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TOPOLOGIES ON PRODUCT AND COPRODUCT FRÖLICHER SPACES

Abstract. In this paper, the topologies underlying a product Frölicher space and a coproduct Frölicher space are defined and compared. It is shown that the product topology, which is equal to the one induced by structure functions, is the weakest one in which all projections are continuous. On the other hand, it is proved that the three topologies arising from the coproduct structure are equal.

1. Introduction

The product topology underlying a Cartesian product of sets endowed with a differential structure is of particular interest for the study of some objects of differential topology and geometry. The spaces under consideration in our study are those endowed with a smooth structure introduced by Alfred Frölicher in a series of works [13], [14], [15], [16], and also in [17] and [18]. They were called *Frölicher spaces* for the first time in Cherenack [8, 2000] (work submitted in 1996), then later in Kriegl and Michor [19, 1997] and in many other publications thereafter. In Batubenge and Tshilombo [3], the authors studied and compared the topologies on a quotient and on a subspace of a Frölicher space. The present follow-up work is a comparative study on the three topologies arising from the product spaces as well as the coproduct spaces, which will help in investigating the topological nature of sheaves, fibre bundles, tori, cylinders or other useful geometric objects. Once more, we would like to point out that for Frölicher spaces, unlike smooth manifolds and differential spaces, the topologies are induced by the existing smooth structure given by functions and curves.

2. Preliminaries

Let M be a nonempty set, $\mathbb{R}^M$ the set of real-valued functions (outputs) on M and $M^{\mathbb{R}}$ the set of curves (inputs) into M. Consider the power sets $\mathcal{P}(\mathbb{R}^M)$

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and $\mathcal{P}(M^{\mathbb{R}})$ ordered by inclusion. Hence both $\mathcal{P}(\mathbb{R}^M)$ and $\mathcal{P}(M^{\mathbb{R}})$, together with the identity (I) and inclusion (i) maps, are categories (see [21]) which we shall denote by $\mathbf{C_f}$ and $\mathbf{C_c}$, respectively. That is, for any $A, A' \in Ob(\mathbf{C_f})$, either $Hom_{\mathbf{C_f}}(A, A') = \{I\}$ or $Hom_{\mathbf{C_f}}(A, A') = \{i\}$ only. The category $\mathbf{C_c}$ is described similarly.

Now, let $\Gamma : \mathbf{C_f} \to \mathbf{C_c}$ and $\Phi : \mathbf{C_c} \to \mathbf{C_f}$ be given by

$$\Gamma(\mathcal{F}) = \{c : \mathbb{R} \to M; f \circ c \in C^{\infty}(\mathbb{R}, \mathbb{R}) \text{ for all f} \in \mathcal{F}\},$$

$$\Phi(\mathcal{C}) = \{f : M \to \mathbb{R}; f \circ c \in C^{\infty}(\mathbb{R}, \mathbb{R}) \text{ for all c} \in \mathcal{C}\},$$

where $\mathcal{F} \in Ob(\mathbf{C_f})$ and $\mathcal{C} \in Ob(\mathbf{C_c})$, respectively. Dropping the brackets, we write

$$\Gamma Ob(\mathbf{C_f}) = Ob(\mathbf{C_c}),$$

$$\Phi Ob(\mathbf{C_c}) = Ob(\mathbf{C_f}).$$

That is, any set $\mathcal{F}$ of functions on M determines a set $\Gamma\mathcal{F}$ of curves. Similarly, any set $\mathcal{C}$ of curves determines a set $\Phi\mathcal{C}$ of functions. The following lemma, the proof of which is straightforward (see [17, p.2]), states that Γ and Φ are contravariant functors.

$$\begin{array}{ccc} A & \xrightarrow{I_A} & A \\ \downarrow \Phi & & \downarrow \Phi \\ B & \xrightarrow{I_B} & B \end{array}$$

Now, let $A \subseteq A'$ in $\mathcal{P}(M^{\mathbb{R}})$. On the sets B and B' of functions generated by Φ, the inclusion occurs the other way around.

$$\begin{array}{ccc} A & \overset{i_A}{\hookrightarrow} & A' \\ \downarrow \Phi & & \downarrow \Phi \\ B = \Phi(A) & \overset{i_{B'}}{\hookleftarrow} & B' = \Phi(A') \end{array}$$

Similar diagrams can be drawn with Γ. That is, the functors Φ and Γ are order reversing.

Lemma 2.1. *Given $\mathcal{F}_0$, $\mathcal{F}_1 \in \mathcal{P}(\mathbb{R}^M)$ and given $\mathcal{C}_0, \mathcal{C}_1 \in \mathcal{P}(M^{\mathbb{R}})$.*

(i) *If $\mathcal{F}_0 \subseteq \mathcal{F}_1$ then $\Gamma\mathcal{F}_0 \supseteq \Gamma\mathcal{F}_1$, $\mathcal{F}_0 \subseteq \Phi\Gamma\mathcal{F}_0$ and $\Gamma\mathcal{F}_0 = \Gamma\Phi\Gamma\mathcal{F}_0$.*
(ii) *If $\mathcal{C}_0 \subseteq \mathcal{C}_1$ then $\Phi\mathcal{C}_0 \supseteq \Phi\mathcal{C}_1$, $\mathcal{C}_0 \subseteq \Gamma\Phi\mathcal{C}_0$ and $\Phi\mathcal{C}_0 = \Phi\Gamma\Phi\mathcal{C}_0$.*

A *Frölicher structure* on a set M is a pair $(\mathcal{C}_M, \mathcal{F}_M)$, where $\mathcal{C}_M \in \mathcal{P}(M^{\mathbb{R}})$ and $\mathcal{F}_M \in \mathcal{P}(\mathbb{R}^M)$ such that the *duality* conditions $\Gamma\mathcal{F}_M = \mathcal{C}_M$, $\Phi\mathcal{C}_M = \mathcal{F}_M$ hold. *A Frölicher space* is a triple $(M, \mathcal{C}_M, \mathcal{F}_M)$, where M is a set and $(\mathcal{C}_M, \mathcal{F}_M)$ is a Frölicher structure on M. One may simply say Frölicher space M instead of Frölicher space $(M, \mathcal{C}_M, \mathcal{F}_M)$ when there is no risk of confusion.

