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On weighted total least squares adjustment for solving the nonlinear problems

Abstract: In the classical geodetic data processing, a nonlinear problem always can be converted to a linear least squares adjustment. However, the errors in Jacob matrix are often not being considered when using the least square method to estimate the optimal parameters from a system of equations. Furthermore, the identity weight matrix may not suitable for each element in Jacob matrix. The weighted total least squares method has been frequently applied in geodetic data processing for the case that the observation vector and the coefficient matrix are perturbed by random errors, which are zero mean and statistically independent with inequality variance. In this contribution, we suggested an approach that employ the weighted total least squares to solve the nonlinear problems and to mitigate the affection of noise in Jacob matrix. The weight matrix of the vector from Jacob matrix is derived by the law of nonlinear error propagation. Two numerical examples, one is the triangulation adjustment and another is a simulation experiment, are given at last to validate the feasibility of the developed method.

Keywords: nonlinear adjustment; nonlinear error propagation; weighted total least squares

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1 Introduction

The nonlinear problem widely existed in geodetic, such as triangulation network adjustment, polynomial regression, et al. These nonlinear systems are always being processed by weighted least squares adjustment after they are linearized, where the Jacob matrix is always considered as the coefficient matrix. Either the Gauss-Newton adjustment and its improved methods or the classical least squares adjustment does not consider the case that the coefficient matrix is also perturbed by random errors, when they are being exploited to compute the parameters of a nonlinear over-determined system. Total least squares, coined by Golub and Van Loan (1980), has been frequently discussed in geodetic (Acar et al. (2006); Davis(1999)) and geographic information science (Yaron A Felus (2004); Felus and Schaffrin (2005)) to solving this problem where both the input variables (coefficient matrix) and the output variables (observation vector) are contaminated by random error. After the nonlinear model being linearized, although the original observations either in left vector or coefficient matrix is homoscedastic, the elements in Jacob matrix may not be as still as normally distributed with identical variance, because each elements may be calculated with different kinds of nonlinear elementary function. Therefore, both in the homoscedastic and heteroscedastic case, the weighted total least squares adjustment (WTLS) is more suitable for the linearized system of nonlinear model. In geodesy, Schaffrin and Wieser (2008) proposed an iterative algorithm for WTLS, where the covariance matrix of elements in coefficient matrix is defined as a Kronecker product of two variance matrices. Obviously, the covariance is constrained with a special structure, namely the elements in each row holds the identical variance. However, this algorithm can be used to solving this problem that the EIV model with singular covariance. In order to overcome this restriction, two improved iterative algorithm were being proposed by Mahboub (2011) and Tong et al. (2011), respectively. An alternative iterative algorithm for Schaffrin-Wieser algorithm also can be found in Shen et al. (2011). Aims to solve the heteroscedastic multivariate model for reference frame transformations, a weighted multivariate TLS method is proposed by Schaffrin and Wieser(2009). In addition, for these models, a number of approaches inherit from TLS have been widely discussed by geodesy

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scientist, such as constrained TLS by Schaffrin and Felus (2009), TLS with condition equations by Schaffrin and Wieser (2011) and robust TLS by Amiri-Simkooei and Jazaeri (2013) .There an alternative solution of WTLS derived by standard least squares method without using the Lagrange approach one can find in Jazaeri et al. (2013) and Amiri-Simkooei and Jazaeri (2012). Moreover, the law of linear error propagation used to calculate the covariance of coefficient matrix is also discussed by Jazaeri et al. (2013).

In this contribution, the weighted total least squares method is introduced to solving nonlinear geodetic problems. The law of nonlinear error propagation is used to calculate the covariance matrix of Jacob matrix. Two experiments are utilized to demonstrate the usefulness of proposed method.

2 A simple summary to the WTLS adjustment

Let the WTLS observation model being described as

$$\mathbf{b} - \mathbf{r} = (\mathbf{A} - \mathbf{E})x. \tag{1}$$

Where $\mathbf{b}$ is the $m \times 1$ observation vector (or left vector) perturbed by a $m\times 1$ random noise vector $\mathbf{r}$; $\mathbf{A}$ is a $m\times n(m > n)$ coefficient matrix (or right matrix) perturbed by a $m \times n$ random noise matrix $\mathbf{E}$; x is a $n \times 1$ parameters vector. Following the discussion in Mahboub (2011), Schaffrin and Wieser (2008) and Shen et al. (2011), the stochastic model for WTLS can be described as

$$\begin{bmatrix} \mathbf{r} \\ \mathbf{e} \end{bmatrix} := \begin{bmatrix} \mathbf{r} \\ vec(\mathbf{E}) \end{bmatrix} \sim \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \sigma_0^2 \begin{bmatrix} \mathbf{Q}_b & \mathbf{O} \\ \mathbf{O} & \mathbf{Q}_{\mathrm{A}} \end{bmatrix} \right) \tag{2}$$

