DE GRUYTER
OPEN

DEMONSTRATIO MATHEMATICA
Vol. XLVII No 4 2014

S.S. Shukla, Akhilesh Yadav*

RADICAL TRANSVERSAL LIGHTLIKE SUBMANIFOLDS OF INDEFINITE PARA-SASAKIAN MANIFOLDS

Abstract. In this paper, we study radical transversal lightlike submanifolds and screen slant radical transversal lightlike submanifolds of indefinite para-Sasakian manifolds giving some non-trivial examples of these submanifolds. Integrability conditions of distributions D and $RadTM$ on radical transversal lightlike submanifolds and screen slant radical transversal lightlike submanifolds of indefinite para-Sasakian manifolds, have been obtained. We also study totally contact umbilical radical transversal lightlike submanifolds of indefinite para-Sasakian manifolds.

1. Introduction

The theory of lightlike submanifolds of a semi-Riemannian manifold was introduced by Duggal and Bejancu [2]. A submanifold M of a semi-Riemannian manifold $\overline{M}$ is said to be lightlike submanifold if the induced metric g on M is degenerate, i.e. there exists a non-zero $X \in \Gamma(TM)$ such that $g(X,Y) = 0$, $\forall Y \in \Gamma(TM)$. In 2003, Duggal and Jin [3] studied the geometry of totally umbilical lightlike submanifolds of a semi-Riemannian manifold. The notion of totally contact umbilical lightlike submanifolds of a semi-Riemannian manifold was considered by several geometers ([7], [8], [15]).

In 2006, Duggal and Sahin [5] studied invariant lightlike submanifolds of an indefinite Sasakian manifold. In 2009, Sahin [10] studied screen slant lightlike submanifolds. In 2010, Yildirim and Sahin [15] defined and studied radical transversal lightlike submanifolds of an indefinite Sasakian manifold. In [12], authors introduced the concept of an ϵ-para-Sasakian structure with some examples. The value of ϵ is not definite, it is either 1 or -1, according as the structure vector field V on $\overline{M}$ is spacelike or timelike.

2010 *Mathematics Subject Classification*: 53C15, 53C40, 53C50.
Key words and phrases: semi-Riemannian manifold, degenerate metric, radical distribution, screen distribution, screen transversal vector bundle, lightlike transversal vector bundle, Gauss and Weingarten formulae.
*Corresponding author.

DOI: 10.2478/dema-2014-0080

In this paper, we study radical transversal lightlike submanifolds of an indefinite para-Sasakian manifold. The paper is arranged as follows. There are some basic results in section 2 . In section 3, we study radical transversal lightlike submanifolds of an indefinite para-Sasakian manifold, giving some examples. Section 4 is devoted to the study of totally contact umbilical radical transversal lightlike submanifolds of an indefinite para-Sasakian manifold. In section 5, we study screen slant radical transversal lightlike submanifolds of an indefinite para-Sasakian manifold and obtain integrability conditions of distributions D and $RadTM$.

2. Preliminaries

A submanifold (M^m, g) immersed in a semi-Riemannian manifold $(\overline{M}^{m+n}, \overline{g})$ is called a lightlike submanifold [2] if the metric g induced from $\overline{g}$ is degenerate and the radical distribution $RadTM$ is of rank r, where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $RadTM$ in TM, that is

$$TM = RadTM \oplus_{orth} S(TM). \tag{2.1}$$

Now consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $RadTM$ in $TM^{\perp}$. Since for any local basis $\{\xi_i\}$ of $RadTM$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$. Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\overline{M}|_M$. Then

$$tr(TM) = ltr(TM) \oplus_{orth} S(TM^{\perp}), \tag{2.2}$$

$$T\overline{M}|_M = TM \oplus tr(TM), \tag{2.3}$$

$$T\overline{M}|_M = S(TM) \oplus_{orth} [RadTM \oplus ltr(TM)] \oplus_{orth} S(TM). \tag{2.4}$$

Following are four cases of a lightlike submanifold $\big(M, g, S(TM), S(TM^{\perp})\big)$:

Case.1 r-lightlike if $r < \min(m, n)$,
Case.2 co-isotropic if $r = n < m$, $S\left(TM^{\perp}\right) = \{0\}$,
Case.3 isotropic if $r = m < n$, $S\left(TM\right) = \{0\}$,
Case.4 totally lightlike if $r = m = n$, $S(TM) = S(TM^{\perp}) = \{0\}$.

The Gauss and Weingarten formulae are given as

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \qquad \forall X, Y \in \Gamma(TM), \tag{2.5}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \qquad \forall V \in \Gamma(tr(TM)), \tag{2.6}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X,Y), \nabla^t_X V\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively. The second fundamental form h is a symmetric $F(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$.

From (2.5) and (2.6), we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \qquad \forall X, Y \in \Gamma(TM), \tag{2.7}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X(N) + D^s(X,N), \qquad \forall N \in \Gamma(ltr(TM)), \tag{2.8}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X(W) + D^l(X,W), \qquad \forall W \in \Gamma(S(TM^\perp)), \tag{2.9}$$

where $h^l(X,Y) = L(h(X,Y))$, $h^s(X,Y) = S(h(X,Y))$, $D^l(X,V) = L(\nabla^t_X V)$, $D^s(X,V) = S(\nabla^t_X V)$. L and S are the projection morphisms of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$, respectively. $\nabla^l$and ∇^s are linear connections on $ltr(TM)$ and $S(TM^\perp)$ called the lightlike connection and screen transversal connection on M, respectively.