We recall that Frölicher spaces form a category which we denote by **FRL**, the morphisms of which are those maps $\varphi : M \to N$ that satisfy the property

$$\varphi_*(\mathcal{C}_M) := \{\varphi \circ c;\ c \in \mathcal{C}\} \subset \mathcal{C}_N$$

or, equivalently,

$$\varphi^*(\mathcal{F}_N) := \{f \circ \varphi;\ f \in \mathcal{F}\} \subset \mathcal{F}_M,$$

and are called **FRL**-morphisms or $\mathbb{F}$-maps. It is easy to see that the maps $c \in \Gamma\mathcal{F}_M$ and $f \in \Phi\mathcal{C}_M$ are **FRL**-morphisms, that is, they are smooth in the structure $(\mathcal{C}_M, \mathcal{F}_M)$ which they determine (see [3]). In the works by Frölicher and Kriegl [17], Kriegl and Michor [19], as well as Cherenack [8] it is proved that this category has interesting properties:

(1) It is complete, that is, arbitrary limits exist. The underlying set is formed as in the category of sets as a certain subset of the Cartesian product, and the smooth Frölicher structure is generated by smooth functions on the factors.
(2) It is cocomplete, that is, arbitrary colimits exist. The underlying set is formed as in the category of sets, that is, as a certain quotient of the disjoint union, and the smooth functions are exactly those which induce smooth functions on the cofactors.
(3) It is Cartesian closed. This follows from a theorem of Lawvere, Schanuel and Zame [20] on smooth spaces, referred to by Frölicher and Kriegl in [18, p. 217]. We restate it as follows. For any Frölicher spaces X, Y and Z, the set $C^\infty(Y, Z)$ of all **FRL**-morphisms from Y to Z carries a canonical smooth Frölicher structure following the exponential law

$$C^\infty(X \times Y, Z) \cong C^\infty(X, C^\infty(Y, Z)).$$

Now, if $X = \mathbb{R}$ in this formula, we construct the set $\mathcal{C}_{Y,Z}$ of curves $c : \mathbb{R} \to C^\infty(Y, Z)$ by requiring that the map $\tilde{c} : \mathbb{R} \times Y \longrightarrow Z$, where $\tilde{c}(t, y) := c(t)(y)$, is smooth. Then, using the functors Φ and Γ, we shall generate a Frölicher structure on $C^\infty(Y, Z)$.
(4) It is topological over **Set**. (See [5], [8], [19]). That is, the forgetful functor **FRL** $\to$ **Set** to the category of sets is faithful and topological, which means that the category **FRL** behaves like the category **Top** of topological spaces. On new sets constructed in **Set**, one defines induced $\mathbb{F}$-structures and induced topologies. So, subsets, quotients, equalizers, coequalizers, products and coproducts exist in **FRL** as limits or colimits lifted from the category of sets. The products and coproducts, the topologies of which are studied in this work, are so constructed.

Note that for a fixed set M, the structures on M are ordered in the sense that $(\mathcal{C}, \mathcal{F})$ is said to be *finer* than $(\mathcal{C}', \mathcal{F}')$ if the identity map of M is a morphism from $(M, \mathcal{C}, \mathcal{F})$ to $(M, \mathcal{C}', \mathcal{F}')$. It follows from Lemma 2.1 above that for any set $\mathcal{C}_0 \in \mathcal{P}(M^{\mathbb{R}})$ there is a unique structure $(\mathcal{C}, \mathcal{F})$ on M such that $\mathcal{C}_0 \subset \mathcal{C}$. It is called the *structure generated by curves* $\mathcal{C}_0$, which is obtained by setting the duality conditions $\mathcal{F} = \Phi\mathcal{C}_0$, $\mathcal{C} = \Gamma\mathcal{F}$. In a similar way one defines the *structure generated by a set of functions* $\mathcal{F}_0 \in \mathcal{P}(\mathbb{R}^M)$. It is the unique structure $(\mathcal{C}, \mathcal{F})$ on M with $\mathcal{F}_0 \subset \mathcal{F}$, obtained by setting $\mathcal{C} = \Gamma\mathcal{F}_0$, $\mathcal{F} = \Phi\mathcal{C}$.

Now we recall that the topology of the smooth space $(M, \mathcal{C}_M, \mathcal{F}_M)$ is defined to be the initial topology generated by structure functions $f \in \mathcal{F}_M$, with subbase $S = \{f^{-1}(0,1)\}_{f \in \mathcal{F}_M}$. We need to investigate the topologies arising from the categorical construction for initial and final objects in **FRL**. We recall that each of the objects $(M, \mathcal{C}_M, \mathcal{F}_M)$ carries two natural topologies, and that initial as well as final objects rather carry more than two topologies, which we shall study and compare. We shall distinguish between both topologies on a Frölicher space by the using following notations: $\tau_{\mathcal{C}_M}$, the topology induced by curves, is the collection of all subsets $\mathcal{U} \subset M$ such that $c^{-1}(\mathcal{U}) \in \tau_{\mathbb{R}}$ for all curves $c \in \mathcal{C}_M$. And $\tau_{\mathcal{F}_M}$, the topology induced by functions, is the collection of all subsets $\mathcal{O}$ that are pre-images $f^{-1}(V)$, for $f \in \mathcal{F}_M$, of open sets V of the standard topology $\tau_{\mathbb{R}}$ of $\mathbb{R}$. By *topology of a Frölicher space* or *Frölicher topology* we shall mean $\tau_{\mathcal{F}_M}$, as it is easy to see that $\tau_{\mathcal{F}_M} \subseteq \tau_{\mathcal{C}_M}$. This inclusion is just an equality for some Frölicher spaces, which A. Cap [6] called *balanced spaces*. In Batubenge and Tshilombo [3], the authors proved that **FRL**-morphisms are continuous in both the topologies of curves and functions.