Here, $\mathbf{Q}_b$ and $\mathbf{Q}_{\mathrm{A}}$ are the covariance of $\mathbf{r}$ and $vec(\mathbf{E})$ with the sizes of $m \times m$ and $mn \times mn$, respectively; σ_0^2 is the unknown variance component. Hence the weight matrix is computed by $\mathbf{P_b} = \mathbf{Q_b^{-1}}$ and $\mathbf{P_A} = \mathbf{Q_A^{-1}}$. In the absence of correlation between $\mathbf{r}$ and $\mathbf{e}$, WTLS seeks to solve the optimization problem as follows (Jazaeri et al. (2013)),

$$\mathbf{r^T P_b r} + \mathbf{e^T P_A e} = \min \tag{3}$$

subject to Eq. (1).

In geodetic, iterative algorithmic is the most frequently approach being used to solving WTLS. Schaffrin and Wieser (2008), in previous years, proposed an iterative algorithm, which derived by traditional Lagrange approach, to optimize the following target function,

$$\Phi(\mathbf{r}, \mathbf{e}, \lambda, x) = \mathbf{r^T P_b r} + \mathbf{e^T P_A e} + 2\lambda^{\mathbf{T}}(\mathbf{b} - \mathbf{r} - (\mathbf{A} - \mathbf{E})x) \tag{4}$$

Schaffrin and Wieser (2008) defined the weight matrix

$$\mathbf{P_A} = \mathbf{Q_A^{-1}} = (\mathbf{Q}_0 \otimes \mathbf{Q}_a)^{-1} \tag{5}$$

where $\mathbf{Q}_0$ has size $n \times n$, and $\mathbf{Q}_a$ has size $m \times m$, which restricted on a particular structure. An alternative iterative algorithm was developed by Mahboub (2011) with a more relax weight matrix, more suitable for structured coefficient matrix case. Following above conditions, a series of necessary equations can be obtained as follows:

$$0.5 \times \left.\frac{\partial \Phi}{\partial \mathbf{r}}\right|_{\frown{\mathbf{r}},\frown{\mathbf{e}},\hat{\lambda},\hat{x}} = \mathbf{P_b}\frown{\mathbf{r}} - \hat{\lambda} = 0 \tag{6}$$

$$0.5 \times \left.\frac{\partial \Phi}{\partial \mathbf{e}}\right|_{\frown{\mathbf{r}},\frown{\mathbf{e}},\hat{\lambda},\hat{x}} = \mathbf{P_A}\frown{\mathbf{e}} + \left(\hat{x} \otimes \mathbf{I}_m\right)\hat{\lambda} = 0 \tag{7}$$

$$0.5 \times \left.\frac{\partial \Phi}{\partial \lambda}\right|_{\frown{\mathbf{r}},\frown{\mathbf{e}},\hat{\lambda},\hat{x}} = \mathbf{b} - \frown{\mathbf{r}} - (\mathbf{A} - \frown{\mathbf{E}})\hat{x} = 0 \tag{8}$$

$$0.5 \times \left.\frac{\partial \Phi}{\partial x}\right|_{\frown{\mathbf{r}},\frown{\mathbf{e}},\hat{\lambda},\hat{x}} = -(\mathbf{A} - \frown{\mathbf{E}})^T\hat{\lambda} = 0 \tag{9}$$

Hats indicate estimated vectors and frowns indicate predicted ones. As derived in Mahboub (2011), if the inverse of $\mathbf{R}_1$ existed, the parameters vector can be computed by

$$\hat{x} = \left(\mathbf{A^T R_1 A} + \mathbf{R_2 A}\right)^{-1} \left(\mathbf{A^T R_1} + \mathbf{R_2}\right)\mathbf{b} \tag{10}$$

$$\mathbf{R}_1 = \left(\mathbf{Q_b} + (\hat{x}^T \otimes \mathbf{I}_m)\mathbf{Q_A}(\hat{x} \otimes \mathbf{I}_m)\right)^{-1} \tag{11}$$

$$\mathbf{R_2} = \left((\mathbf{I}_n \otimes \hat{\lambda}^T)\mathbf{Q_A}(\hat{x} \otimes \mathbf{I}_m)\right)\mathbf{R_1} \tag{12}$$

With above formulates, the following algorithm has been proposed by Mahboub (2011) as

