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# The Case of the Homogeneous Errors-In-Variables Model

**Abstract:** Recently, it has been claimed that the Homogeneous Errors-In-Variables (HEIV) Model, where the left-hand side (LHS) vector is allowed to be multiplied with an unknown scale factor, would represent a *generalization* of the regular EIV-Model for which a number of efficient algorithms already exist. Unfortunately, due to the *forced rank deficiency* in the case of the HEIV-Model, an additional constraint needs to be introduced to guarantee uniqueness of the TLS solution ("datum constraint"). If this constraint is *linear*, a simple manipulation will reduce the HEIV-Model with one constraint to the regular EIV-Model. But also in the case of a *non-linear* datum constraint, by introducing parameter ratios as unknowns, an EIV-Model may result that can be treated by standard TLS adjustment, followed by a solution of the datum constraint for the additional LHS scale parameter. This approach will be applied to an example where the datum constraint is chosen to be quadratic.

**Keywords:** Errors-In-Variables Model: Homogeneous vs. regular, Total Least-Squares adjustment: With and without datum constraint, model equivalency

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## Introduction

In recent contributions by Yeredor and De Moor (2004), Inkilä (2005), and Matei and Meer (2006) the authors claim or imply that the Homogeneous Errors-In-Variables (HEIV) Model is more general than the regular EIV-Model insofar as an additional unknown scale factor is introduced. By doing this, however, the HEIV-Model will have, at least, a rank deficiency of 1, which needs to be compensated by one additional "datum constraint" that must be taken into account when generating the (unique) TLS solution in this case.

Here it will be argued that such a procedure is unnecessarily cumbersome as two alternative approaches may be followed that are based on standard TLS adjustments in the regular EIV-Model. First, if the datum constraint is *linear*, it can be solved for the additional scale parameter which then may be back-substituted into the HEIV-Model so that a regular EIV-Model results.

In case of a *nonlinear* datum constraint, parameter ratios may be introduced to form a regular EIV-Model in which a standard TLS adjustment can be performed. Introducing the estimated ratios into the constraint, allows the additional scale parameter to be solved for.

After a short review of the regular EIV-Model, the HEIV-Model with a datum constraint will be introduced, and the unique Total Least-Squares (TLS) solution will be presented in chapter 1. Then, chapter 2 will be devoted to two alternative approaches to this solution via TLS adjustment within EIV-Models. In chapter 3, a simple example for an HEIV-Model with a *quadratic* datum constraint will be presented before some conclusions will be drawn in chapter 4.

## 1 A review of various Errors-In-Variables Models

### 1.1 The regular EIV-Model

Let the regular EIV-Model be defined through

$$\boldsymbol{y} - \boldsymbol{e}_y = \underbrace{(\boldsymbol{A} - \boldsymbol{E}_A)}_{n\times m}\boldsymbol{\xi},\ \operatorname{rk}\boldsymbol{A} = m < n, \tag{1a}$$

$$\begin{bmatrix} \boldsymbol{e}_y \\ \boldsymbol{e}_A := \operatorname{vec}\boldsymbol{E}_A \end{bmatrix} \sim \left( \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}, \sigma_0^2 \begin{bmatrix} \boldsymbol{Q}_y & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q}_A \end{bmatrix} = \sigma_0^2 \begin{bmatrix} \boldsymbol{P}_y^{-1} & 0 \\ 0 & \boldsymbol{P}_A^{-1} \end{bmatrix} \right), \tag{1b}$$

where

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| | |
|---|---|
| $\boldsymbol{y}$ | denotes the $n \times 1$ observation vector, |
| $\boldsymbol{A}$ | the $n \times m$ data matrix of full column rank, |
| $\boldsymbol{e}_y$ | the $n \times 1$ (unknown) random error vector associated with $\boldsymbol{y}$, |
| $\boldsymbol{E}_A$ | the $n \times m$ (unknown) random error matrix associated with $\boldsymbol{A}$, |
| $\boldsymbol{\xi}$ | the $m \times 1$ (unknown) parameter vector, |
| $\boldsymbol{e}_A$ | the $nm \times 1$ vectorized form of the random error matrix $\boldsymbol{E}_A$, |
| $\sigma_0^2$ | the (unknown) variance component, only one for the sake of simplicity, |
| $\boldsymbol{Q}_y = \boldsymbol{P}_y^{-1}$ | the $n \times n$ symmetric positive-definite cofactor matrix of $\boldsymbol{e}_y$, and |
| $\boldsymbol{Q}_A = \boldsymbol{P}_A^{-1}$ | the $nm \times nm$ symmetric positive-definite cofactor matrix of $\boldsymbol{e}_A := \operatorname{vec} \boldsymbol{E}_A$. |

Here, vec denotes the operation that transforms the $n \times m$ matrix $\boldsymbol{E}_A$ into the $nm \times 1$ vector $\boldsymbol{e}_A$ by stacking all the columns underneath each other going from left to right.

The (weighted) Total Least-Squares (TLS) solution is now based on the objective

$$\boldsymbol{e}_y^T \boldsymbol{P}_y \boldsymbol{e}_y + \boldsymbol{e}_A^T \boldsymbol{P}_A \boldsymbol{e}_A = \min \quad \text{subject to Eq. (1a)}, \tag{2}$$

or, equivalently, on the Lagrange function

$$\Phi(\boldsymbol{e}_y, \boldsymbol{e}_A, \boldsymbol{\xi}, \boldsymbol{\lambda}) := \boldsymbol{e}_y^T \boldsymbol{P}_y \boldsymbol{e}_y + \boldsymbol{e}_A^T \boldsymbol{P}_A \boldsymbol{e}_A + 2\boldsymbol{\lambda}^T \left[\boldsymbol{y} - \boldsymbol{e}_y - (\boldsymbol{A} - \boldsymbol{E}_A)\boldsymbol{\xi}\right], \tag{3}$$

that needs to be made stationary. Consequently, the necessary Euler-Lagrange conditions are obtained as