Note finally, that for arbitrary sets of the functions $\mathcal{C}_0 \subset M^{\mathbb{R}}$ and $\mathcal{F}_0 \subset \mathbb{R}^M$, the functors Γ and Φ given above induce important inclusions in the process of generating a Frölicher structure, and provide us with obvious examples of Frölicher spaces.

1. $(\mathbb{R}^n, \mathcal{C}, \mathcal{F})$ is called the canonical $\mathbb{F}$-space. The smooth structure is induced by Boman's lemma [4]. That is, $\mathcal{C} = C^\infty(\mathbb{R}, \mathbb{R}^n)$ and $\mathcal{F} = C^\infty(\mathbb{R}^n, \mathbb{R})$, so that the differentiability of elements of $\mathcal{F}$ is tested by $\mathcal{C}$ and the other way around.
2. Let M be a linear space and M' its algebraic dual. The Frölicher structure, that is linearly generated on M, is given by $(\Gamma\mathcal{F}_o, \Phi\Gamma\mathcal{F}_o)$, where $\mathcal{F}_o \subseteq M'$ separates points in M.
3. The Frölicher space $(\mathbb{Q}, \mathcal{C}_{\mathbb{Q}}, \mathcal{F}_{\mathbb{Q}})$, with the canonical structure generated by the canonical inclusion in $\mathbb{R}$ (which, in fact, is a smooth map) is a balanced space with discrete topologies $\tau_{\mathcal{F}_{\mathbb{Q}}}$ and $\tau_{\mathcal{C}_{\mathbb{Q}}}$.
4. Let $(\mathbb{R}, \mathcal{C}_{\mathbb{R}}, \mathcal{F}_{\mathbb{R}})$ be the Frölicher space generated by constant curves. By a similar reasoning, one has $\mathcal{F}_{\mathbb{R}} = \mathbb{R}^{\mathbb{R}}$. The resulting structure curves are also constant. The topologies $\tau_{\mathcal{F}_M} = \tau_{\mathcal{C}_M}$ are discrete.

As indicated before, there are two natural topologies on an **FRL**-object M. These are $\tau_{\mathcal{F}_M}$ and $\tau_{\mathcal{C}_M}$. Now let us point out that, when forming a new Frölicher space out of a collection of others, the new initial or final **FRL**-object carries three topologies. These are the two arising from the smooth structure, and a third that is simply the usual product or coproduct topology.

In order to present this comparative study, the paper is organised in two main sections according to the new types of **FRL**-objects under consideration: the product and the coproduct spaces.

3. Topologies on a Frölicher product space

The $\mathbb{F}$-product space in the category **FRL** is the initial object obtained by lifting the product in the category **Set** to **FRL**.

Let $M^* := \prod_{i\in I} M_i$ denote the product in **Set**. The initial structure on M^* in **FRL** is the $\mathbb{F}$-structure generated by the family $(f \circ p_i : M^* \to \mathbb{R})$; where $f : M_i \longrightarrow \mathbb{R}$, and $p_i : M^* \to M_i$ are projections in **Set**. Now let $\mathcal{F}_{oM_i}$ generate the $\mathbb{F}$-structure $(\mathcal{C}_{M_i}, \mathcal{F}_{M_i})$ on each M_i. The resulting Frölicher product structure $(\mathcal{C}_{M^*}, \mathcal{F}_{M^*})$ is generated by the collection $\mathcal{F}_o$ of arbitrary unions of functions of the form $f_i \circ p_i$ for all $f_i \in \mathcal{F}_{oM_i}$.

Hence,

$$\mathcal{C}_{M^*} = \Gamma\mathcal{F}_o = \{c : \mathbb{R} \to M^* \mid g \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all g} \in \mathcal{F}_0\}$$

and

$$\mathcal{F}_{M^*} = \Phi\mathcal{C}_{M^*} = \Phi\Gamma\mathcal{F}_o = \{f : M^* \to \mathbb{R} \mid f \circ (c_i)_{i\in I} \in C^\infty(\mathbb{R}, \mathbb{R}),\ c_i \in \mathcal{C}_{M_i}\}.$$

It is easy to show that a smooth curve c into the product $\mathcal{C}_{M^*}$ has the form $c = (c_i)_{i\in I}$ for all $c_i \in \mathcal{C}_{M_i}$, $i \in I$.

Definition 3.1. The $\mathbb{F}$-space $(M^*, \mathcal{C}_{M^*}, \mathcal{F}_{M^*})$ is called the product of M_i or the product of $\mathbb{F}$-spaces $(M_i, \mathcal{C}_{M_i}, \mathcal{F}_{M_i})$. The pair $(\mathcal{C}_{M^*}, \mathcal{F}_{M^*})$ is the initial product structure such that all the projections maps p_i are smooth maps. The topologies $\tau_{\mathcal{F}_{M^*}}$ and $\tau_{\mathcal{C}_{M^*}}$ induced by the Frölicher structure, are called $\mathbb{F}$-topologies on M^*, or $\mathbb{F}$-product topologies.

