left(\frac{1}{n}\right)^{(\alpha-\beta)s}, \tag{13}$$

and

$$\|Q_n x_i\| \le 2/\sqrt{3} f_{i,n+1} \cdot \ldots \cdot f_{i,n-s+1} \le B \left(\frac{1}{n}\right)^{(\alpha-\beta)(s+1)}, \tag{14}$$

where $A, B > 0$ depend on α, β, γ and s, but are independent on i and n satisfing $1 \le i \le rn$, $n \ge N_1$ and N_1 is given by (12).

Proof. Let $1 \le i \le rn$ and $n \ge N_1$. Because of the choice of N_1, if $m \in \{n, ..., n-s\}$ then $m \ge n_1$ and $1 \le i \le rn \le r'm$. So, we apply lemma 3.1 to obtain the following estimates:

$$|x_{i,n}| \le f_{i,n}|x_{i,n-1}| \le f_{i,n}f_{i,n-1}|x_{i,n-2}| \le \cdots \le f_{i,n}f_{i,n-1} \cdot \ldots \cdot f_{i,n-(s-1)}|x_{i,n-s}| \le f_{i,n}f_{i,n-1} \cdot \ldots \cdot f_{i,n-(s-1)}.$$

Using (10) we finally obtain (13).
To prove (14) notice that

$$|x_{i,k}| \le f_{i,k}f_{i,k-1} \cdot \ldots \cdot f_{i,n-(s-1)},$$

for $k \ge n$, so

$$\begin{aligned}\|Q_n x_i\|^2 = \sum_{k=n+1}^{\infty} |x_{i,k}|^2 &\le (f_{i,n-s+1}f_{i,n-s+2} \cdot \ldots \cdot f_{i,n+1})^2 \left(1 + f_{i,n+2}^2 + f_{i,n+2}^2 f_{i,n+3}^2 + \cdots\right) \\ &\le (f_{i,n-s+1}f_{i,n-s+2} \cdot \ldots \cdot f_{i,n+1})^2 \left(1 + \frac{1}{4} + (\frac{1}{4})^2 + \cdots\right) \le \frac{4}{3}(f_{i,n-s+1} \cdot \ldots \cdot f_{i,n+1})^2.\end{aligned}$$

Using (10) again we finish the proof of (14). □

Let $[a] = \max\{q \in \mathbb{N} : \ q \le a\}$ be an integer part of the real number a.
Let $n \ge N_1$ and $k \le rn$, then

$$\|L^{(k,n)}\| \le \sum_{i=1}^{k} \|Q_n x_i\|^2 \le \sum_{i=1}^{[rn]} \|Q_n x_i\|^2 \le rnB^2 \left(\frac{1}{n}\right)^{2(\alpha-\beta)(s+1)} \le rB^2 \left(\frac{1}{n}\right)^{2(\alpha-\beta)(s+1)-1} \le B^2 \left(\frac{1}{n}\right)^{\gamma},$$

where the last inequality holds because of (11).
Lemma 3.2 yields the following estimate:

$$|c_n|(\sum_{i=1}^{k} |x_{i,n+1}|^2)^{1/2} \le Sn^{\beta}(\sum_{i=1}^{[rn]} |x_{i,n+1}|^2)^{1/2} \le Sn^{\beta}\left(rnA^2\left(\frac{1}{n}\right)^{2(\alpha-\beta)s}\right)^{1/2} = Sr^{1/2}A\left(\frac{1}{n}\right)^{(\alpha-\beta)s-1/2-\beta} \le SA\left(\frac{1}{n}\right)^{\gamma/2}$$

and, similarly, we prove:

$$(\sum_{j=1}^{k} |x_{j,n}|^2)^{1/2} \le A\left(\frac{1}{n}\right)^{\gamma/2},$$

if $k \le rn$ and $n \ge N_1$.
Now we are going to estimate $\sum_{i=1}^{k} |\lambda_k - \lambda_i|^2 \|Q_n x_i\|^2$. Proposition (2.1) implies that if $k > i$ then

$$\lambda_k - \lambda_i \le \lambda_k - \lambda_1 \le \delta(k^{\alpha} + p_k) + 2q_k - \lambda_1 \le C_2 k^{\alpha},$$

where $C_2 > 0$ is a constant large enough. Let take $n \geq N_1$ and $1 \leq k \leq rn$; we may write the inequalities:

$$\sum_{i=1}^{k} |\lambda_k - \lambda_i|^2 \|Q_n x_i\|^2 \leq \sum_{i=1}^{k} C_2^2 k^{2\alpha} \|Q_n x_i\|^2 \leq rn C_2^2 (rn)^{2\alpha} B^2 \left(\frac{1}{n}\right)^{2(\alpha-\beta)(s+1)}$$
$$\leq r^{1+2\alpha} C_2^2 B^2 \left(\frac{1}{n}\right)^{2(\alpha-\beta)(s+1)-2\alpha-1} \leq C_2^2 B^2 \left(\frac{1}{n}\right)^{\gamma}.$$