1st step:

$$\hat{x}^{(0)} = (\mathbf{A^T P_b A})^{-1}\mathbf{A^T P_b b} \tag{13}$$

2nd step:

$$\mathbf{R}_1^{(i)} = \left(\mathbf{Q_b} + (\hat{x}^{(i-1)T} \otimes \mathbf{I}_m)\mathbf{Q_A}(\hat{x}^{(i-1)} \otimes \mathbf{I}_m)\right)^{-1} \tag{14}$$

$$\hat{\lambda}^{(i)} = \mathbf{R}_1^{(i)}(\mathbf{b} - \mathbf{A}\hat{x}^{(i-1)}) \tag{15}$$

$$\mathbf{R}_2^{(i)} = \left((\mathbf{I}_n \otimes \hat{\lambda}^{(i)T})\mathbf{Q_A}(\hat{x}^{(i-1)} \otimes \mathbf{I}_m)\right)\mathbf{R}_1^{(i)} \tag{16}$$

$$\hat{x}^{(i)} = \left(\mathbf{A^T R}_1^{(i)}\mathbf{A} + \mathbf{R}_2^{(i)}\mathbf{A}\right)^{-1}\left(\mathbf{A^T R}_1^{(i)} + \mathbf{R}_2^{(i)}\right)\mathbf{b} \tag{17}$$

3rd step: until $\left\|\hat{x}^{(i)} - \hat{x}^{(i-1)}\right\| \leq \varepsilon$, ε is the given tolerance factor; calculate the estimated variance component and the residual vector as follows,

$$\hat{\sigma}_0^2 = \frac{\left(\hat{\lambda}^{(i)}\right)^T \left(\mathbf{R}_1^{(i)}\right)^{-1} \hat{\lambda}^{(i)}}{m - n} \tag{18}$$

$$\widehat{r} = \mathbf{Q}_b \hat{\lambda}^{(i)} \quad (19)$$

$$\widehat{e} = -\mathbf{Q}_A \left(\hat{\xi}^{(i)} \otimes \mathbf{I}_m\right) \hat{\lambda}^{(i)} \quad (20)$$

3 Compute the covariance matrix

The observation function of a nonlinear problem can be expressed as

$$\mathbf{L} \approx f(\mathbf{X}, \xi), \quad (21)$$

and the corresponding error functions can be represented as

$$\mathbf{L} + \mathbf{V} = f(\mathbf{X}, \xi). \quad (22)$$

Where, $\mathbf{L}$ is $m \times n$ observation vector noised by error vector $\mathbf{V}(m \times n)$; $\mathbf{X}$ often is the $m \times n$ input vector, which noised by random error in some regression models, but here we assume it is error free; ξ is the $q \times 1$ parameters vector. Therefore, as long as the approximation value of ξ is given, the nonlinear model can be linearized as

$$f_j(\mathbf{X}_j, \hat{\xi}^{\mathrm{T}}) = f_j + \left(\frac{df_j}{d\xi_1}\Delta\xi_1 + \frac{df_j}{d\xi_2}\Delta\xi_2 + \ldots + \frac{df_j}{d\xi_t}\Delta\xi_t\right) \quad (23)$$

where, $f_j = f_j(\mathbf{X}_j, \xi^0)$ and $\frac{df_j}{d\xi_k} = \left.\frac{df_j}{d\xi_k}\right|_{\xi=\xi^0}$, as well as $j = 1, 2, \ldots, m$. Submitting Eq. (23) into Eq. (22) obtain

$$\mathbf{b} - \mathbf{r} = (\mathbf{A} - \mathbf{E})\Delta\xi. \quad (24)$$

Here, $\mathbf{b} = \mathbf{L} - \mathbf{F}$, $\mathbf{r} = -\mathbf{V}$ and $\mathbf{F} = (f_1, f_2, \ldots, f_m)^T$; $\mathbf{A}$ is the Jacob matrix affected by the error matrix $\mathbf{E}$ which raise from the residual of approximation value ξ^0. Therefore, Eq. (24) can be solved by weighted total least squares approach if known the covariance of Jacob matrix.

According to the law of nonlinear error propagation, we obtain the following covariance matrix

$$\widehat{\mathbf{Q}}_{\mathbf{A}} = \mathbf{B}\mathbf{Q}_{\hat{\xi}\hat{\xi}}\mathbf{B}^{\mathbf{T}}. \quad (25)$$

where $\mathbf{B} = \left.\frac{\partial\{vec(\mathbf{A})\}}{\partial\xi}\right|_{\xi=\xi^i}$ with the size $mn \times q$. In the first step we does not know the definitely value of $\mathbf{Q}_{\hat{\xi}\hat{\xi}}$, thus it should be computed by the WLS results. However, when the value of $\hat{\xi}$ being estimated by WTLS, according to the results of Amiri-Simkooei and Jazaeri (2012) and Jazaeri et al. (2013), $\mathbf{Q}_{\hat{\xi}\hat{\xi}}$ is calculated by

$$\mathbf{Q}_{\hat{\xi}\hat{\xi}} = ((\mathbf{A}\text{-}\mathbf{E})^{\mathrm{T}}\mathbf{R}_1(\mathbf{A}\text{-}\mathbf{E}))^{-1}. \quad (26)$$

Only the second partial existed and nonzero the cofactor matrix can hold the mathematical sense, and only the residual of parameters being taken into account here.