Lemma 3.1. [3] *Let M^* be a topological product of $\mathbb{F}$-spaces M_i with the topological product denoted by τ_Π. Let $p_j : M^* \to M_j$ be the jth projection and let $f_{ji} \in \mathcal{F}_{M_j}$ be an arbitrary collection of structure functions on M_j (for a fixed j). Let $f_{ji}^{-1}(0, 1)$ be a subbasic open set in $\tau_{\mathcal{F}_{M_j}}$, for each $i \in I$. Then $p_j^{-1}(f_{ji}^{-1}(0, 1))$ is also a subbasic open set in τ_Π.*

This is a well-known result from point-set topology. Also it implies that the canonical projection p_j is a continuous, onto and open map for τ_Π. We shall make use of the fact that a smooth map on a Frölicher space is continuous in both $\tau_{\mathcal{F}_M}$ and $\tau_{\mathcal{C}_M}$, as proved in the authors' work (see [3]), and notice that $\tau_{\mathcal{F}_{M^*}}$ and $\tau_{\mathcal{C}_{M^*}}$ on M^* are, respectively, the smallest and the largest ones, in which all smooth functions from M^* and all smooth curves into M^* are continuous. Now, we can compare the usual topology τ_Π of the product of $\mathbb{F}$-spaces with the two topologies arising from the $\mathbb{F}$-product structure, keeping in mind that τ_{Π} is the topology induced by the topologies $\tau_{\mathcal{F}_{M_i}}$ of factors such that if $\mathcal{U}_{j_k} \in \tau_{\mathcal{F}_{M_{j_k}}}$ is open in the coordinate space M_{j_k} and $p_{j_k} : M^* \to M_{j_k}$ is the projection, then $p_{j_k}^{-1}(\mathcal{U}_{i_k}) = \Pi\{M_i : i \neq j_k\} \times U_{j_k}$ forms a subbase in τ_{Π}, and the set $B = \{\bigcap_{j=1}^{n} p_j^{-1}(\mathcal{U}_j) \,|\, \mathcal{U}_j \in \tau_{\mathcal{F}_{M_j}}\}$ is its base.

Theorem 3.1. *Let $\tau_{\mathcal{F}_{M^*}}$ and $\tau_{\mathcal{C}_{M^*}}$ be the $\mathbb{F}$-topologies induced by the Frölicher structure on the countable product $M^* = \Pi M_i$ of Frölicher spaces. Let $\tau_{\prod}$ be the (Tychonoff) product topology on M^*. Then $\tau_{\prod} = \tau_{\mathcal{F}_{M^*}} \subseteq \tau_{\mathcal{C}_{M^*}}$.*

Proof. (a) We first prove the inclusion $\tau_{\mathcal{F}_{M^*}} \subset \tau_{\Pi}$. Assume $V = f^{-1}(0,1)$ for $f \in \mathcal{F}_{M^*}$ (or an arbitrary union of these) is a $\tau_{\mathcal{F}_{M^*}}$-subbasic open set. Referring to the characterisation of open sets, we have the following. For each $x \in V = f^{-1}(0,1) \subset M^*$, $x \in f^{-1}(t)$, for some $t \in (0,1)$ such that $f(x) = t$ and $x = (x_i)_i$ with $x_i \in M_i$ for i ranging in a countable set I. Hence, there is an open set $\mathcal{U}_{i_0}$ such that $x_{i_o} \in \mathcal{U}_{i_o} \subset M_{i_o}$ with $\mathcal{U}_{i_o} \neq M_k$ for $k = 1, \ldots$ and $\mathcal{U}_j = M_j$ for $j \neq 1, \ldots$. That is, $(x_k) \in \prod_{k=1}^{n} \mathcal{U}_k$ and $(x_j)_{j \neq 1,\ldots,n} \in \prod M_j$. Thus, $f = f_i \circ p_i$ yields

$$t = f(x_i)_{i \in I} = (f_i \circ p_i)((x_i)_{i \in I}) = f_i(x_i).$$

It follows that

$$x = (x_i)_{i \in I} \in f^{-1}(t) = p_k^{-1} \circ f_k^{-1}(t)$$

and moreover,

$$x \in f^{-1}(0,1) = p_k^{-1} \circ f_k^{-1}(0,1) \quad \text{for } k = 1, \ldots.$$

Hence, there exists $\mathcal{U} \in \tau_{\prod}$, where $x \in \mathcal{U} = \prod_{k=1}^{\infty} \mathcal{U}_k \times \prod_{j \neq 1,\ldots} M_j$ as gleaned from the definition of a $\tau_{\prod}$ basic open set. And since p_k is (open and) smooth, hence continuous, we have

$$\mathcal{U} = \bigcap_{k=1}^{\infty} p_k^{-1}(\mathcal{U}_k) \;\text{ and }\; \mathcal{U}_k = f_k^{-1}(0,1).$$

Recalling that f is among the g_k, one has

$$\mathcal{U} = \bigcap_{k=1}^{\infty} p_k^{-1} f_k^{-1}(0,1) = \bigcap_{k=1}^{\infty} (g_k^{-1}(0,1)) \subset f^{-1}(0,1) = V.$$

Therefore, $\mathcal{U} \subset V$. That is, V contains a basic open set $\mathcal{U}$ of $\tau_{\prod}$. Thus, $V \in \tau_{\prod}$, which gives the required inclusion $\tau_{\mathcal{F}_{M^*}} \subset \tau_{\prod}$.