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{e}_y} = \boldsymbol{P}_y \tilde{\boldsymbol{e}}_y - \hat{\boldsymbol{\lambda}} \doteq \boldsymbol{0} \qquad \Rightarrow \tilde{\boldsymbol{e}}_y = \boldsymbol{Q}_y \hat{\boldsymbol{\lambda}}, \tag{4a}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{e}_A} = \boldsymbol{P}_A \tilde{\boldsymbol{e}}_A + (\hat{\boldsymbol{\xi}} \otimes \boldsymbol{I}_n)\hat{\boldsymbol{\lambda}} \doteq \boldsymbol{0} \qquad \Rightarrow \tilde{\boldsymbol{e}}_A = -\boldsymbol{Q}_A(\hat{\boldsymbol{\xi}} \otimes \hat{\boldsymbol{\lambda}}), \tag{4b}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{\xi}} = \left(\boldsymbol{A} - \tilde{\boldsymbol{E}}_A\right)^T \hat{\boldsymbol{\lambda}} \doteq \boldsymbol{0} \qquad \Rightarrow \boldsymbol{A}^T \hat{\boldsymbol{\lambda}} = \\ = \tilde{\boldsymbol{E}}_A^T \hat{\boldsymbol{\lambda}} = \operatorname{vec}(\tilde{\boldsymbol{E}}_A^T \hat{\boldsymbol{\lambda}}) = \operatorname{vec}(\hat{\boldsymbol{\lambda}}^T \tilde{\boldsymbol{E}}_A) = \left(\boldsymbol{I}_m \otimes \hat{\boldsymbol{\lambda}}\right)^T \tilde{\boldsymbol{e}}_A, \tag{4c}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{\lambda}} = \boldsymbol{y} - \tilde{\boldsymbol{e}}_y - \boldsymbol{A}\hat{\boldsymbol{\xi}} + \left(\hat{\boldsymbol{\xi}}^T \otimes \boldsymbol{I}_n\right)\tilde{\boldsymbol{e}}_A \doteq \boldsymbol{0} \Rightarrow \boldsymbol{y} - \boldsymbol{A}\hat{\boldsymbol{\xi}} = \\ = \left[\boldsymbol{Q}_y + \left(\hat{\boldsymbol{\xi}} \otimes \boldsymbol{I}_n\right)^T \boldsymbol{Q}_A \left(\hat{\boldsymbol{\xi}} \otimes \boldsymbol{I}_n\right)\right]\hat{\boldsymbol{\lambda}} =: \boldsymbol{Q}_1 \cdot \hat{\boldsymbol{\lambda}}. \tag{4d}$$

Here, $\otimes$ denotes the Kronecker-Zehfuss product of matrices, which is defined by

$$\boldsymbol{G} \otimes \boldsymbol{H} := [g_{ij} \cdot \boldsymbol{H}] \quad \text{if} \quad \boldsymbol{G} = [g_{ij}], \tag{5a}$$

with the key property being

$$\operatorname{vec}(\boldsymbol{A}\boldsymbol{B}\boldsymbol{C}^T) = (\boldsymbol{C} \otimes \boldsymbol{A}) \operatorname{vec} \boldsymbol{B}. \tag{5b}$$

From Eq. (4d), an estimated vector of the Lagrange multipliers is immediately obtained as

$$\boxed{\hat{\boldsymbol{\lambda}} = \boldsymbol{Q}_1^{-1}\left(\boldsymbol{y} - \boldsymbol{A}\hat{\boldsymbol{\xi}}\right) = \left[\boldsymbol{Q}_y + (\hat{\boldsymbol{\xi}} \otimes \boldsymbol{I}_n)^T \boldsymbol{Q}_A (\hat{\boldsymbol{\xi}} \otimes \boldsymbol{I}_n)\right]^{-1}(\boldsymbol{y} - \boldsymbol{A}\hat{\boldsymbol{\xi}}).} \tag{6}$$

It is noted that the matrix $\boldsymbol{Q}_1$ is a function of the solution vector $\hat{\boldsymbol{\xi}}$, and so is the "compound residual vector"

$$\boldsymbol{y} - \boldsymbol{A}\hat{\boldsymbol{\xi}} = \tilde{\boldsymbol{e}}_y - (\hat{\boldsymbol{\xi}} \otimes \boldsymbol{I}_n)^T \tilde{\boldsymbol{e}}_A = \tilde{\boldsymbol{e}}_y - \tilde{\boldsymbol{E}}_A \hat{\boldsymbol{\xi}}, \tag{7a}$$

which, however, can be separated in accordance with Eqs. (4a) and (4b) into

$$\tilde{\boldsymbol{e}}_y = \boldsymbol{Q}_y \cdot \boldsymbol{Q}_1^{-1}(\boldsymbol{y} - \boldsymbol{A}\hat{\boldsymbol{\xi}}), \tag{7b}$$

$$\text{and } \tilde{\boldsymbol{e}}_A = -\boldsymbol{Q}_A(\hat{\boldsymbol{\xi}} \otimes \boldsymbol{I}_n) \cdot \boldsymbol{Q}_1^{-1}(\boldsymbol{y} - \boldsymbol{A}\hat{\boldsymbol{\xi}}), \tag{7c}$$

$$\text{respectively } \tilde{\boldsymbol{E}}_A = \operatorname{Invec} \tilde{\boldsymbol{e}}_A, \tag{7d}$$

with Invec as inverse vec-operation.

The solution vector $\hat{\boldsymbol{\xi}}$ thus has to come from Eq. (4c) via

$$\boldsymbol{A}^T \boldsymbol{Q}_1^{-1}\left(\boldsymbol{y} - \boldsymbol{A}\hat{\boldsymbol{\xi}}\right) = \boldsymbol{A}^T \hat{\boldsymbol{\lambda}} = \tilde{\boldsymbol{E}}_A^T \hat{\boldsymbol{\lambda}} = \tilde{\boldsymbol{E}}_A^T \boldsymbol{Q}_1^{-1}(\boldsymbol{y} - \boldsymbol{A}\hat{\boldsymbol{\xi}}). \tag{8a}$$

After ordering terms in $\boldsymbol{y}$ and in $\hat{\boldsymbol{\xi}}$, while subtracting identical terms on both sides, this leads to the *non-linear normal equations*

$$\boxed{(\boldsymbol{A} - \tilde{\boldsymbol{E}}_A)^T \boldsymbol{Q}_1^{-1}(\boldsymbol{A} - \tilde{\boldsymbol{E}}_A) \cdot \hat{\boldsymbol{\xi}} = (\boldsymbol{A} - \tilde{\boldsymbol{E}}_A)^T \boldsymbol{Q}_1^{-1}(\boldsymbol{y} - \tilde{\boldsymbol{E}}_A \hat{\boldsymbol{\xi}}),} \tag{8b}$$

which generate $\hat{\boldsymbol{\xi}}$ as a function of $\tilde{\boldsymbol{E}}_A$, resp. $\hat{\boldsymbol{\lambda}}$. Here, Eq. (6) and Eq. (8b) are to be solved consecutively by iteration. The system Eq. (8b) can first be found in the PhD thesis of Fang (2011), variations of it also in Mahboub (2012) and Xu et al. (2012), while a simplified version for the case of $\boldsymbol{Q}_A = \boldsymbol{Q}_0 \otimes \boldsymbol{Q}_x$ had been developed by Schaffrin and Wieser (2008).