The above estimates and Lemma 2.2 enable us to estimate the following matrix norm

$$\|M^{(k,n)} + \lambda_k L^{(k,n)}\| \leq SA^2 \left(\frac{1}{n}\right)^{\gamma} + B^2 C_2 \left(\frac{1}{n}\right)^{\gamma} \leq \tilde{C} \left(\frac{1}{n}\right)^{\gamma},$$

where $n \geq N_1$, $1 \leq k \leq rn$ and $\tilde{C}$ is a suitable constant independent of n.
Choose $N \geq N_1$ large enough for the inequality

$$\|L^{(k,n)}\| \leq \frac{B^2}{n^{\gamma}} \leq 1/2$$

to hold for $n \geq N$. By Lemma 2.1, we obtain

$$0 \leq \mu_{k,n} - \lambda_k \leq \tilde{C} \left(\frac{1}{n}\right)^{\gamma} \cdot (1 - 1/2)^{-1} = 2\tilde{C} \frac{1}{n^{\gamma}},$$

where $n \geq N$ and $k \in \{1, ..., [rn]\}$. Finally, the proof is complete.

4. Applications

In this section we consider some consequences of Theorem 1.1. In Theorem 1.1 we descibe the relation between the point spectrum of a Jacobi matrix, that acts in l^2, and appropriate eigenvalues of finite submatrices. Thus we can focus on the problem of finding asymptotic formulas for eigenvalues of finite matrices. We still assume $(A1)$ and $(A2)$. Additionaly, we formulate the following condition

$(A3)$ there exist $\omega \in \mathbb{R}, \rho > 0$ and $\mathrm{n}_0 \geq 1$ such that $d_n - d_{n-1} - |c_n| - 2|c_{n-1}| - |c_{n-2}| \geq \rho n^{\omega}, \quad n \geq n_0.$

We need this condition to investigate where eigenvalues of the finite matrices (2) are located.
Denote

$$R_n = |c_n| + |c_{n-1}|,\ n \geq 1,\ (c_0 = 0), \tag{15}$$

$$K_n = \{x \in \mathbb{R} : |d_n - x| \leq R_n\}. \tag{16}$$

Let

$$M = \max\{d_i + |c_i| + |c_{i-1}|\ :\ 0 \leq i \leq n_0\}.$$

Under $(A1)$ and $(A2)$ $\lim_{n\to\infty}(d_n - |c_n| - |c_{n-1}|) = +\infty$; therefore, there exists $K \in \mathbb{N}$ such that

$$d_n - |c_n| - |c_{n-1}| > M \text{ for } n \geq K.$$

Put

$$N_0 = \max\{K, n_0\} + 2. \tag{17}$$

Remark 4.1.
Under $(A1) - (A3)$, if $n \geq N_0$ then $K_n \cap \left(\bigcup_{m\neq n} K_m\right) = \emptyset$.

4.1. Localization of the point spactrum of J in sense of the Gerschgorin theorem

The Gerschgorin theorem, known for matrices, is generalized by Shivakumar, Williams and Rudraiah in [20] for a wide class of discrete operators acting in the vector spaces l^1 or l^∞. The result below needs very strong assumptions but it can be treated as a consequence of Theorem 1.1 for operators in l^2.

Proposition 4.1.

Let J be an operator in l^2 given by Jacobi matrix (1) and R_n and K_n be given by (15) and (16), respectively. Under conditions (A1) *and* (A2)

$$\sigma_p(J) \subset \bigcup_{n=1}^{\infty} K_n.$$

Proof. Let $\lambda_k \in \sigma_p(J)$. We apply Theorem 1.1 for $r = \frac{1}{2}$ and $\gamma = 1$. There exists $C > 0$ such that $|\lambda_k - \mu_{k,n}| \leq \frac{C}{n}$ for $n > 2k$.
Let $s_0 \geq 1$ be such that

$$\delta n^\alpha - (\delta p_n + R_n) \geq \frac{\delta}{2} n^\alpha, \quad \text{for} \quad n \geq s_0.$$

Using Proposition 2.1, we claim that there exists $M > 0$ such that

$$\lambda_k + \frac{C}{n} \leq \delta k^\alpha + (\delta p_k + 2q_k) + 1 \leq M\delta k^\alpha, \quad \text{for} \quad n \geq k_0 = \max\{C, 2k\}.$$