The reverse inclusion is a classical result in point-set topology. We would verify its validity in the category **FRL** as follows:

(b) Let $V \in \tau_{\prod}$. For some countable indexing set $J \subset \mathbb{N}$, we know that $\{p_j^{-1}(U_{ij})\}_{(i,j)} \in I \times J$, where $U_{ij} \in \tau_{M_j}$ is a subbase for $\tau_{\prod}$, and p_j is continuous in $\tau_{\prod}$ since they are $\mathbb{F}$-smooth by definition. Hence, $\bigcap_{j=1}^{n}\{p_j^{-1}(U_{ij})\}$ is a base for $\tau_{\mathcal{F}_{M*}}$. Therefore,

$$V = \bigcup_{k\in\Lambda}\bigcap_{j=1}^{n}\{p_j^{-1}(U_{ij})\}_{i\in I}.$$

But for all $f_j \in \mathcal{F}_{M_j}$, $\{f_j^{-1}(0,1)\}$ is a subbase for τ_{M_j}, so that $\bigcap_{j=1}^{n}\{f_j^{-1}(0,1)\}$ is a base and

$$p_j^{-1}\Big(\bigcap_{j=1}^{n}\{f_j^{-1}(0,1)\Big) = \bigcap_{j=1}^{n}\{p_j^{-1}(f_{ij}^{-1}(0,1))\} = \bigcap_{j=1}^{n}(f_{ij}\circ p_j)^{-1}(0,1)$$

is a basic open set in $\tau_{\mathcal{F}_{M*}}$. Now let $f_{ij}\circ p_j = g_i$, which is a generator for the Frölicher product structure. We have

$$\bigcup_{k\in\Lambda}\bigcap_{j=1}^{n}(g_i^{-1}(0,1)) \in \tau_{\mathcal{F}_{M*}}.$$

Thus, $V \in \tau_{\mathcal{F}_{M*}}$, which shows that $\tau_{\prod} \subseteq \tau_{\mathcal{F}_{M*}}$. The equality $\tau_{\prod} = \tau_{\mathcal{F}_{M*}}$ is proved.

(c) We finally show that $\tau_{\mathcal{F}_{M*}}$ is weaker than $\tau_{\mathcal{C}_{M*}}$. Let $V \in \tau_{\mathcal{F}_{M*}}$. That is, $V = \bigcup_{\lambda\in\Lambda}\bigcap_{j=1}^{n} p_j^{-1}(U_{j_k})$, with $U_{j_k} \in \tau_{M_j}$. Notice that V will be in $\tau_{\mathcal{C}_{M*}}$ if $c^{-1}(V) \in \tau_{\mathbb{R}}$ for all $c \in \mathcal{C}_{M*}$. Recall that $c \in \mathcal{C}_{M*}$ is such that $c = (c_i)_{i\in I}$ where $c_i \in \mathcal{C}_{M_i}$ for all i. We have

$$\begin{aligned} c^{-1}(V) = c^{-1}\bigcup_{\lambda\in\Lambda}\bigcap_{j=1}^{n} p_j^{-1}(U_{j_k}) &= \bigcup_{\lambda\in\Lambda}\bigcap_{j=1}^{n}(c^{-1}(p_j^{-1}(U_{j_k}))) \\ = \bigcup_{\lambda\in\Lambda}\bigcap_{j=1}^{n}(p_j\circ c)^{-1}(U_{j_k}) &= \bigcup_{\lambda\in\Lambda}\bigcap_{j=1}^{n} c_j^{-1}(U_{j_k}), \end{aligned}$$

where U_{j_k} is a $\tau_{\mathcal{C}_{M_j}}$ open set, and $c_j \in \mathcal{C}_{M_j}$. Therefore,

$$\bigcup_{\lambda\in\Lambda}\bigcap_{j=1}^{n} c_j^{-1}(U_{j_k}) \in \tau_{\mathbb{R}}.$$

Thus $V \in \mathcal{C}_{M*}$ and the inclusion $\tau_{\mathcal{F}_{M*}} \subseteq \tau_{\mathcal{C}_{M*}}$ holds true. ∎

4. Topologies on a coproduct of Frölicher spaces

The coproduct space in the category **FRL** is the final object obtained by lifting the coproduct in the category **Set** to **FRL**.

Let $\bar{M} = \amalg_{i\in I} M_i$ denote the coproduct in **Set**. There are natural inclusion maps $(s_i : M_i \to \bar{M})_{i\in I}$ such that $f\circ s_i = f_i$, where $f : \bar{M} \to \mathbb{R}$ is defined by $f = (f_i)_{i\in I}$. The Frölicher structure $(\mathcal{C}_{\bar{M}}, \mathcal{F}_{\bar{M}})$ is the final one that is generated by the set $\mathcal{C}_o$ of curves, given by

$$\mathcal{C}_o = \bigcup_{i \in I} \{s_i \circ c_{ij}\},$$

c_{ij} running over $\mathcal{C}_{oM_i}$, which is the generating set for the smooth structure $(\mathcal{C}_{M_i}, \mathcal{F}_{M_i})$ on each M_i. It follows that $\mathcal{F}_{\bar{M}} = \Phi\mathcal{C}_o$, where $f = (f_i)_{i \in I} : \bar{M} \to \mathbb{R}$ such that $f_{|M_i} = f_i \in \mathcal{F}_{M_i}$ and

$$\mathcal{C}_{\bar{M}} = \Gamma\mathcal{F}_{\bar{M}} = \Gamma\Phi\mathcal{C}_o,$$

where $c : \mathbb{R} \to \bar{M} \mid c = s_i \circ c_i,\ c_i \in \mathcal{C}_{M_i}$. Thus, the pair $(\mathcal{C}_{\bar{M}}, \mathcal{F}_{\bar{M}})$ is the coproduct structure in **FRL**.

Definition 4.1. The $\mathbb{F}$-space $(\bar{M}, \mathcal{C}_{\bar{M}}, \mathcal{F}_{\bar{M}})$ is called an $\mathbb{F}$-coproduct space of M_i or a coproduct of $\mathbb{F}$-spaces M_i. Also the pair $(\mathcal{C}_{\bar{M}}, \mathcal{F}_{\bar{M}})$ is the coproduct structure such that all s_i are smooth maps.