4 An improved WTLS algorithm for nonlinear problem

Following the designed algorithm by Mahboub (2011), Schaffrin and Wieser (2008) and the formulas derived in previous section, an improved algorithm for nonlinear problem can be designed as follows.

1st step: Given the initial value $\xi^{(0)}$, calculate

$$\Delta\hat{\xi}^{(0)} = (\mathbf{A}^{(0)\mathbf{T}}\mathbf{P_b}\mathbf{A}^{(0)})^{-1}\mathbf{A}^{(0)\mathbf{T}}\mathbf{P_b}\mathbf{b}^{(0)} \quad (27)$$

and

$$\widehat{\mathbf{Q}}_{\mathbf{A}}^{(0)} = \mathbf{B}^{(0)}(\mathbf{A}^{(0)\mathbf{T}}\mathbf{P_b}\mathbf{A}^{(0)})^{-1}\mathbf{B}^{(0)\mathbf{T}} \quad (28)$$

2nd step:

$$\mathbf{R}_1^{(i)} = \left(\mathbf{Q_b} + (\Delta\hat{\xi}^{(i-1)T} \otimes \mathbf{I}_m)\widehat{\mathbf{Q}}_{\mathbf{A}}^{(i-1)}(\Delta\hat{\xi}^{(i-1)} \otimes \mathbf{I}_m)\right)^{-1} \quad (29)$$

$$\hat{\lambda}^{(i)} = \mathbf{R}_1^{(i)}(\mathbf{b}^{(i-1)} - \mathbf{A}^{(i-1)}\Delta\hat{\xi}^{(i-1)}) \quad (30)$$

$$\mathbf{R}_2^{(i)} = \left((\mathbf{I}_n \otimes \hat{\lambda}^{(i)T})\widehat{\mathbf{Q}}_{\mathbf{A}}^{(i-1)}(\Delta\hat{\xi}^{(i-1)} \otimes \mathbf{I}_m)\right)\mathbf{R}_1^{(i)} \quad (31)$$

$$\Delta\hat{\xi}^{(i)} = \left(\mathbf{A}^{(i-1)\mathrm{T}}\mathbf{R}_1^{(i)}\mathbf{A}^{(i-1)} + \mathbf{R}_2^{(i)}\mathbf{A}^{(i-1)}\right)^{-1}\left(\mathbf{A}^{(i-1)\mathrm{T}}\mathbf{R}_1^{(i)} + \mathbf{R}_2^{(i)}\right)\mathbf{b}^{(i-1)} \quad (32)$$

$$\widehat{r}^{(i)} = \mathbf{Q}_b\hat{\lambda}^{(i)} \quad (33)$$

$$\widehat{e}^{(i)} = -\tilde{\mathbf{Q}}_A^{(i-1)}\left(\Delta\hat{\xi}^{(i)} \otimes \mathbf{I}_m\right)\hat{\lambda}^{(i)} \quad (34)$$

3rd step: Calculate the newly parameters with $\hat{\xi}^{(i)} = \hat{\xi}^{(i-1)} + \Delta\hat{\xi}^{(i)}$ and refresh the cofactor matrix $\widehat{\mathbf{Q}}_{\mathbf{A}}$ with

$$\widehat{\mathbf{Q}}_A^{(i)} = \mathbf{B}^{(i-1)}\left((\mathbf{A}^{(i-1)}\text{-}\mathbf{E}^{(i)})^{\mathrm{T}} \times \left(\mathbf{Q_b} + (\Delta\hat{\xi}^{(i)T} \otimes \mathbf{I}_m)\widehat{\mathbf{Q}}_{\mathbf{A}}^{(i-1)}(\Delta\hat{\xi}^{(i)} \otimes \mathbf{I}_m)\right)^{-1} \times (\mathbf{A}^{(i-1)}\text{-}\mathbf{E}^{(i)})\right)^{-1}\mathbf{B}^{(i-1)\mathbf{T}}, \quad (35)$$

as well as update the Jacob matrix $\mathbf{A}$ and the left vector $\mathbf{b}$ with $\hat{\xi}^{(i)}$.

4th step: repeat the second step and the third step until $\Delta\hat{\xi}^{(i)} < \tau$, which is a given tolerance.