The value $\mu_{k,n}$ is an eigenvalue of the real symmetric matrix J_n, so $\mu_{k,n} \in K_{s_n}$ for some $1 \leq s_n \leq n$, by the Gershgorin theorem applied to J_n. Then we observe that $d_{s_n} - R_{s_n} \leq \mu_{k,n} \leq \lambda_k + \frac{C}{n}$, for $n \geq k_0$. Thus $s_n \leq s_0$ or

$$M\delta k^\alpha \geq \mu_{k,n} \geq d_{s_n} - R_{s_n} \geq \delta s_n^\alpha - (\delta p_{s_n} + R_{s_n}) \geq \frac{\delta}{2} s_n^\alpha.$$

It means that $s_n \leq \max\{s_0, (2M)^{1/\alpha} k\}$. So, $\{s_n\}$ is a bounded sequence of integers and it contains a constant subsequence $s_{n_l} = s$, for $l \geq 1$. Then

$$d_s - R_s \leq \mu_{k,n_l} \leq d_s + R_s, \quad l \geq 1$$

and sending l to $+\infty$ ($n_l \to +\infty$), we obtain $\lambda_k = \lim_{l\to\infty} \mu_{k,n_l} \in K_s$. □

Proposition 4.2.

Assume (A1), (A2) *and* (A3). *Then $\lambda_n \in K_n$ for $n \geq N_0$, where N_0 is defined by (17).*

Proof. Choose an integer $p \geq 2$ and $n \geq N_0$. Consider the finite matrix J_N, where $N \geq pn$. It is clear $\sigma_p(J_N) = \{\mu_{1,N}, \mu_{2,N}, ..., \mu_{N,N}\} \subset \bigcup_{s=1}^{N} K_s$, because of the Gerschgorin theorem. Let $D = \bigcup_{i=1}^{m} K_{s_i}$ and we assume D is disjoint with all other Gerschgorin intervals (discs) K_j, for $j \neq s_i$, $i = 1, ..., m$. Applying Theorem 3.12 from the book of Y. Saad ([19]), we affirm D contains exactly m eigenvalues of J_N. Due to Remark 4.1, if $n \geq N_0$ then the Gerschgorin interval K_n is disjoint from all other Gerschgorin intervals, so it contains exactly one eigenvalue. Moreover, we observe that $\{\mu_{1,N}, \mu_{2,N}, ..., \mu_{n-1,N}\} \subset \sum_{s=1}^{n-1} K_s$ and $\mu_{n,N} \in K_n$ ($n \geq N_0$, $N \geq pn$). Thus, applying Theorem 1.1, we obtain $\lambda_n = \lim_{N\to\infty} \mu_{n,N} \in K_n$, for $n \geq N_0$. □

Remark 4.2.

Under (A1), (A2) and (A3), if $N \geq n \geq N_0$ then $\mu_{n,N} \in K_n$.

Proof. (Y. Saad [19]). □

4.2. Asymptotic behaviour of eigenvalues of J

As a corollary of Theorem 1.1 we may obtain some results related to the asymptotic behaviour of eigenvalues $\{\lambda_n\}$, with $n\to\infty$. The methods, applied for example in [4–6, 12, 14, 16, 18] and [25] use the idea of diagonalization. We are going to show an alternative approach to the problem of asymptotics.

Proposition 4.3.
Assume that $p\geq 2$ is an integer and $\gamma>0$. If J is an operator in l^2 given by matrix (1), that satisfies $(A1)$ and $(A2)$, then

$$|\mu_{n,pn}-\lambda_n| = O\left(\frac{1}{n^\gamma}\right),\quad n\to\infty,$$

where λ_n is the n-th eigenvalue of J, $\mu_{n,pn}$ is the n-th eigenvalue of the finite matrix J_{pn} of dimension $pn\times pn$ and the constant in "O" depends on J, p, γ but is independent on n.

Proof. Applying Theorem 1.1 for $\tilde{n}=pn$ and $r=\frac{1}{p}$ we have

$$0\leq \mu_{n,pn}-\lambda_n = \mu_{[r\tilde{n}],\tilde{n}}-\lambda_{[r\tilde{n}]} \leq \frac{\tilde{C}}{\tilde{n}^\gamma} = \frac{\tilde{C}}{p^\gamma}\left(\frac{1}{n}\right)^\gamma .$$