Note that the Total Sum of Squared Residuals (TSSR) is easily determined from Eqs. (4a) and (4b) by

$$\text{TSSR} = \tilde{\boldsymbol{e}}_y^T \boldsymbol{Q}_y^{-1} \tilde{\boldsymbol{e}}_y + \tilde{\boldsymbol{e}}_A^T \boldsymbol{Q}_A^{-1} \tilde{\boldsymbol{e}}_A = \\ = \hat{\boldsymbol{\lambda}}^T \left[\boldsymbol{Q}_y + (\hat{\boldsymbol{\xi}} \otimes \boldsymbol{I}_n)^T \boldsymbol{Q}_A (\hat{\boldsymbol{\xi}} \otimes \boldsymbol{I}_n)\right]\hat{\boldsymbol{\lambda}} = \\ = (\boldsymbol{y} - \boldsymbol{A}\hat{\boldsymbol{\xi}})^T \boldsymbol{Q}_1^{-1}(\boldsymbol{y} - \boldsymbol{A}\hat{\boldsymbol{\xi}}) = \hat{\boldsymbol{\lambda}}^T(\boldsymbol{y} - \boldsymbol{A}\hat{\boldsymbol{\xi}}), \tag{9a}$$

so that a (not necessarily unbiased) estimate of the variance component can be obtained through

$$\boxed{\hat{\sigma}_0^2 = \hat{\boldsymbol{\lambda}}^T(\boldsymbol{y} - \boldsymbol{A}\hat{\boldsymbol{\xi}})/(n - m).} \tag{9b}$$

Furthermore, Amiri-Simkooei and Jazaeri (2012) proposed a first-order approximation for the dispersion matrix of $\hat{\boldsymbol{\xi}}$ as

$$D\{\hat{\boldsymbol{\xi}}\} = \sigma_0^2 \cdot \left[(\boldsymbol{A} - \tilde{\boldsymbol{E}}_A)^T \boldsymbol{Q}_1^{-1}(\boldsymbol{A} - \tilde{\boldsymbol{E}}_A)\right]^{-1}, \tag{10a}$$

which, in our opinion, requires further substantiation, as their underlying assumption

$$D\{\boldsymbol{y} - \tilde{\boldsymbol{E}}_A\hat{\boldsymbol{\xi}}\} = D\{\boldsymbol{y} - (\hat{\boldsymbol{\xi}} \otimes \boldsymbol{I}_n)\tilde{\boldsymbol{e}}_A\} \approx \approx \sigma_0^2 \cdot \left[\boldsymbol{Q}_y + (\hat{\boldsymbol{\xi}} \otimes \boldsymbol{I}_n)^T \boldsymbol{Q}_A (\hat{\boldsymbol{\xi}} \otimes \boldsymbol{I}_n)\right] = \sigma_0^2 \boldsymbol{Q}_1 \tag{10b}$$

may *not be valid* since $D\{\tilde{\boldsymbol{e}}_A\} \neq \sigma_0^2 \boldsymbol{Q}_A = D\{\boldsymbol{e}_A\}$.

## 1.2 The Homogeneous EIV-Model with datum constraint

Now, let the LHS of Eq. (1a) only be given up to an unknown scale factor $\xi_0$ (say). Then, due to

$$\boldsymbol{y} - \boldsymbol{e}_y \in \mathcal{R}(\boldsymbol{A} - \boldsymbol{E}_A), \tag{11}$$

the HEIV-Model will read

$$\boldsymbol{0} = \underbrace{\left[\boldsymbol{y} - \boldsymbol{e}_y \mid \boldsymbol{A} - \boldsymbol{E}_A\right]}_{n\times(m+1)} \begin{bmatrix}\xi_0 \\ \boldsymbol{\xi}\end{bmatrix}, \ \mathrm{rk}\left[\boldsymbol{y} - \boldsymbol{e}_y \mid \boldsymbol{A} - \boldsymbol{E}_A\right] = m, \tag{12a}$$

$$\begin{bmatrix}\boldsymbol{e}_y \\ \boldsymbol{e}_A = \mathrm{vec}\,\boldsymbol{E}_A\end{bmatrix} \sim \left(\begin{bmatrix}\boldsymbol{0} \\ \boldsymbol{0}\end{bmatrix}, \sigma_0^2 \begin{bmatrix}\boldsymbol{Q}_y & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q}_A\end{bmatrix} = \sigma_0^2 \begin{bmatrix}\boldsymbol{P}_y^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}_A^{-1}\end{bmatrix}\right), \tag{12b}$$

which needs one additional *datum constraint*

$$f(\xi_0, \boldsymbol{\xi}) = 1, \quad f : \mathbb{R}^{m+1} \to \mathbb{R} \text{ differentiable}, \tag{12c}$$

in order to guarantee a *unique TLS solution* that might be derived from the Lagrange function

$$\begin{aligned}\Phi(\boldsymbol{e}_y, \boldsymbol{e}_A, \boldsymbol{\xi}, \xi_0, \boldsymbol{\lambda}, \mu) &:= \\ &:= \boldsymbol{e}_y^T \boldsymbol{P}_y \boldsymbol{e}_y + \boldsymbol{e}_A^T \boldsymbol{P}_A \boldsymbol{e}_A + 2\boldsymbol{\lambda}^T \left[\boldsymbol{y} - \boldsymbol{e}_y \mid \boldsymbol{A} - \boldsymbol{E}_A\right] \begin{bmatrix}\xi_0 \\ \boldsymbol{\xi}\end{bmatrix} \\ &\quad - \mu \cdot [f(\xi_0, \boldsymbol{\xi}) - 1]\end{aligned} \tag{13}$$

once it is made stationary. Note that, for the sake of simplicity, only one variance component is introduced in Eq. (12b). The necessary Euler-Lagrange conditions then result in