Note that $s_i \circ\ \mathcal{C}_{M_i} = \mathcal{C}_{s_i(M_i)}$ if, and only if

$$\mathcal{F}_{\bar{M}} \circ s_i = \mathcal{F}_{M_i} \simeq \mathcal{F}_{s_i(M_i)} = \mathcal{F}_{\bar{M}}|s_i(M_i).$$

In what follows, we intend to study and compare the topologies underlying a $\mathbb{F}$-coproduct space.

Definition 4.2. The topologies $\tau_{\mathcal{F}_{\bar{M}}}$ and $\tau_{\mathcal{C}_{\bar{M}}}$, induced by smooth functions and smooth curves, are called $\mathbb{F}$-topologies on $\bar{M}$ or $\mathbb{F}$-coproduct topologies.

These are respectively the smallest and the strongest topologies in which the canonical inclusions are continuous. The subbase for the topology $\tau_{\mathcal{F}_{\bar{M}}}$ is the collection $\mathcal{S} = \{f^{-1}(0,1)\}_{f \in \mathcal{F}_{\bar{M}}}$. Moreover, the $\mathbb{F}$-coproduct space carries third topology in the usual sense, that is, the topology of the space is considered as a coproduct space. Next, we shall compare these topologies, keeping in mind that the topology of a Frölicher space is the one which is generated by structure functions.

Definition 4.3. The topological coproduct space $\bar{M}$ is the coproduct of the family $(M_i)_{i \in I}$ in **Sets**, endowed with the topology in which open sets are unions of $s_i(\mathcal{U}_i)$, $i \in I$, and where $\mathcal{U}_i$ is an arbitrary $\tau_{\mathcal{F}_{M_i}}$ open set in M_i. We shall denote the topology on $\bar{M}$ by $\tau_{\amalg}$, and call it the coproduct topology for $\mathbb{F}$-topological spaces M_i.

Although the properties of the natural injections that follow are well-known in point-set topology, their proofs intertwine with the ones on the Frölicher coproduct structure.

Lemma 4.1. *Let $\bar{M}$ be the coproduct of the family $(M_i)_{i \in I}$ and let $\tau_{\amalg}$ be its coproduct topology. Then:*

(1) *s_i is a continuous map for $\tau_{\amalg}$,*

(2) *the family* $\mathcal{B} = \{s_i(\mathcal{U}_i) \mid \mathcal{U}_i \in \tau_{\mathcal{F}_{M_i}}$ *for all* $i \in I\}$ *is a basis for* $\tau_{\amalg}$*,*
(3) $s_i(M_i)$ *is a basic open set in* $\tau_{\amalg}$ *for all* $i \in I$*,*
(4) s_i *is an open map for* $\tau_{\amalg}$*.*

Proof. (1) Let $\mathcal{U} \in \tau_{\amalg}$, that is, $\mathcal{U} = \bigcup_{j\in J} s_j(\mathcal{U}_j)$ with $\mathcal{U}_j = \bigcup_{f_j\in\mathcal{F}_{M_j}} f_j^{-1}(0,1)$. In the sequel, let us fix i and vary j. Since $\bar{M}$ is partitioned M_j, and s_i is an injective map for all i, it follows that:

$$\begin{aligned} {s_i}^{-1}(\mathcal{U}) &= {s_i}^{-1}\Big[\bigcup_{j\in J} s_j(\mathcal{U}_j)\Big] = {s_i}^{-1}\Big[\bigcup_{j\in J}\bigcup_{f_j\in\mathcal{F}_{M_j}} s_j(f_j^{-1}(0,1))\Big] \\ &= \Big[\bigcup_{j\in J}\bigcup_{f_j\in\mathcal{F}_{M_j}} ({s_i}^{-1}(s_j(f_j^{-1}(0,1))))\Big] \\ &= \Big[\bigcup_{j\neq i}({s_i}^{-1}s_jf_j^{-1}(0,1))\Big] \cup \Big[\bigcup_{j=i}({s_i}^{-1}s_jf_j^{-1}(0,1))\Big] \\ &= \emptyset \cup \Big[\bigcup_{j=i} {s_i}^{-1}s_jf_j^{-1}(0,1)\Big] \\ &= \Big[\bigcup_{f_i\in\mathcal{F}_{M_i}} f_i^{-1}(0,1)\Big] \in \tau_{\mathcal{F}_{M_i}} \end{aligned}$$

One may verify that, whenever $i \neq j$,

$$\begin{aligned} {s_i}^{-1}[s_jf_j^{-1}(0,1)] &= \{x \in M_i | s_i(x) \in s_jf_j^{-1}(0,1) \subset s_j(M_j)\,, i \neq j\} \\ &= \{x \in M_i | s_i(x) \in s_i(M_i) \cap s_j(M_j) = \emptyset\, i \neq j\} = \emptyset. \end{aligned}$$

Hence, s_i is a continuous map for $\tau_{\amalg}$.

(2) This is a consequence of the definition of the coproduct topology given above.

(3) Since $M_i \in \tau_{\mathcal{F}_{M_i}}$ then, by assumption, $s_i(M_i) \in \mathcal{B}$ for all $i \in I$. That is $s_i(M_i)$ is a $\tau_{\amalg}$-basic open set.