□

From Proposition 4.3 we derive

$$\lambda_n = \mu_{n,2n}+O\left(\frac{1}{n^\gamma}\right),\quad n\to\infty,$$

where $\gamma>0$ is properly chosen. So, the asymptotic behaviour for the sequence $\{\lambda_n\}_{n=1}^\infty$ of the eigenvalues of J can be discovered by a suitable estimate for $\mu_{n,2n}$, which is an eigenvalue of the finite matrix J_{2n}. We are going to see that this is possible under assumptions $(A1)-(A3)$. Denote

$$J_l^k = \begin{pmatrix} d_k & c_k & & \\ c_k & d_{k+1} & \ddots & \\ & & \ddots & c_{l-1} \\ & \ddots & c_{l-1} & d_l \end{pmatrix},\quad D_l^k(\lambda)=\det(J_l^k-\lambda),\ k\leq l,$$

$$P_n(\lambda)=\frac{D_{n-3}^1(\lambda)}{D_{n-2}^1(\lambda)},\quad S_n(\lambda)=\frac{D_{2n}^{n+3}(\lambda)}{D_{2n}^{n+2}(\lambda)} \tag{18}$$

and

$$\rho_n = |c_{n+1}|+|c_n|+|c_{n-1}|+|c_{n-2}| = R_{n+1}+R_{n-1}. \tag{19}$$

Define

$$\alpha_n = d_n-\frac{c_{n-1}^2}{d_{n-1}-d_n}-\frac{c_n^2}{d_{n+1}-d_n},\quad n>1. \tag{20}$$

Lemma 4.1.
Assume $(A1)-(A3)$. Let $\mu_{n,2n}$ be the n-th eigenvalue of J_{2n}.
Then

$$|\mu_{n,2n}-\alpha_n|\leq c_{n-1}^2\frac{R_n+c_{n-2}^2/(d_n-d_{n-2}-\rho_{n-1})}{(d_n-d_{n-1}-R_n-c_{n-2}^2/(d_n-d_{n-2}-\rho_{n-1}))(d_n-d_{n-1})}$$
$$+c_n^2\frac{R_n+c_{n+1}^2/(d_{n+2}-d_n-\rho_{n+1})}{(d_{n+1}-d_n-R_n-c_{n+1}^2/(d_{n+2}-d_n-\rho_{n+1}))(d_{n+1}-d_n)},$$

where $n>N_0$ (N_0 is determined by (17)).

Proof. Let $\lambda = \mu_{n,2n}$, then $\lambda \in K_n$ due to Remark 4.2. J^1_{n-2} is a real symmetric matrix, so $\|J^1_{n-2}\| = \mu_{n-2,n-2} \in K_{n-2}$. Let $x \in \mathbb{R}^{n-2}$, $\|x\| = 1$ and $n > N_0$, then

$$\begin{aligned}\|(J^1_{n-2} - \lambda)x\| &\geq \lambda - \|J^1_{n-2}x\| \geq \lambda - \|J^1_{n-2}\| \geq d_n - R_n - d_{n-2} - R_{n-2} \\ &= d_n - d_{n-2} - \rho_{n-1} = d_n - d_{n-1} + d_{n-1} - d_{n-2} - \rho_{n-1} \geq \rho n^\omega + \rho(n-1)^\omega + 2R_{n-1} > 0,\end{aligned} \tag{21}$$

because of (17) and (*A*3). Thus

$$\|(J^1_{n-2} - \lambda)^{-1}\| \leq (d_n - d_{n-2} - \rho_{n-1})^{-1}.$$

Now, notice that if we denote $(J^1_{n-2} - \lambda)^{-1} = (b_{i,j})^{n-2}_{i,j=1}$ then $P_n(\lambda) = b_{n-2,n-2}$ and, obviously,

$$|P_n(\lambda)| = |b_{n-2,n-2}| \leq \|(J^1_{n-2} - \lambda)^{-1}\| \leq (d_n - d_{n-2} - \rho_{n-1})^{-1}. \tag{22}$$

Similarly we estimate

$$\|(J^1_{n-1} - \lambda)x\| \geq \rho n^\omega \|x\|, \;\; x \in \mathbb{R}^{n-1}.$$

Thus both of the matrices $J^1_{n-2} - \lambda$ and $J^1_{n-1} - \lambda$ are invertible and $D^1_{n-1}(\lambda) \neq 0$, $D^1_{n-2}(\lambda) \neq 0$.
The estimation for $S_n(\lambda)$ can be similarly derived. Indeed, notice that

$$\mu^* = \min\{\mu : \; \mu \text{ is an eigenvalue of } J^{n+2}_{2n}\} = \min\{(J^{n+2}_{2n}w, w) : \; w \in \mathbb{R}^{n-1}, \; \|w\| = 1\} \in K_{n+2}.$$