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{e}_y} = \boldsymbol{P}_y \tilde{\boldsymbol{e}}_y - \hat{\boldsymbol{\lambda}} \cdot \hat{\xi}_0 \doteq \boldsymbol{0} \ \Rightarrow \ \tilde{\boldsymbol{e}}_y = \boldsymbol{Q}_y \hat{\boldsymbol{\lambda}} \cdot \hat{\xi}_0, \tag{14a}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{e}_A} = \boldsymbol{P}_A \tilde{\boldsymbol{e}}_A - (\hat{\boldsymbol{\xi}} \otimes \boldsymbol{I}_n)\hat{\boldsymbol{\lambda}} \doteq \boldsymbol{0} \ \Rightarrow \ \tilde{\boldsymbol{e}}_A = \boldsymbol{Q}_A(\hat{\boldsymbol{\xi}} \otimes \hat{\boldsymbol{\lambda}}), \tag{14b}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial [\xi_0 \mid \boldsymbol{\xi}^T]^T} = \begin{bmatrix}(\boldsymbol{y} - \tilde{\boldsymbol{e}}_y)^T \\ (\boldsymbol{A} - \tilde{\boldsymbol{E}}_A)^T\end{bmatrix} \hat{\boldsymbol{\lambda}} - \frac{\partial f(\hat{\xi}_0, \hat{\boldsymbol{\xi}})}{\partial [\xi_0 \mid \boldsymbol{\xi}^T]^T} \cdot \hat{\mu} \doteq \boldsymbol{0} \tag{14c}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{\lambda}} = (\boldsymbol{y} - \tilde{\boldsymbol{e}}_y)\hat{\xi}_0 + (\boldsymbol{A} - \tilde{\boldsymbol{E}}_A)\hat{\boldsymbol{\xi}} \doteq \boldsymbol{0}, \tag{14d}$$

$$\frac{\partial \Phi}{\partial \mu} = 1 - f(\hat{\xi}_0, \hat{\boldsymbol{\xi}}) \doteq \boldsymbol{0}. \tag{14e}$$

Using Eqs. (14a) and (14b) in Eq. (14d) yields

$$\boldsymbol{y} \cdot \hat{\xi}_0 + \boldsymbol{A}\hat{\boldsymbol{\xi}} = \left[\hat{\xi}_0^2 \cdot \boldsymbol{Q}_y + (\hat{\boldsymbol{\xi}} \otimes \boldsymbol{I}_n)^T \boldsymbol{Q}_A (\hat{\boldsymbol{\xi}} \otimes \boldsymbol{I}_n)\right] \cdot \hat{\boldsymbol{\lambda}} =: \boldsymbol{Q}_{10} \cdot \hat{\boldsymbol{\lambda}} \tag{15a}$$

and thus (if $\hat{\xi}_0 \neq 0$)

$$\boxed{\hat{\boldsymbol{\lambda}} = \boldsymbol{Q}_{10}^{-1} \cdot (\boldsymbol{y} \cdot \hat{\xi}_0 + \boldsymbol{A}\hat{\boldsymbol{\xi}})} \tag{15b}$$

for the estimated vector of Lagrange multipliers. Again, it is obvious that $\boldsymbol{Q}_{10}$ depends on both $\hat{\xi}_0$ and $\hat{\boldsymbol{\xi}}$. In addition, Eq. (14c) leads first to

$$\boldsymbol{y}^T \hat{\boldsymbol{\lambda}} - (\partial f / \partial \xi_0) \cdot \hat{\mu} = \tilde{\boldsymbol{e}}_y^T \hat{\boldsymbol{\lambda}} = \hat{\xi}_0 \cdot \hat{\boldsymbol{\lambda}}^T \boldsymbol{Q}_y \hat{\boldsymbol{\lambda}} \tag{16a}$$

and

$$\begin{aligned}\boldsymbol{A}^T \hat{\boldsymbol{\lambda}} - (\partial f / \partial \boldsymbol{\xi}) \cdot \hat{\mu} &= \mathrm{vec}(\hat{\boldsymbol{\lambda}}^T \tilde{\boldsymbol{E}}_A) = (\boldsymbol{I}_m \otimes \hat{\boldsymbol{\lambda}}^T)\tilde{\boldsymbol{e}}_A = \\ &= (\boldsymbol{I}_m \otimes \hat{\boldsymbol{\lambda}})^T \boldsymbol{Q}_A (\hat{\boldsymbol{\xi}} \otimes \boldsymbol{I}_n) \cdot \hat{\boldsymbol{\lambda}}\end{aligned} \tag{16b}$$

and then, with Eq. (15b), to the (non-symmetric) *normal equations*

$$\left(\begin{bmatrix}(\boldsymbol{y} - \boldsymbol{Q}_y \hat{\boldsymbol{\lambda}} \cdot \hat{\xi}_0)^T \\ \left[\boldsymbol{A} - (\hat{\boldsymbol{\xi}} \otimes \boldsymbol{I}_n)^T \boldsymbol{Q}_A (\boldsymbol{I}_m \otimes \hat{\boldsymbol{\lambda}})\right]^T\end{bmatrix} \boldsymbol{Q}_{10}^{-1} \left[\boldsymbol{y} \mid \boldsymbol{A}\right]\right) \cdot \begin{bmatrix}\hat{\xi}_0 \\ \hat{\boldsymbol{\xi}}\end{bmatrix} = = \frac{\partial f(\hat{\xi}_0, \hat{\boldsymbol{\xi}})}{\partial [\xi_0 \mid \boldsymbol{\xi}^T]^T} \cdot \hat{\mu} \tag{17}$$

that are to be solved subject to Eq. (14e). This is the formulation by Zhou et al. (2014) who restrict themselves to a *quadratic* constraint though. An *implicit formulation* in analogy to Fang (2011) would instead lead to the (quasi) symmetric system

$$\boxed{\begin{aligned}&\begin{bmatrix}(\boldsymbol{y} - \tilde{\boldsymbol{e}}_y)^T \\ (\boldsymbol{A} - \tilde{\boldsymbol{E}}_A)^T\end{bmatrix} \boldsymbol{Q}_{10}^{-1} \left[\boldsymbol{y} - \tilde{\boldsymbol{e}}_y \middle| \boldsymbol{A} - \tilde{\boldsymbol{E}}_A\right] \begin{bmatrix}\hat{\xi}_0 \\ \hat{\boldsymbol{\xi}}\end{bmatrix} = \frac{\partial f(\hat{\xi}_0, \hat{\boldsymbol{\xi}})}{\partial [\xi_0, \boldsymbol{\xi}^T]^T} \cdot \hat{\mu} \\ &- \begin{bmatrix}(\boldsymbol{y} - \tilde{\boldsymbol{e}}_y)^T \\ (\boldsymbol{A} - \tilde{\boldsymbol{E}}_A)^T\end{bmatrix} \boldsymbol{Q}_{10}^{-1} \left(\tilde{\boldsymbol{e}}_y \cdot \hat{\xi}_0 + \tilde{\boldsymbol{E}}_A \cdot \hat{\boldsymbol{\xi}}\right), \qquad (18a) \\ &f(\hat{\xi}_0, \hat{\boldsymbol{\xi}})) = 1, \qquad (18b)\end{aligned}}$$

where $\tilde{\boldsymbol{e}}_y$ and $\tilde{\boldsymbol{E}}_A$ stem from Eqs. (14a) and (14b) using $\hat{\boldsymbol{\lambda}}$ from Eq. (15b). Note that the matrix in Eq. (18a) may well be singular, owing to the rank deficiency of the matrix $[\boldsymbol{y} - \boldsymbol{e}_y \mid \boldsymbol{A} - \boldsymbol{E}_A]$ because of Eq. (11). But this should as well be true for the system Eq. (17) of Zhou et al. (2014).