(4) Since $f_i^{-1}(0,1)$ is a $\tau_{\mathcal{F}_{M_i}}$-basic open set, where $f_i \in \mathcal{F}_{M_i}$, it is also a $\tau_{\mathcal{F}_{M_i}}$ open set. Therefore, $s_i(f_i^{-1}(0,1)) \in \mathcal{B}$. Hence, s_i sends a basic open of $\tau_{\mathcal{F}_{M_i}}$ to a basic open of $\tau_{\amalg}$. Thus, each $s_i(\mathcal{U}_i)$, where $\mathcal{U}_i \in \tau_{\mathcal{F}_{M_i}}$, is an open set in $\tau_{\amalg}$. Therefore, s_i is an open map. ∎

Lemma 4.2. *Let* $\mathcal{U} \in \tau_{\amalg}$*. Then* $\mathcal{U}$ *is a* $\tau_{\amalg}$*-open (closed) set in* $\bar{M}$ *if and only if* $\mathcal{U} \cap s_i(M_i)$ *is a* $\tau_{\amalg}$*-open (closed) set in* $\bar{M}$ *for all* $i \in I$*.*

Proof. $\Rightarrow$: Let $\mathcal{U} \in \tau_{\amalg}$. That is, $\mathcal{U} = \bigcup_{j\in J} s_j(U_j)$ with $\mathcal{U}_j \in \tau_{\mathcal{F}_{M_j}}$ for some j. But from Lemma 4.1 and the definition of $\mathcal{B}$, it follows that more than one term of the union may be in $s_i(M_i)$. Keeping i fixed, we use the partition on $\bar{M}$ as follows:

Let $V = \mathcal{U} \cap s_i(M_i)$. Thus,

$$V = \Big[\bigcup_{j\in J} s_j(\mathcal{U}_j)\Big] \cap s_i(M_i) = \bigcup_{j\in J}[s_j(\mathcal{U}_j) \cap s_i(M_i)]$$
$$= \Big[\bigcup_{j\neq i}(s_j(\mathcal{U}_j) \cap s_i(M_i))\Big] \cup \Big[\bigcup_{j=i} s_i(\mathcal{U}_i) \cap s_i(M_i)\Big].$$

It follows that $V = \emptyset \cup (\bigcup_{j=1} s_i(\mathcal{U}_i)) = \mathcal{V}_i$, where $\mathcal{V}_i \subset M_i$. Hence, V is a $\tau_\amalg$-open set in $\bar{M}$. Now, let $\mathcal{U}$ be $\tau_\amalg$-closed in $\bar{M}$. However, $s_i(M_i)$ is $\tau_\amalg$-closed in $\bar{M}$ as well. Thus $\mathcal{U} \cap s_i(M_i)$ is $\tau_\amalg$-closed.

$\Leftarrow$: Let $\mathcal{U} \cap s_i(M_i)$ be a $\tau_\amalg$-open (closed) set in $\bar{M}$. Since $s_i(M_i)$ is an open (closed) set in $\bar{M}$, it follows that $\mathcal{U} \cap s_i(M_i) \subset \mathcal{U}$ is an open (closed) set for the trace topology of $\tau_\amalg$ on $\mathcal{U}$, which we denote by $\tau_\amalg(\mathcal{U})$. By the assumption $\mathcal{U} \cap s_i(M_i) \subset \mathcal{U} \subset \bar{M}$, it follows that $\mathcal{U} \cap s_i(M_i)$ is a $\tau_\amalg$-open (closed) set in $\bar{M}$. Also, $\mathcal{U} \cap s_i(M_i)$ is a $\tau_\amalg(\mathcal{U})$-open (closed) set. Thus $\mathcal{U} \in \tau_\amalg$ or $\mathcal{U}$ is a $\tau_\amalg$-closed set in $\bar{M}$. ■

Now, we have

Lemma 4.3. *The topology $\tau_\amalg$ is the finest one in which all the canonical inclusions $s_i : M_i \to \bar{M}$ are continuous, and we also have $\tau_{\mathcal{F}_{\bar{M}}} \subset \tau_{\mathcal{C}_{\bar{M}}} \subset \tau_\amalg$.*

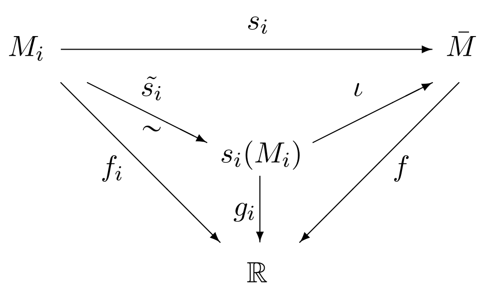
Proof. Let τ be an arbitrary topology on $\bar{M}$ for which all s_i are continuous, that is $s_i^{-1}(\mathcal{U}) = \mathcal{U}_i$ is an open set in M_i for all $i \in I$ and all $\mathcal{U} \in \tau$. Applying s_i to both sides yields

$$s_i(\mathcal{U}_i) = s_i(s_i^{-1}(\mathcal{U})) = \mathcal{U} \cap s_i(M_i) \in \tau_\amalg.$$

But $s_i(\mathcal{U}_i)$ is a $\tau_\amalg$-basic open set. Hence, from Lemma 4.2 it follows that $\mathcal{U}$ is a $\tau_\amalg$-open set in $\bar{M}$. Thus $\tau \subset \tau_\amalg$. In particular, if $\tau = \tau_{\mathcal{F}_{\bar{M}}}$ or $\tau = \tau_{\mathcal{C}_{\bar{M}}}$, we have the inclusion $\tau_{\mathcal{F}_{\bar{M}}} \subset \tau_{\mathcal{C}_{\bar{M}}} \subset \tau_\amalg$. ■

Lemma 4.4. *Let $\tau_\amalg$ be the coproduct topology and $\tau_{\mathcal{F}_{\bar{M}}}$ the $\mathbb{F}$-topology on $\bar{M}$. Then $\tau_\amalg = \tau_{\mathcal{F}_{\bar{M}}}$.*

Proof. First of all, let us draw the diagram below:



This diagram commutes in all its components. It is worth noticing that M_i and $\bar{M}$ are endowed with their $\mathbb{F}$-structures, whereas $s_i(M_i)$ is equipped with the $\mathbb{F}$-subspace structure of $\bar{M}$. Moreover, the maps are related as follows:

$$f \circ \iota = g_i, \;\; g_i \circ \tilde{s_i} = f_i = f \circ s_i,$$

with f, g_i and f_i the structure functions, and ι, s_i, $\tilde{s_i}$ smooth and injective maps such that $\tilde{s_i}$ is a diffeomorphism (see Batubenge [2, p. 20], also Kriegl and Michor [19, p. 80]).