Then, for $x \in \mathbb{R}^{n-1}$, $\|x\| = 1$,

$$\|(J^{n+2}_{2n} - \lambda)x\| \geq \|J^{n+2}_{2n}x\| - \lambda \geq \mu^* - \lambda \geq d_{n+2} - d_n - \rho_{n+1} > 0$$

and

$$\|(J^{n+2}_{2n} - \lambda)^{-1}\| \leq (d_{n+2} - d_n - \rho_{n+1})^{-1}.$$

In this case, if $(J^{n+2}_{2n} - \lambda)^{-1} = (\tilde{b}_{i,j})^{n-1}_{i,j=1}$ then

$$|S_n(\lambda)| = |\tilde{b}_{1,1}| \leq \|(J^{n+2}_{2n} - \lambda)^{-1}\| \leq (d_{n+2} - d_n - \rho_{n+1})^{-1}. \tag{23}$$

The analogous argument we use to obtain

$$\|(J^{n+1}_{2n} - \lambda)x\| \geq (d_{n+1} - R_{n+1} - d_n - R_n)\|x\| \geq \rho(n+1)^\omega \|x\|$$

for all $x \in \mathbb{R}^n$. Then $J^{n+2}_{2n} - \lambda$ and $J^{n+1}_{2n} - \lambda$ are also invertible and $D^{n+1}_{2n}(\lambda) \neq 0$, $D^{n+2}_{2n}(\lambda) \neq 0$ for $n \geq N_0$.
From the Laplace formula, applied to $J_{2n} = J^1_{2n}$, we derive

$$D^1_{2n}(\lambda) = (d_n - \lambda)D^1_{n-1}(\lambda)D^{n+1}_{2n}(\lambda) - c^2_{n-1}D^1_{n-2}(\lambda)D^{n+1}_{2n}(\lambda) - c^2_n D^1_{n-1}(\lambda)D^{n+2}_{2n}(\lambda).$$

Notice that $\lambda = \mu_{n,2n}$ is an eigenvalue of J^1_{2n}, so $D^1_{2n}(\lambda) = 0$ and, equivalently, λ satisfies

$$\lambda = d_n - c^2_{n-1}\frac{D^1_{n-2}(\lambda)}{D^1_{n-1}(\lambda)} - c^2_n \frac{D^{n+2}_{2n}(\lambda)}{D^{n+1}_{2n}(\lambda)}. \tag{24}$$

We apply the Laplace formula again to obtain

$$D^1_{n-1}(\lambda) = (d_{n-1} - \lambda)D^1_{n-2}(\lambda) - c^2_{n-2}D^1_{n-3}(\lambda),$$

$$D_{2n}^{n+1}(\lambda) = (d_{n+1} - \lambda)D_{2n}^{n+2}(\lambda) - c_{n+1}^2 D_{2n}^{n+3}(\lambda).$$

Using (18), we can rewrite (24) as

$$\lambda = d_n - \frac{c_{n-1}^2}{d_{n-1} - \lambda - c_{n-2}^2 P_n(\lambda)} - \frac{c_n^2}{d_{n+1} - \lambda - c_{n+1}^2 S_n(\lambda)}. \tag{25}$$

Using (15), (19) and (21) we have

$$0 \le \frac{|c_{n-2}|}{d_n - d_{n-2} - \rho_{n-1}} \le \frac{|c_{n-2}|}{\rho n^\omega + \rho(n-1)^\omega + 2R_{n-1}} \le \frac{|c_{n-2}|}{R_{n-1}} \le 1, \ n > N_0.$$

Next, from $(A3)$ and the estimates above we derive

$$\begin{aligned} d_n - d_{n-1} - R_n - c_{n-2}^2(d_n - d_{n-2} - \rho_{n-1})^{-1} &\ge d_n - d_{n-1} - R_n - |c_{n-2}| \\ = d_n - d_{n-1} - R_n - R_{n-1} + |c_{n-1}| &\ge \rho n^\omega + |c_{n-1}| > 0 \end{aligned} \tag{26}$$

for all $n > N_0$. From (21) and (A3) we deduce $d_{n+2} - d_n - \rho_{n+1} > 2R_{n+1}$, so

$$0 \le \frac{|c_{n+1}|}{d_{n+2} - d_n - \rho_{n+1}} \le \frac{|c_{n+1}|}{2R_{n+1}} \le 1.$$

Then for $n > N_0$ the following estimates can be applied:

$$\begin{aligned} d_{n+1} - d_n - R_n - c_{n+1}^2(d_{n+2} - d_n - \rho_{n+1})^{-1} &\ge d_{n+1} - d_n - R_n - |c_{n+1}| \\ = d_{n+1} - d_n - R_n - R_{n+1} + |c_n| &\ge \rho(n+1)^\omega + |c_n| > 0. \end{aligned} \tag{27}$$