In any case, the TSSR value should be unaffected by this singularity and can best be computed via

$$\begin{aligned}\mathrm{TSSR} &= \tilde{\boldsymbol{e}}_y^T \boldsymbol{Q}_y^{-1} \tilde{\boldsymbol{e}}_y + \tilde{\boldsymbol{e}}_A^T \boldsymbol{Q}_A^{-1} \tilde{\boldsymbol{e}}_A = \\ &= \hat{\boldsymbol{\lambda}}^T \left[\hat{\xi}_0^2 \cdot \boldsymbol{Q}_y + (\hat{\boldsymbol{\xi}} \otimes \boldsymbol{I}_n)^T \boldsymbol{Q}_A (\hat{\boldsymbol{\xi}} \otimes \boldsymbol{I}_n)\right] \hat{\boldsymbol{\lambda}} = \\ &= (\boldsymbol{y} \cdot \hat{\xi}_0 + \boldsymbol{A}\hat{\boldsymbol{\xi}})^T \boldsymbol{Q}_{10}^{-1} (\boldsymbol{y} \cdot \hat{\xi}_0 + \boldsymbol{A}\hat{\boldsymbol{\xi}}) = \hat{\boldsymbol{\lambda}}^T (\boldsymbol{y} \cdot \hat{\xi}_0 + \boldsymbol{A}\hat{\boldsymbol{\xi}})\end{aligned} \tag{19a}$$

so that a suitable variance component estimate may be obtained from

$$\boxed{\hat{\sigma}_0^2 = \hat{\boldsymbol{\lambda}}^T(\boldsymbol{y} \cdot \hat{\xi}_0 + \boldsymbol{A}\hat{\boldsymbol{\xi}})/(n-m),} \tag{19b}$$

as the rank deficiency of 1 in Eq. (12a) is compensated by the datum constraint in Eq. (12c).

Obviously, the *regular EIV-Model* is equivalent to the HEIV-Model if and only if the datum constraint is *linear* with $\xi_0 = -1$ or, equivalently, with

$$\kappa_0 \cdot \xi_0 + \boldsymbol{\kappa}^T\boldsymbol{\xi} = 1, \quad \underset{(m+1)\times 1}{\left[\kappa_0 \mid \boldsymbol{\kappa}^T\right]^T} := [-1 \mid 0, \ldots, 0]^T, \tag{20a}$$

or – up to the sign – if it is *quadratic* with $\xi_0^2 = 1$, resp.

$$\begin{aligned} &\left[\xi_0 \mid \boldsymbol{\xi}^T\right]\boldsymbol{M}\begin{bmatrix}\xi_0\\ \boldsymbol{\xi}\end{bmatrix} = 1, \\ &\underset{(m+1)\times(m+1)}{\boldsymbol{M}} = \begin{bmatrix}\kappa_0\\ \boldsymbol{\kappa}\end{bmatrix}\left[\kappa_0 \mid \boldsymbol{\kappa}^T\right] = \mathrm{Diag}(1, 0, \ldots, 0), \ \mathrm{rk}\,\boldsymbol{M} = 1. \end{aligned} \tag{20b}$$

Alternative treatments for the HEIV-Models with a linear or quadratic datum constraint will be presented in the following chapter. For regular EIV-Models with constraints, see, for instance, Schaffrin and Felus (2005) and Schaffrin (2006).

# 2 Transforming the HEIV-Model into an equivalent EIV-Model

## 2.1 The HEIV-Model with a linear datum constraint

In this case the HEIV-Model of Eqs. (12a) and (12b) is augmented with a *linear constraint* of type ($\kappa_0 \neq 0$)

$$\left[\kappa_0 \mid \boldsymbol{\kappa}^T\right]\begin{bmatrix}\xi_0\\ \boldsymbol{\xi}\end{bmatrix} = 1, \quad \text{resp. } \kappa_0 \cdot \xi_0 = 1 - (\boldsymbol{\kappa}^T\boldsymbol{\xi}). \tag{21}$$

Consequently, the scale parameter $\xi_0$ can be replaced by a linear combination of $\boldsymbol{\xi}$, thus leading to

$$\begin{aligned} \boldsymbol{y} - \boldsymbol{e}_y &= (\boldsymbol{y} - \boldsymbol{e}_y)(\kappa_0\xi_0) + (\boldsymbol{y} - \boldsymbol{e}_y)\cdot(\boldsymbol{\kappa}^T\boldsymbol{\xi}) = && (22a)\\ &= -\kappa_0(\boldsymbol{A} - \boldsymbol{E}_A)\boldsymbol{\xi} + (\boldsymbol{y} - \boldsymbol{e}_y)\cdot(\boldsymbol{\kappa}^T\boldsymbol{\xi}) = && (22b)\\ &= \left[(\kappa_0\boldsymbol{A} - \boldsymbol{y}\boldsymbol{\kappa}^T) - (\kappa_0\boldsymbol{E}_A - \boldsymbol{e}_y\boldsymbol{\kappa}^T)\right]\cdot(-\boldsymbol{\xi}) =: \\ &=: (\bar{\boldsymbol{A}} - \bar{\boldsymbol{E}}_A)\cdot(-\boldsymbol{\xi}) && (22c) \end{aligned}$$

with

$$\begin{aligned} &\begin{bmatrix}\boldsymbol{e}_y\\ \bar{\boldsymbol{e}}_A = \mathrm{vec}\,\bar{\boldsymbol{E}}_A\end{bmatrix} \sim \left(\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{0}\end{bmatrix},\right. \\ &\left.\sigma_0^2\left[\begin{array}{c|c} \boldsymbol{Q}_y & -\boldsymbol{Q}_y(\boldsymbol{\kappa}\otimes\boldsymbol{I}_n)^T \\ -(\boldsymbol{\kappa}\otimes\boldsymbol{I}_n)\boldsymbol{Q}_y & \kappa_0^2\boldsymbol{Q}_A + (\boldsymbol{\kappa}\otimes\boldsymbol{I}_n)\boldsymbol{Q}_y(\boldsymbol{\kappa}\otimes\boldsymbol{I}_n)^T \end{array}\right]\right), \end{aligned} \tag{22d}$$

which represents an EIV-Model with *correlation* between the random vectors $\boldsymbol{e}_y$ and $\bar{\boldsymbol{e}}_A$. Such a model can be treated along the lines of Fang (2011) or Snow (2012) where suitable algorithms have been provided.