5th step: calculate the estimated variance component as follows,

$$\hat{\sigma}_0^2 = \frac{\left(\hat{\lambda}^{(i)}\right)^T\left(\mathbf{R}_1^{(i)}\right)^{-1}\hat{\lambda}^{(i)}}{m-n} \quad . \quad (36)$$

5 Numerical examples

5.1 Triangulation adjustment

Triangulation adjustment is a typical nonlinear problem in geodetic. A trilateration network as described in Fig. 1 will being used to test our algorithm, its corresponding measured distances and weights are listed in Table 1.

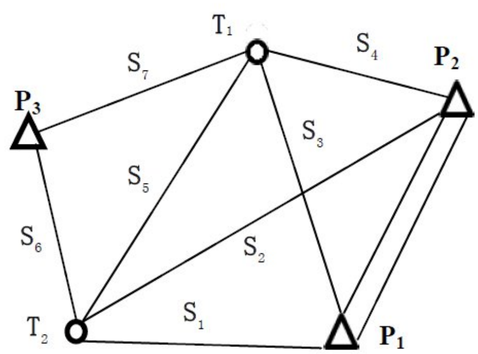

Fig. 1. Trilateration network

In Fig. 1, P_1 ,P_2 and P_3 are the given points, they are being listed in the second and the third column of Table 2, T_1 and T_2 are the unknown points needs us to estimate, its approximation value listed in the fifth and the sixth column of Table 2.

Table 1. The measured distance and corresponding weight

No.	Distance(m)	Weights
S_1	249.124	1
S_2	380.916	50
S_3	317.414	200
S_4	226.939	350
S_5	321.157	650
S_6	215.116	800
S_7	194.847	1500

As we know, any measured distance in trilateration network can be presented as

$$S_i - e_i = \sqrt{\left(\hat{x}_j - \hat{x}_k\right)^2 + \left(\hat{y}_j - \hat{y}_k\right)^2} \quad (37)$$

Where, j is the start points and k is the end point of side S_i. Therefore, this nonlinear function should be linearized by Taylor series expansion method as follows

$$S_i - S_{jk}^0 = -\frac{\Delta x_{jk}^0}{S_{jk}^0}\delta x_j - \frac{\Delta y_{jk}^0}{S_{jk}^0}\delta y_j + \frac{\Delta x_{jk}^0}{S_{jk}^0}\delta x_k + \frac{\Delta y_{jk}^0}{S_{jk}^0}\delta y_k \quad (38)$$

Where, $S_{jk}^0 = \sqrt{\left(\hat{x}_j^0 - \hat{x}_k^0\right)^2 + \left(\hat{y}_j^0 - \hat{y}_k^0\right)^2} = \sqrt{\left(\Delta\hat{x}_{jk}^0\right)^2 + \left(\Delta\hat{y}_{jk}^0\right)^2}$ is the approximation distance. In this paper, the given point is the start point if it included in one side. Consequently, we can employ these formulas to organize the coefficient matrix **A** and matrix **B** as description in section 3.

The estimated parameters by our algorithm are listed in Table 3.

From the expression in Table 3, the estimated parameters after the second iteration do not have any remarkable changes. The parameters are also estimated by least squares, the results equal to the value being listed in the second column of Table 3, namely, the first iteration.

The computed residuals and the estimated component variance are listed in the Table 4. Of course, the estimated residual r and the variance component $\hat{\sigma}_0^2$ from LS adjustment still equal to the value being listed in second column of Table 4, but in which not included the residual $\widehat{e}$. After five iterate the algorithm converged at a stable point, the residual of Jacob will decreased swiftly. On the other hand, this proposed algorithm will ensure the constant elements in coefficient matrix do not have any residual.

This experiment validated that the suggested algorithm can be used to compute the parameters in practical geodetic and can obtain more reasonable results than general nonlinear least squares.

5.2 Simulation experiment

In order to further investigate the affections of the suggested method on the solution of a nonlinear problem, a nonlinear regression being discussed in this section.

The nonlinear regression model as follows,

$$y = f(\mathbf{X}, \xi) = \frac{e^{\xi_1 x} + e^{\xi_2 x}}{\xi_2 x^{\xi_1} + 2} \quad (39)$$

where, the given value of parameters $[\xi_1, \xi_2]^T$ are respectively 2.3618 and 0.7631, and the true value of x and y data are given in Table 5.The case 1 is mainly designed to test affection in different standard deviation conditions. The initial value effect will be examined by the case 2.