Let $n > N_0$ and $r_n = \lambda - d_n = \mu_{n,2n} - d_n \in [-R_n, R_n]$, using by turns (25), (20), (22), (23), (26) and (27) we obtain

$$\begin{aligned} |\lambda - \alpha_n| &\le c_{n-1}^2 \left| \frac{-r_n - c_{n-2}^2 P_n(\lambda)}{(d_{n-1} - d_n - r_n - c_{n-2}^2 P_n(\lambda))(d_{n-1} - d_n)} \right| + c_n^2 \left| \frac{-r_n - c_{n+1}^2 S_n(\lambda)}{(d_{n+1} - d_n - r_n - c_{n+1}^2 S_n(\lambda))(d_{n+1} - d_n)} \right| \\ &\le c_{n-1}^2 \frac{|r_n| + c_{n-2}^2 |P_n(\lambda)|}{(|d_{n-1} - d_n| - |r_n| - c_{n-2}^2 |P_n(\lambda)|)|d_{n-1} - d_n|} + c_n^2 \frac{|r_n| + c_{n+1}^2 |S_n(\lambda)|}{(|d_{n+1} - d_n| - |r_n| - c_{n+1}^2 |S_n(\lambda)|)(d_{n+1} - d_n)} \\ &\le c_{n-1}^2 \frac{R_n + c_{n-2}^2 (d_n - d_{n-2} - \rho_{n-1})^{-1}}{(d_n - d_{n-1} - R_n - c_{n-2}^2 (d_n - d_{n-2} - \rho_{n-1})^{-1})(d_n - d_{n-1})} \\ &\quad + c_n^2 \frac{R_n + c_{n+1}^2 (d_{n+2} - d_n - \rho_{n+1})^{-1}}{(d_{n+1} - d_n - R_n - c_{n+1}^2 (d_{n+2} - d_n - \rho_{n+1})^{-1})(d_{n+1} - d_n)}. \end{aligned}$$

This completes the proof. □

Example 4.1.

Let J be an operator in l^2 such that $d_n = n$, $n \ge 1$ and $\lim_{n\to\infty} c_n = 0$. Assume $\lambda_1, \lambda_2, \lambda_3, \dots$ are the non-decreasingly ordered eigenvalues of J. It is clear that $(A1) - (A3)$ are satisfied with $\alpha = 1$ and, e.g., $\beta = \omega = 0$. By Proposition 4.2 we can observe $|\lambda_n - n| \le |c_n| + |c_{n-1}|$, $n > N_0$. Now, assume that $\{c_n\}$ converges to 0 not very fast, i.e., $\liminf_{n\to\infty} n^\gamma \rho_n^3 > 0$ for some $\gamma > 0$, where ρ_n is given by (19). Moreover, $|\mu_{n,2n} - \lambda_n| = O\left(\frac{1}{n^\gamma}\right)$, due to Proposition 4.3. Then Lemma 4.1 implies that there exists $M > 0$ such that

$$|\mu_{n,2n} - \alpha_n| \le M\rho_n^3, \ n \ge 1,$$

where, according to (20),

$$\alpha_n = n + c_{n-1}^2 - c_n^2.$$

Finally, we obtain the following asymptotics for the point spectrum of J :

$$\lambda_n = n + c_{n-1}^2 - c_n^2 + O(\rho_n^3), \ n \to \infty.$$

Example 4.2.

In this example, let J be associated with Jacobi matrix (1), for which $d_n = \delta n^2(1+\delta_n)$, $|c_n| \le Cn^{\beta}$, $n \ge 1$. We assume $\beta < \frac{2}{3}$ and $\delta_n = o(1)$, $\delta_n - \delta_{n-1} = o(\frac{1}{n})$, $n \to \infty$, where the notation "small o" ($o(1)$, $o(\frac{1}{n})$) has an usual asymptotic meaning. Then from Lemma 4.1, Proposition 4.2 and Proposition 4.3 we derive the asymptotics for the eigenvalues of J

$$\lambda_n = d_n - \frac{c_{n-1}^2}{d_{n-1}-d_n} - \frac{c_n^2}{d_{n+1}-d_n} + O\left(\frac{1}{n^{2-3\beta}}\right), \ n \to \infty.$$

Also, it can be interesting to observe that if $\beta = 0$ then

$$\lambda_n = d_n + \frac{c_{n-1}^2 - c_n^2}{2\delta n} + o\left(\frac{1}{n}\right), \ n \to \infty.$$

The conditions, supposed in this example, are satisfy by concrete Jacobi matrices, which appear in various applications (see [4],[5],[14],[23]).