## 2.2 The HEIV-Model with a quadratic datum constraint

Here, the HEIV-Model Eqs. (12a) and (12b) is augmented with a *quadratic* constraint of type ($\xi_0 \neq 0$).

$$\left[\xi_0 \mid \boldsymbol{\xi}^T\right]\boldsymbol{M}\begin{bmatrix}\xi_0\\ \boldsymbol{\xi}\end{bmatrix} = 1, \quad \text{resp. } \left[1 \mid \xi_0^{-1}\boldsymbol{\xi}^T\right]\boldsymbol{M}\begin{bmatrix}1\\ \boldsymbol{\xi}\xi_0^{-1}\end{bmatrix} = \xi_0^{-2}. \tag{23}$$

In analogy, the model Eq. (12a) can also be rewritten as

$$\boxed{\boldsymbol{y} - \boldsymbol{e}_y = (\boldsymbol{A} - \boldsymbol{E}_A)\cdot(-\boldsymbol{\xi}\xi_0^{-1}),} \tag{24}$$

which, under the specification of Eq. (12b), generates the weighted TLS solution for the vector $(-\boldsymbol{\xi}\xi_0^{-1})$ of (negative) parameter ratios. This TLS solution, $\widehat{\boldsymbol{\xi}\xi_0^{-1}}$, can now be inserted into the second part of Eq. (23) to yield the *TLS estimate of the scale parameter* via

$$\boxed{\hat{\xi}_0 = \pm\sqrt{\left(\left[1 \mid (\widehat{\boldsymbol{\xi}\xi_0^{-1}})^T\right]\boldsymbol{M}\left[1 \mid (\widehat{\boldsymbol{\xi}\xi_0^{-1}})^T\right]^T\right)^{-1}}} \tag{25a}$$

and the final TLS estimate of the parameter vector via

$$\hat{\boldsymbol{\xi}} = (\widehat{\boldsymbol{\xi}\xi_0^{-1}})\cdot\hat{\xi}_0. \tag{25b}$$

Obviously, the sign in Eq. (25a) has to be chosen in such a way that the TSSR – in analogy to Eq. (9a) – is minimal in the end.

It is noted that this approach can oftentimes be applied to the general datum constraint in Eq. (12c) as well, namely by setting

$$f(1, \boldsymbol{\xi}\xi_0^{-1}) = f_0(\xi_0), \quad f_0 : \mathbb{R} \to \mathbb{R} \quad \text{invertible}, \tag{26}$$

and by making sure that the TLS solution to Eq. (24) fulfills this constraint for a certain scale estimate $\hat{\xi}_0$.

# 3 An example from Inkilä (2005): Fitting a straight line to points in 2-D space

In this example, a straight line in 2D-space is fitted to $n = 3$ measured coordinate pairs that have iid random errors. The data are taken from Inkilä (2005) and are listed in Table 1 below.

**Table 1:** Measured 2-D coordinates $(x_i, y_i)$

| $i$-th point | $x_i$ | $y_i$ |
|---|---|---|
| 1 | 1 | 2 |
| 2 | 2 | 6 |
| 3 | 6 | 1 |

The *homogeneous model* for the line can be written as

$$(y_i - e_{y_i}) \cdot \xi_0 + (x_i - e_{x_i}) \cdot \xi_1 + \xi_2 = 0, \quad i = 1, 2, 3,$$
$$\left[e_{y_1}, \ldots, e_{x_3}\right]^T \sim (\mathbf{0}, \sigma_0^2 \boldsymbol{I}_6). \tag{27}$$

This HEIV model can be transformed equivalently to the following *regular EIV model* according to Eq. (24):

$$(y_i - e_{y_i}) = (x_i - e_{x_i}) \cdot (-\xi_1/\xi_0) + (-\xi_2/\xi_0), \quad i = 1, 2, 3,$$
$$\left[e_{y_1}, \ldots, e_{x_3}\right]^T \sim (\mathbf{0}, \sigma_0^2 \boldsymbol{I}_6), \tag{28}$$

where the three original unknowns have been reduced to a $2 \times 1$ vector of negative parameter ratios $-\boldsymbol{\xi} \cdot \xi_0^{-1} := -[\xi_1, \xi_2]^T \cdot \xi_0^{-1}$.

## 3.1 The standard constraint

The term *standard constraint* is used for the case where the scale parameter $\xi_0$ is forced to take on a value of $\pm 1$ by using $\boldsymbol{M} = \mathrm{Diag}(1, 0, 0)$ in the quadratic constraint of Eq. (23).

A system of normal equations in the form of Eq. (8b) can be solved for the vector of parameter ratios in Eq. (28), where the $3 \times 2$ *data matrix* is defined as $\boldsymbol{A} := [\boldsymbol{x} \quad \mathbf{1}_n]$, with $\mathbf{1}_n$ as an $n \times 1$ "summation vector" ($n = 3$). The estimation requires initial parameter approximations, $\xi_0^{(0)}$ and $\boldsymbol{\xi}^{(0)}$, which can be obtained by setting $\xi_0^{(0)} = -1$ and using the Ordinary Least-Squares solution $\tilde{\boldsymbol{\xi}}_{OLS} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{y} = [-0.5, 4.5]^T$ for $\boldsymbol{\xi}^{(0)}$, where the $x$-coordinates appearing in the $3 \times 2$ matrix $\boldsymbol{A} := [\boldsymbol{x} \quad \mathbf{1}_n]$ are treated as independent, errorless variables, and $\boldsymbol{y} = [y_1, y_2, y_3]^T$ are treated as observations having iid random errors.