Case 1: We assume x data is known exactly without any errors, and y data is noised by random errors. This case is to validate the feasibility of the developed algorithm and

Table 2. The given and approximation value of coordinates

Points	Given Coordinates(m)		Points	Approximation coordinates(m)	
	x	y		x	y
P_1	486.642	388.325	T_1	780	267
P_2	676.209	468.991	T_2	485	138
P_3	695.367	91.879			

Table 3. The estimated parameters at each iteration

Iterations Parameters	1	2	3	4	5
$\hat{x}_{T_1}$	780.1332	780.1334	780.1334	780.1334	780.1334
$\hat{y}_{T_1}$	267.3075	267.3076	267.3076	267.3076	267.3076
$\hat{x}_{T_2}$	485.5550	485.5586	485.5586	485.5586	485.5586
$\hat{y}_{T_2}$	139.3434	139.3388	139.3388	139.3388	139.3388

to examine the results under different variance assumption with weighted least squares (WLS) and WTLS, respectively. Therefore, the random errors with standard deviation 1, 5 and 10 are added in y data, respectively. Then we independently implement the iterative algorithm and WLS adjustment 1000 times and compute the differences of parameters between the estimated value and the true value, which are presented in Fig. 2 and Fig. 3 respect to the two parameters. In this test, the initial value of parameters β_1^0=2.36 and β_2^0 = 0.76.

Following the two figures, it is easily to find that although the observation vector y perturbed by different amount of errors, the mean of the differences from WLS and WTLS are equal to zero.

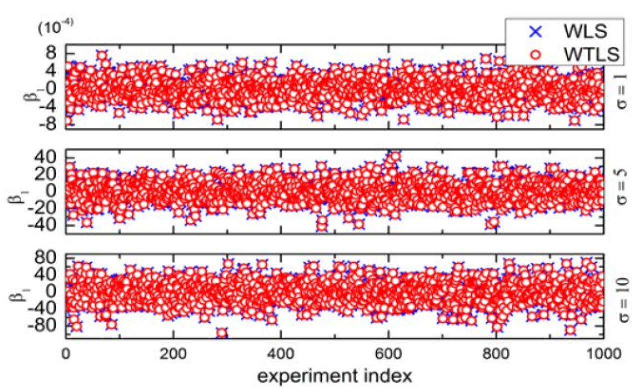

Fig. 2. The difference of β_1 between the estimate value and the true value respect to WLS and WTLS; the '*blue cross*' and the '*red circle*' denotes the results of WLS and WTLS respectively; each subfigure from top to button are corresponding to the case of that the standard deviation 1,5 and 10, respectively.

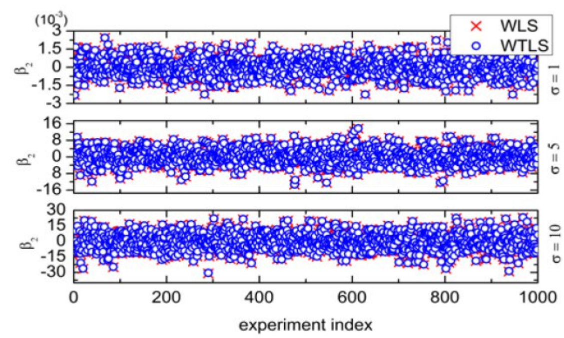

Fig. 3. The difference of β_2 between the estimate value and the true value respect to WLS and WTLS; the '*red cross*' and the '*blue circle*' denotes the results of WLS and WTLS, respectively; each subfigure from top to button are corresponding to the case of that the standard deviation 1,5 and 10, respectively.

Case 2: There still assume that x data is known exactly and only y data subjected to the random error, and the standard deviation is constantly fixed at 2. However, the initial parameters are designated as β_0^1 = 2.33, β_2^0 = 0.73 and β_0^1 = 2.39,β_2^0 = 0.79, running this test 1000 by WLS and WTLS, respectively. Under the first designation, the difference of parameters between the estimated value and the true value are illustrated in Fig. 4 and Fig. 5 respect to β_1 and β_2, respectively.

Under the second designation, similarly, the differences are represented in Fig. 6 and Fig. 7 for β_1 and β_2, respectively.