Example 4.3.

Now assume a weaker condition for $\{c_n\}$ then in the previous example. Let

$$|c_n| \le Cn^{\frac{2}{3}}(\ln(n+1))^{-1}, \ n \ge 1.$$

As before, let $d_n = \delta n^2(1+\delta_n)$, $\delta_n \to 0$, and $\delta_n - \delta_{n-1} = o(\frac{1}{n})$, $n \to \infty$. Clearly, $(A1), (A2)$ are satisfied with $\alpha = 2$ and $\beta = \frac{2}{3}$. Condition $(A3)$ is satisfied with $\omega = 1$. So, we apply Proposition 4.3 and Lemma 4.1 and obtain the following asymptotic formula for the point spectrum of the operator J

$$\lambda_n = d_n - \frac{c_{n-1}^2}{d_{n-1}-d_n} - \frac{c_n^2}{d_{n+1}-d_n} + O\left(\frac{1}{(\ln(n+1))^3}\right), \ n \to \infty.$$

Example 4.4.

Let consider a general case of $\{d_n\}$, $d_n = \delta n^{\alpha}(1+\delta_n)$, $\alpha > 1$. We assume $(A1)$, $(A2)$ and, additionally, $\alpha > \beta + 1$. The assumption $\delta_n - \delta_{n-1} = o(\frac{1}{n})$, $n \to \infty$, is also needed. Then $(A3)$ is satisfied with $\omega = \alpha - 1$. We apply Proposition 4.3 with $p = 2$ and $\gamma > 0$ large enough. Let $\lambda = \mu_{n,2n}$ be the n-th eigenvalue of J_{2n}. From Lemma 4.1 we derive

$$|\lambda - \alpha_n| \le Mn^{-2(\alpha-1-\frac{3}{2}\beta)}, \quad n \ge N_0, \tag{28}$$

where α_n is defined by (20) and M is a constant independent on n. Thus we obtain the asymptotic behaviour for the eigenvalues of J

$$\lambda_n = \alpha_n + O(n^{-2(\alpha-1-\frac{3}{2}\beta)}), \ n \to \infty. \tag{29}$$

Notice that this asymptotic is true without any additional assumption on the sign of the expression $\alpha - 1 - \frac{3}{2}\beta$. For n large enough ($n > N_0 + 2$, where N_0 is given by (17)) we can write an expanded version of equation (24)

$$\lambda = d_n - c_{n-1}^2\left(d_{n-1} - \lambda - c_{n-2}^2/(d_{n-2} - \lambda - c_{n-3}^2\tilde{P}_n(\lambda))\right)^{-1} - c_n^2\left(d_{n+1} - \lambda - c_{n+1}^2/(d_{n+2} - \lambda - c_{n+2}^2\tilde{S}_n(\lambda))\right)^{-1}, \tag{30}$$

where $\tilde{P}_n(\lambda) = \frac{D_{n-4}^1(\lambda)}{D_{n-3}^1(\lambda)}$ and $\tilde{S}_n(\lambda) = \frac{D_{2n}^{n+4}(\lambda)}{D_{2n}^{n+3}(\lambda)}$. Using the method from the proof of Lemma 4.1 we obtain the estimates

$$|\tilde{P}_n(\lambda)| \le \frac{C}{n^{\alpha-1}}, \quad |\tilde{S}_n(\lambda)| \le \frac{C}{n^{\alpha-1}}, \quad n > N_0 + 2. \tag{31}$$

Denote

$$\tilde{\alpha}_n = d_n - \frac{c_{n-1}^2}{d_{n-1} - \alpha_n - c_{n-2}^2/(d_{n-2} - \alpha_n)} - \frac{c_n^2}{d_{n+1} - \alpha_n - c_{n+1}^2/(d_{n+2} - \alpha_n)}, \tag{32}$$

where α_n is given by (20). Elementary calculations and (28)-(32) lead to

$$|\lambda - \tilde{\alpha}_n| \le M_1 n^{-4(\alpha-1-\frac{5}{4}\beta)}, \quad n > N_0 + 2,$$

where $M_1 > 0$ is a constant independet on n. If we combine this with Proposition 4.3, we obtain the following asymptotics for the point spectrum of J

$$\lambda_n = \tilde{\alpha}_n + O(n^{-4(\alpha-1-\frac{5}{4}\beta)}), \; n \to \infty.$$

The above asymptotics is more exact then that one in (29). This procedure could be repeated to obtain even better exactness of the asymptotic behaviour of the eigenvalues of the operator J.