Based on Eq. (8b), the Total Least-Squares solution for the (negative) estimated parameter ratios is

$$-\widehat{\boldsymbol{\xi}\xi_0^{-1}} = -[\widehat{\xi_1/\xi_0}, \widehat{\xi_2/\xi_0}]^T = -[1, -6]^T. \tag{29}$$

Equation (25a) is used to compute $\hat{\xi}_0$, with $\boldsymbol{M} = \mathrm{Diag}(1, 0, 0)$, resulting in a choice between $\tilde{\xi}_0^{(1)} = -1$ or $\tilde{\xi}_0^{(2)} = 1$, where the value that leads to the smallest *Total Sum of Squared Residuals* (TSSR) must be chosen. It turns out that $\tilde{\xi}_0^{(1)} = -1$ leads to $\mathrm{TSSR}_1 = 7.0$, and $\tilde{\xi}_0^{(2)} = 1$ leads to $\mathrm{TSSR}_2 = 75.0$. Therefore, we must choose $\hat{\xi}_0 = -1$, leading to the final solution

$$\hat{\xi}_0 = -1, \ \hat{\xi}_1 = -1, \ \hat{\xi}_2 = 6, \tag{30}$$

which coincides with the slope ($\hat{\xi}_1$) and intercept ($\hat{\xi}_2$) parameters estimated by Inkilä (2005). Using Eq. (9b), the estimated variance component $\hat{\sigma}_0^2$ is computed by

$$\hat{\sigma}_0^2 = \mathrm{TSSR}/(n - m) = 7.0/(3 - 2) = 7.0, \tag{31}$$

where $m = 2$ is the number of estimated parameter ratios. The data with the fitted line are shown in Figure 1, and the list of residuals (predicted errors), computed from Eqs. (7b) and (7c), follows in Table 2.

**Table 2:** Residuals (predicted errors)

| $i$-th point | $\tilde{e}_{x_i}$ | $\tilde{e}_{y_i}$ |
|---|---|---|
| 1 | -1.5 | -1.5 |
| 2 | 1.0 | 1.0 |
| 3 | 0.5 | 0.5 |

## 3.2 The Hesse normal form

The Hesse normal form of a 2-D line parametrizes the line in terms of a perpendicular vector that begins at the origin of the coordinate system (see Bôcher, 1915, chapters 25 and 26). A line $AB$ is shown in Figure 2 with a perpendicular vector $OQ$ of magnitude $\rho$ and direction $\alpha$ as reckoned counter-clockwise from the positive $x$-axis, $0 \le \alpha < \pi$.

The line $AB$ can be expressed in standard form, analogous to Eq. (27), with parameters $a$, $b$, and $c$ as

$$ax + by + c = 0. \tag{32}$$

Following Bôcher (1915), the same line can be represented by *Hesse's normal form* in terms of parameters $\alpha$ and $\rho$ as

$$x \cos\alpha + y \sin\alpha - \rho = 0. \tag{33}$$

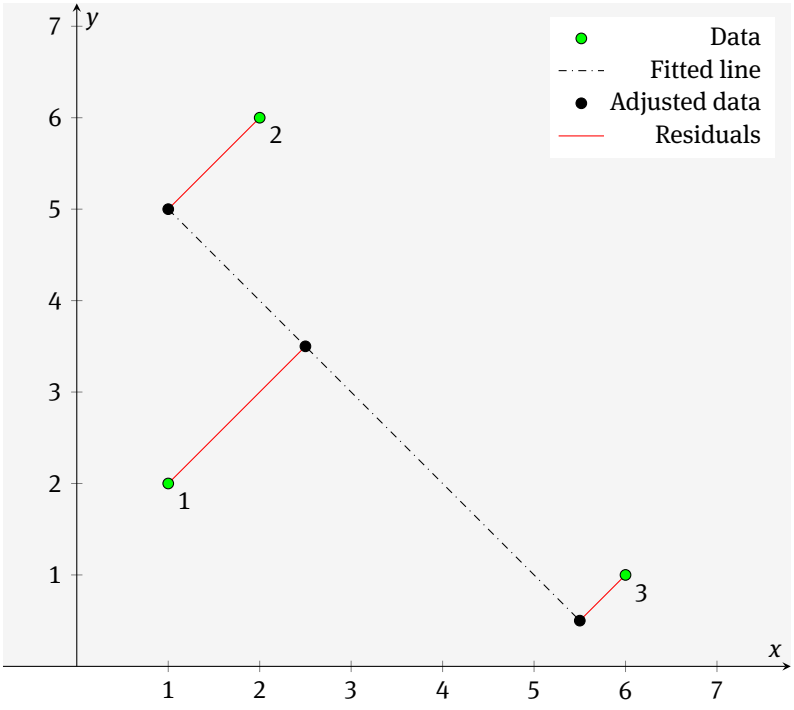


**Figure 1:** Fitted 2-D line

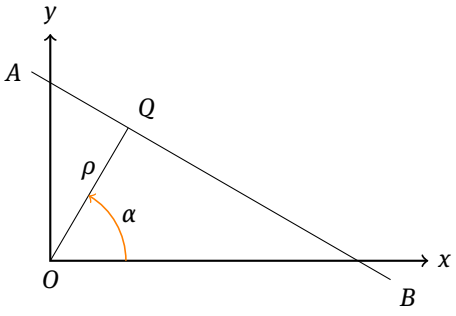


**Figure 2:** Line $AB$ with normal $OQ$

Since both Eqs. (32) and (33) represent the same line, they are equivalent up to a scaling factor $r$, so that

$$ra = \cos\alpha, \quad rb = \sin\alpha, \quad rc = -\rho, \tag{34a}$$

and therefore

$$r^2(a^2 + b^2) = \cos^2\alpha + \sin^2\alpha = 1, \tag{34b}$$

or

$$r = \frac{\pm 1}{\sqrt{a^2 + b^2}}. \tag{34c}$$

Thus, Eq. (32) can be transformed to Hesse's normal form by simply dividing it by $\pm\sqrt{a^2 + b^2}$, with the consequence that

$$\cos\alpha = \frac{\pm a}{\sqrt{a^2 + b^2}}, \quad \sin\alpha = \frac{\pm b}{\sqrt{a^2 + b^2}}, \quad \rho = \frac{\mp c}{\sqrt{a^2 + b^2}}. \tag{35}$$

Note that $\rho$ needs to be chosen as positive in Eq. (35) for $\alpha$ to be as shown in Figure 2; otherwise, if $\rho$ is taken as negative, $\alpha$ will have to be augmented by $\pi$. See Bôcher (1915) chapters 25 and 26 for further discussion.