As described in Fig. 4 to Fig. 7, the results from WTLS are closer to the real value than WLS adjustment when the initial parameters are known inexactly. The means of difference from WTLS are almost equal to zero under the two different designations, in the meanwhile, the mean differ-

Table 4. The computed residual and the component of variance at each iteration

Iterations Items	1	2	3	4	5
$\hat{r}$	0.1406	1.4859	0.1309	0.1355	0.1355
	0.1057	1.5465	0.1014	0.1036	0.1036
	-0.0481	-0.0542	-0.0487	-0.0485	-0.0485
	0.0549	0.2671	0.0546	0.0547	0.0547
	-0.0136	0.7851	-0.0119	-0.0132	-0.0132
	0.0069	0.2471	0.0114	0.0068	0.0068
	0.0125	-0.3223	0.0122	0.0124	0.0124
$\hat{e}$	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000
	-0.0255E-5	-0.0558E-7	-0.0240E-11	-0.1336E-15	0.0329E-18
	0.0002E-5	0.0004E-7	0.0001E-11	0.0006E-15	-0.0001E-18
	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000
	-0.0467E-5	-0.0768E-7	-0.0662E-11	-0.2408E-15	0.0536E-18
	0.0270E-5	0.0443E-7	0.0383E-11	0.1392E-15	-0.0310E-18
	0.0034E-5	0.0014E-7	0.0074E-11	0.0276E-15	-0.0061E-18
	0.0082E-5	0.0035E-7	0.0180E-11	0.0670E-15	-0.0148E-18
	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000
	-0.0334E-5	-0.0707E-7	-0.0273E-11	-0.2151E-15	0.0550E-18
	-0.0172E-5	-0.0363E-7	-0.0141E-11	-0.1109E-15	0.0283E-18
	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0206E-5	0.0298E-7	0.0362E-11	0.0731E-15	-0.0127E-18
	-0.0471E-5	-0.0681E-7	-0.0834E-11	-0.1683E-15	0.0292E-18
	-0.0206E-5	-0.0298E-7	-0.0362E-11	-0.0731E-15	0.0127E-18
	0.0471E-5	0.0681E-7	0.0834E-11	0.1683E-15	-0.0292E-18
	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0283E-5	0.0399E-7	0.0469E-11	0.1479E-15	-0.0313E-18
	0.1290E-5	0.1819E-7	0.2073E-11	0.6538E-15	-0.1386E-18
	-0.0769E-5	-0.1357E-7	-0.0850E-11	-0.5227E-15	0.1289E-18
	0.0372E-5	0.0656E-7	0.0411E-11	0.2526E-15	-0.0623E-18
	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000
$\hat{\sigma}_0^2$	0.8300	250.8700	0.8232	0.8171	0.8171

Table 5. The given set of x and y data

i	*x*	*y*	*i*	*x*	*y*
1	5.9000	21464.0387	6	3.7000	333.2986
2	5.4000	8051.8135	7	2.8000	70.5047
3	5.2000	5465.2431	8	2.8000	70.5047
4	4.6000	1740.9123	9	2.4000	36.8200
5	3.5000	233.7155	10	1.5000	9.4531

ences from WLS are equal to 0.0029, 0.0022 and 0.0055, 0.0044, which are corresponding to β_1 and β_2, respectively.

6 Conclusions

To solving the nonlinear problem with weighted total lest squares method, an improved algorithm is investigated in this article. A triangulation adjustment example is used to test the applicable of our suggested algorithm. Because the residual of parameters in Jacob matrix is taken into account, the estimated component variance from the developed algorithm is smaller than WLS. Through the simula-

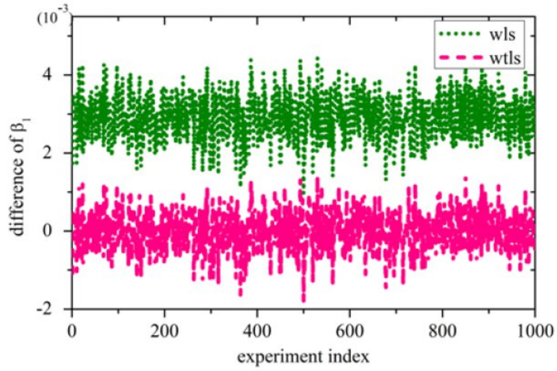

Fig. 4. The difference of β_1 between the estimate value and the true value respect to WLS and WTLS; the '*green dot*' and the '*pink dash*' denotes the results of WLS and WTLS, respectively; where the initial value of β_1 is 2.33.

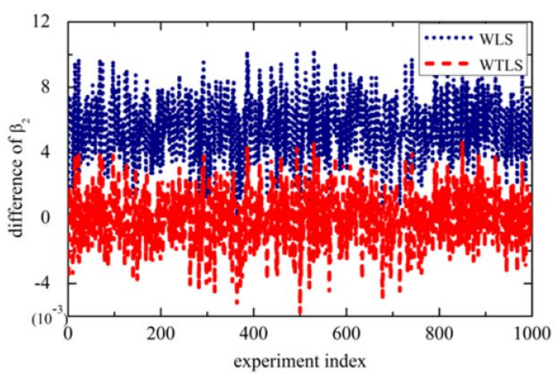

Fig. 5. The difference of β_2 between the estimate value and the true value respect to WLS and WTLS; the '*blue dot*' and the '*red dash*' denotes the results of WLS and WTLS, respectively; where the initial value of β_2 is 0.73.