To obtain the result concerning an asymptotic behaviour of the point spectrum of the operator J we use $p = 2$ but if we choose $p > 2$ there is no difference in the final result. Notice that in the left-hand site of the inequality in Lemma 4.1 we may put $\mu_{n,pn}$ instead of $\mu_{n,2n}$, but the formula on the right remains without any change. Also the parameter γ, that appears in Proposition 4.3, is arbitrary chosen and is independent on p. A constant, which can be associated with the expression $O(\frac{1}{n^\gamma})$, depends on p, but this fact does not change the asymptotics as $n \to \infty$.

4.3. Approximation for eigenvectors of J

Proposition 4.4.
Let J be an operator in l^2 associated with (1). Assume (A1), (A2) and (A3) hold. Let $r \in (0, 1)$ and $\gamma > 0$. Let λ_k be the k-th eigenvalue of the operator J and $x_k \in l^2$ be an eigenvector that corresponds to λ_k. Let $y_{k,n} \in \mathbb{R}^n$, and $\mu_{k,n}$ be the k-th eigenvector and eigenvalue of J_n, given by (2),respectively, and assume that (5) holds. Then there exists $M > 0$ such that

$$\sup_{1\le k\le rn} \|P_n x_k - y_{k,n}\| \le M n^{-\frac{1}{2}(\gamma+1)}, \; n \ge 1.$$

Proof. This proof is based on lemma 2.3. Without loos of generality, we can assume $c_n \neq 0$, $n \ge 1$, and $\gamma > \max\{\omega', -1 + 2\omega'\}$, where $\omega' = \max\{0, -\omega\}$ and ω is determined in $(A3)$. Then the multiplicities of all eigenvalues of J and J_n equal one. From Proposition 4.2 and $(A3)$ we derive

$$\lambda_{k+1} - \lambda_k \ge d_{k+1} - d_k - |c_{k+1}| - 2|c_k| - |c_{k-1}| \ge \rho k^\omega, \quad k \ge N_0$$

and, obviously,

$$\lambda_{k+1} - \lambda_k \ge \min\{\lambda_{i+1} - \lambda_i : i \in \{1, ..., N_0\}\} = \rho_1 > 0, \; k \le N_0.$$

If $1 \le i < k \le rn$ then

$$|\mu_{i,n} - \lambda_k| \ge \lambda_k - \lambda_i - |\mu_{i,n} - \lambda_i| \ge \lambda_k - \lambda_{k-1} - \frac{C}{n^\gamma},$$

because of Theorem 1.1. If $i > k$ then

$$\mu_{i,n} - \lambda_k \ge \mu_{k+1,n} - \lambda_k \ge \lambda_{k+1} - \lambda_k.$$

Thus

$$|\mu_{i,n} - \lambda_k| \ge \min\{\rho k^\omega, \rho(k+1)^\omega, \rho_1\} - \frac{C}{n^\gamma} \ge \rho_2 k^{-\omega'} - \frac{C}{n^\gamma},$$

where $i \neq k$, $1 \le i \le n$, $1 \le k \le rn$, $k \ge 1$, and $\rho_2 > 0$ is a suitable constant. Because $\gamma > \omega'$, we observe there exist $\tilde{\rho} > 0$ and N large enough such that if $1 \le k \le rn$ then

$$\Delta_{k,n} = \max\{|\mu_{i,n} - \lambda_k|^{-1} : 1 \le i \le n, \; i \neq k\} \le \tilde{\rho}^{-1} n^{\omega'},$$

where $n \geq N$. Let $s \in \mathbb{N}$ satisfy (11). Additionally, we enlarge s to satisfy the inequality

$$2(\alpha-\beta)(s+1) - 2(|\beta| + \omega') > \gamma + 1.$$

From lemmata 2.3, 3.1 and 3.2 we derive the estimates

$$\Delta_{k,n}^2 K_{k,n}^2 \leq \Delta_{k,n}^2 c_n^2 f_{k,n+1}^2 |x_{k,n}|^2 \leq \tilde{\rho}^{-2} n^{2\omega'} S^2 n^{2\beta} a^2 A n^{-2(\alpha-\beta)(s+1)} \leq C' n^{-2(\alpha-\beta)(s+1)+2\beta+2\omega'} \leq C' n^{-\gamma-1},$$

for $n > \max\{N, N_1\}$, where N_1 is taken from lemma 3.2. Finally, from lemmata 2.3 and 3.2 we derive

$$\|P_n x_k - y_{k,n}\|^2 \leq C' n^{-\gamma-1} + (B^2 n^{-2(\alpha-\beta)(s+1)} + C' n^{-\gamma-1})^2 \leq M n^{-\gamma-1},$$

where $1 \leq k \leq rn$. □

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