Now let us assume that $\mathcal{U} \in \tau_{\amalg}$. So $\mathcal{U} = s_i(\mathcal{U}_i)$ with $\mathcal{U}_i \in \tau_{\mathcal{F}_{M_i}}$ for some $i \in I$. It follows that $\mathcal{U}_i = \bigcup_{j\in J} f_{ji}^{-1}(0,1)$, with f_{ji} running over $\mathcal{F}_{M_i}$ and j describing structures functions on M_i. Therefore,

$$\mathcal{U} = \bigcup_{i\in I} s_i\Big(\bigcup_{f_{ij}\in\mathcal{F}_{M_i}} f_{ji}^{-1}(0,1)\Big) = \bigcup_{i\in I}\bigcup_{f_{ij}\in\mathcal{F}_{M_i}} (s_i\, f_{ji}^{-1}(0,1)).$$

Now,

$$\mathcal{U} = \bigcup_{i\in I}\bigcup_{j\in J}(s_i(\tilde{s_i}^{-1} \circ g_{ji}^{-1}(0,1))) = \bigcup_{(i,j)\in I\times J} s_i\,\tilde{s_i}^{-1}(g_{ji}^{-1}(0,1)).$$

Thus,

$$\mathcal{U} = \bigcup_{(i,j)\in I\times J} (g_{ji}^{-1}(0,+\infty) \cap s_i(M_i)) = \bigcup_{(i,j)\in I\times J} g_{ji}^{-1}(0,1)$$

since $g_{ji}^{-1}(0,1) \subset s_i(M_i)$ for each $i \in I$. If we fix i, then

$$\bigcup_{j\in J} g_{ji}^{-1}(0,1) \in \tau_{\mathcal{F}_{s_i(M_i)}}$$

and since $f = (f_i)_i$, one has

$$\bigcup_{j\in J} g_{ji}^{-1}(0,1) = \bigcup_{j\in J}(\iota^{-1} \circ f_j{}^{-1}(0,1)) = \bigcup_{j\in J}[f_j{}^{-1}(0,1) \cap s_i(M_i)].$$

Hence,

$$\bigcup_{j\in J} g_{ji}^{-1}(0,1) \in \tau_{\mathcal{F}_{\bar{M}}}(s_i(M_i)),$$

the trace topology on $s_{(M_i)}$. Thus, $\bigcup_j g_{ji}^{-1}(0,+\infty) \subset s_i(M_i) \subset \bar{M}$ and $\tau_{\mathcal{F}_{s_i(M_i)}} = \tau_{\mathcal{F}_{\bar{M}}}(s_i(M_i))$ with $f_j \in \mathcal{F}_{\bar{M}}$ such that $f_j \circ \iota = g_{ji}$. That is, $s_i(M_i) \in \tau_{\mathcal{F}_{\bar{M}}}$ and $\bigcup_{j\in J} g_{ji}^{-1}(0,1) \in \tau_{\mathcal{F}_{\bar{M}}}$. Hence $\tau_{\amalg} \subset \tau_{\mathcal{F}_{\bar{M}}}$. It follows from Lemma 4.3 that

$$\tau_{\amalg} = \tau_{\mathcal{F}_{\bar{M}}}.$$

So,

$$\tau_{\amalg} = \tau_{\mathcal{F}_{\bar{M}}} = \tau_{\mathcal{C}_{\bar{M}}}. \;\blacksquare$$

Notice that $\tau_{\mathcal{F}_{\bar{M}}} = \tau_{\amalg} = \tau_{\mathcal{C}_{\bar{M}}}$ allows us to characterise open sets in either context. Also,

$$s_i(f_i^{-1}(0,1)) = s_i(\tilde{s}_i^{-1} \circ g_i^{-1}(0,1)) = g_i^{-1}(0,1),$$

that is, s_i sends basic open sets of $\tau_{\mathcal{F}_{M_i}}$ to basic open sets of $\tau_{\mathcal{F}_{s_i(M_i)}}$. We have $\mathcal{F}_{s_i(M_i)} = \mathcal{F}_{\bar{M}_{|s_i(M_i)}}$ since $s_i(M_i) \in \tau_{\mathcal{F}_{\bar{M}}}$. It is easy to show that the isomorphism $M_i \simeq s_i(M_i)$ in **FRL** induces an isomorphism of rings of functions between $\mathcal{F}_{s_i(M_i)}$ and $\mathcal{F}_{M_i}$. Furthermore, there is a bijection between the bases of $\tau_{\mathcal{F}_{s_i(M_i)}}$ and $\tau_{\mathcal{F}_{M_i}}$.

Corollary 4.1. *Let $\tau_{\amalg}$ be the coproduct topology and let $\tau_{\mathcal{F}_{\bar{M}}}$ be the Frölicher -topology on $\bar{M}$. Then*

$$\tau_{\mathcal{F}_{s_i(M_i)}} = \tau_{\mathcal{F}_{\bar{M}}}(s_i(M_i)) = \tau_{\amalg}(s_i(M_i)),$$

with respect to the trace topology and $s_i(M_i) \in \tau_{\amalg} = \tau_{\mathcal{F}_{\bar{M}}} = \tau_{\mathcal{C}_{\bar{M}}}$.

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