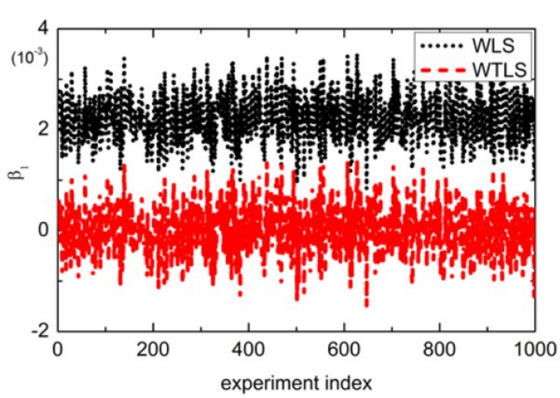

Fig. 6. The difference of β_1 between the estimate value and the true value respect to WLS and WTLS; the '*black dot*' and the '*red dash*' denotes the results of WLS and WTLS, respectively; where the initial value of β_1 is 2.39.

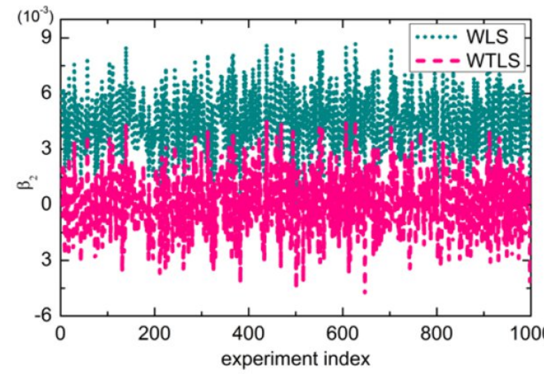

Fig. 7. The difference of β_2 between the estimate value and the true value respect to WLS and WTLS; the '*dark cyan dot*' and the '*pink dash'* denotes the results of WLS and WTLS, respectively; where the initial value of β_2 is 0.79.

tion experiment, the feasibility of the developed algorithm is being validated, and two conclusions can be presented as 1), if the initial value is known very much exactly, the difference of estimated results from WLS and WTLS is quietly small, hence we recommend using the classical WLS adjustment to solving the nonlinear problem in this situations; 2) if we cannot obtain the exactly initial parameters, employing the WTLS method to processing nonlinear problem is better than WLS adjustment.

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References

Acar M., Ozludemir M., Akyilmaz O., Celik R. and Ayan T., 2006, Deformation analysis with Total Least Squares, Nat. Hazard. Earth. Sys, 6,4, 663-669.

Amiri-Simkooei A., and Jazaeri S., 2012, Weighted total least squares formulated by standard least squares theory, J Geod. Sci., 2,2, 113-124.

Amiri-Simkooei A., and Jazaeri S., 2013, Data-snooping procedure applied to errors-in-variables models, Stud. Geophys. Geod, 1-16.

Davis T., 1999, Total least-squares spiral curve fitting, J Surv. Eng., 125,4, 159-176.

Felus Y., 2004, Application of total least squares for spatial point process analysis, J Surv. Eng., 130,3, 126-133.

Felus Y. and Schaffrin B., 2005, A total least-squares approach in two stages for semivariogram modeling of aeromagnetic data, 215-220.

Golub G. H. and Van Loan C., 1980, An analysis of the total least squares problem, SIAM J Num. Ana., 17,6, 883-893.
Jazaeri S., Amiri-Simkooei A. and Sharifi M., 2013, Iterative algorithm for weighted total least squares adjustment, Surv Rev, in press.
Mahboub V., 2011, On weighted total least-squares for geodetic transformations, J Geod., 1-9.
Schaffrin B. and Wieser A., 2008, On weighted total least-squares adjustment for linear regression, J Geod, 82,7, 415-421.
Schaffrin B. and Wieser A., 2009, Empirical affine reference frame transformations by weighted multivariate TLS adjustment, Geod. Ref. Fram., 134, 213-218.
Schaffrin B. and Felus Y., 2009, An algorithmic approach to the total least-squares problem with linear and quadratic constraints, Stud Geophys Geod, 53, 01.
Schaffrin B. and Wieser A., 2011, Total least-squares adjustment of condition equations, Stud. Geophys. Geod., 55,3, 529-536.
Shen Y., Li B. and Chen Y., 2011, An iterative solution of weighted total least-squares adjustment, J Geod, 85,4, 229-238.
Tong X., Jin Y. and Li L., 2011, An improved weighted total least squares method with applications in linear fitting and coordinate transformation, J Surv. Eng., 137,4, 120-128.