In the following, rather than exploiting Eq. (35), the parameters $\alpha$ and $\rho$ of Hesse's normal form of a line shall be estimated directly, using the data from Table 1 of the previous section. More specifically, in order to work with a (quasi)linear system, we estimate the trigonometric functions $\sin\alpha$ and $\cos\alpha$, rather than estimate $\alpha$ directly. Thus our $3\times 1$ parameter vector is defined as $[\xi_0 \mid \boldsymbol{\xi}^T]^T := [\sin\alpha \mid \cos\alpha,\ -\rho]^T$. Then, the homogenous EIV model is written as

$$\begin{aligned} &(y_i - e_{y_i}) \cdot \sin\alpha + (x_i - e_{x_i}) \cdot \cos\alpha - \rho = 0, \ \ i = 1, 2, 3, \\ &\left[e_{y_1}, \ \ldots\ , e_{x_3}\right]^T \sim (\mathbf{0}, \sigma_0^2 \boldsymbol{I}_6), \end{aligned} \tag{36a}$$

which is easily transformed to an equivalent regular EIV model by dividing by $\sin\alpha =: \xi_0$ and rearranging terms:

$$\begin{aligned} &(y_i - e_{y_i}) = (x_i - e_{x_i}) \cdot (-\cos\alpha/\sin\alpha) - (-\rho/\sin\alpha), \ \ i = 1, 2, 3, \\ &\left[e_{y_1}, \ \ldots\ , e_{x_3}\right]^T \sim (\mathbf{0}, \sigma_0^2 \boldsymbol{I}_6), \end{aligned} \tag{36b}$$

thereby forming a vector of (negative) parameter ratios $-\boldsymbol{\xi} \cdot \xi_0^{-1} := -[\cos\alpha,\ \rho]^T \cdot (\sin\alpha)^{-1}$ of size $2 \times 1$.

Following the procedures of the previous section, we may solve a system of non-linear normal equations in the form of Eq. (8b) for the vector of unknown parameter ratios, where the matrix $\boldsymbol{M}$ used in the constraint from

Eq. (23) is defined now as $\boldsymbol{M} := \mathrm{Diag}(1, 1, 0)$. The data matrix $\boldsymbol{A}$ differs from that of the previous section by a sign change in the second column, owing to the subtraction of the parameter $\rho$ in the models of Eqs. (36a) and (36b). Thus we have $\boldsymbol{A} := [\boldsymbol{x} \quad -\mathbf{1}_n]$, with $n = 3$.

Using Eq. (35), the following initial parameter approximations are computed:

$$\left[\cos\alpha \cdot (\sin\alpha)^{-1}\right]^{(0)} = a^{(0)}/b^{(0)} = (-0.5)/(-1.0) = 0.5,$$

$$\left[\rho \cdot (\sin\alpha)^{-1}\right]^{(0)} = -c^{(0)}/b^{(0)} = (-4.5)/(-1.0) = 4.5, \quad \text{so that} \quad \left[-\boldsymbol{\xi} \cdot \xi_0^{-1}\right]^{(0)} = [-0.5, -4.5]^T,$$

where the substituted values $a^{(0)} = -0.5$, $b^{(0)} = -1$, and $c^{(0)} = 4.5$ are taken from the initial OLS approximations computed in the previous section.

Based on Eq. (8b), the Total Least-Squares Solution for the (negative) estimated parameter ratios is

$$-\widehat{\boldsymbol{\xi}\xi_0^{-1}} = -[\widehat{\xi_1/\xi_0},\ \widehat{\xi_2/\xi_0}]^T = -[\widehat{\cos\alpha/\sin\alpha},\ \widehat{\rho/\sin\alpha}]^T = \\ = -[1,\ 6]^T. \tag{37}$$

Using Eq. (25a) to compute $\hat{\xi}_0$, with $\boldsymbol{M} = \mathrm{Diag}(1, 1, 0)$, results in

$$\hat{\xi}_0 = \widehat{\sin\alpha} = \pm\sqrt{\left(\left[1 \,\middle|\, \widehat{\cos\alpha/\sin\alpha},\ \widehat{\rho/\sin\alpha}\right] \mathrm{Diag}(1,1,0) \left[1 \,\middle|\, \widehat{\cos\alpha/\sin\alpha},\ \widehat{\rho/\sin\alpha}\right]^T\right)^{-1}} = \\ = \pm\sqrt{(1+1+0)^{-1}} = \pm\sqrt{1/2} \Rightarrow \hat{\xi}_0 = \sqrt{1/2} \quad \text{since} \quad 0 \le \alpha < \pi. \tag{38}$$

This choice results in:

$$\widehat{\sin\alpha} = \sqrt{1/2}, \quad \widehat{\cos\alpha} = \sqrt{1/2} \quad \Rightarrow \hat{\alpha} = \pi/4, \quad \hat{\rho} = 4.243. \tag{39}$$

Substituting these parameter values back into the starting HEIV model Eq. (36a) and using Eqs. (15a), (15b) and (19a) to compute the TSSR, it is found that the same TSSR value of 7.0 is generated as was in the previous section. Apparently this is so because both solutions represent the same normal line; only the datum constraints differed.

Obviously, since $\alpha$ is the angle of the normal line measured counter-clockwise from the positive $x$-axis (Figure 2), then, by setting $b = -1$ in Eq. (32), the slope $a$ of the line $AB$ is determined by $a = -\cot\alpha$, which is confirmed by Eq. (34a). Now, the $y$-intercept $c$ of $AB$ also can be determined from Eq. (34a) since $r(-1) = \sin\alpha \Rightarrow c = \rho/\sin\alpha$. This results in:

$$\hat{a} = \cot\hat{\alpha} = -1, \quad \hat{c} = \hat{\rho}/\widehat{\sin\alpha} = 6.0 \quad \Rightarrow \ \mathrm{TSSR} = 7.0, \tag{40}$$

which nicely confirms the results of the preceding section. It is noted that an early, more "pedestrian" version of the above algorithm can be found in an article by Linkwitz (1976).

# 4 Conclusions

The present findings of chapters 2.1 and 2.2 are best explained by interpreting the results from the examples in chapters 3.1 and 3.2:

The example of chapter 3.1 demonstrates that the (rank-deficient) Homogeneous EIV-Model with a datum constraint offers no advantages over an equivalent (regular) EIV model. In fact, according to chapter 1.2, we know already that the extra parameter $\xi_0$ required by the Homogeneous EIV-Model, when chosen to be $\pm 1$, will cause the model to be equivalent to a regular EIV-Model in the parameter ratios. Thus, it appears that the (rank-deficient) Homogeneous EIV model gives rise only to an additional "nuisance" parameter $\xi_0$, i.e., one that is of no interest.

The example of chapter 3.2 demonstrates that this choice for $\xi_0$ does not restrict the generality; for, even when an HEIV model is such that the scale parameter $\xi_0 \neq \pm 1$, the model can still be successfully transformed to a standard EIV model with essentially the same parameter ratios, and the actual parameters can be determined after the adjustment by use of a suitable datum constraint. Thus, the "increased generality" of the HEIV model evaporates in view of these findings.

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