Now for any vector field X tangent to M, we put

$$\phi X = PX + FX, \tag{2.10}$$

where PX and FX are tangential and transversal parts of ϕX, respectively.

By using (2.5), (2.7)–(2.9) and metric connection $\overline{\nabla}$, we obtain

$$\overline{g}(h^s(X,Y),W) + \overline{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.11}$$

$$\overline{g}(D^s(X,N),W) = \overline{g}(N, A_W X). \tag{2.12}$$

Denote the projection of TM on $S(TM)$ by $\overline{P}$. Then from the decomposition of the tangent bundle of a lightlike submanifold, we have

$$\nabla_X \overline{P}Y = \nabla^*_X \overline{P}Y + h^*(X, \overline{P}Y), \qquad \forall X, Y \in \Gamma(TM), \tag{2.13}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \qquad \xi \in \Gamma(RadTM). \tag{2.14}$$

By using above equations, we obtain

$$\overline{g}(h^l(X, \overline{P}Y), \xi) = g(A^*_\xi X, \overline{P}Y), \tag{2.15}$$

$$\overline{g}(h^*(X, \overline{P}Y), N) = g(A_N X, \overline{P}Y), \tag{2.16}$$

$$\overline{g}(h^l(X,\xi),\xi) = 0, \qquad A^*_\xi \xi = 0. \tag{2.17}$$

It is important to note that in general ∇ is not a metric connection. Since $\overline{\nabla}$ is metric connection, by using (2.7), we get

$$(\nabla_X g)(Y,Z) = \overline{g}(h^l(X,Y),Z) + \overline{g}(h^l(X,Z),Y). \tag{2.18}$$

A semi-Riemannian manifold $(\overline{M}, \overline{g})$ is called an ϵ-almost paracontact metric manifold [12] if there exists a (1, 1) tensor field ϕ, a vector field V called characteristic vector field and a 1-form η, satisfying

$$\phi^2 X = X - \eta(X)V, \ \eta(V) = \epsilon, \ \eta \circ \phi = 0, \ \phi(V) = 0, \tag{2.19}$$

$$\overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) - \epsilon\eta(X)\eta(Y), \qquad \forall X, Y \in \Gamma(T\overline{M}), \tag{2.20}$$

where $\epsilon = 1$ or -1.

It follows that

$$\overline{g}(V, V) = \epsilon, \tag{2.21}$$

$$\overline{g}(X, V) = \eta(X), \tag{2.22}$$

$$\overline{g}(X, \phi Y) = \overline{g}(\phi X, Y), \forall X, Y \in \Gamma(T\overline{M}). \tag{2.23}$$

Then $(\phi, V, \eta, \overline{g})$ is called an ϵ-almost paracontact metric structure on $\overline{M}$.

An ϵ-almost paracontact metric structure $(\phi, V, \eta, \overline{g})$ is called an indefinite para-Sasakian structure [12] if

$$(\overline{\nabla}_X \phi)Y = -\overline{g}(\phi X, \phi Y)V - \epsilon\eta(Y)\phi^2 X, \qquad \forall X, Y \in \Gamma(T\overline{M}), \tag{2.24}$$

where $\overline{\nabla}$ is Levi-Civita connection with respect to $\overline{g}$.

A semi-Riemannian manifold endowed with an indefinite para-Sasakian structure is called an indefinite para-Sasakian manifold.

From (2.24), we get

$$(\overline{\nabla}_X V) = \phi X, \qquad \forall X \in \Gamma(T\overline{M}). \tag{2.25}$$

Let $(\overline{M}, \overline{g}, \phi, V, \eta)$ be an ϵ-almost paracontact metric manifold. If $\epsilon = 1$, then $\overline{M}$ is said to be a spacelike almost paracontact metric manifold and if $\epsilon = -1$, then $\overline{M}$ is called a timelike almost paracontact metric manifold. In this paper we consider indefinite para-Sasakian manifolds with spacelike characteristic vector field V.

3. Radical transversal lightlike submanifolds

Definition 3.1. Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold, tangent to the structure vector field V, immersed in an indefinite para-Sasakian manifold $(\overline{M}, \overline{g})$. We say that M is radical transversal lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

$$\phi(RadTM) = ltr(TM), \tag{3.1}$$

$$\phi(D) = D, \tag{3.2}$$

where $S(TM) = D \perp \{V\}$ and D is complementary non-degenerate distribution to $\{V\}$ in $S(TM)$.

Let $(\mathbb{R}_q^{2m+1}, \overline{g}, \phi, \eta, V)$ denote the manifold $\mathbb{R}_q^{2m+1}$ with its usual para-Sasakian structure given by

$$\eta = \frac{1}{2}\Big(dz - \sum_{i=1}^{m} y^i dx^i\Big), \qquad V = 2\partial z,$$

$$\overline{g} = \eta \otimes \eta + \frac{1}{4}\Big(-\sum_{i=1}^{\frac{q}{2}} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=\frac{q}{2}+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i\Big),$$

$$\phi\Big(\sum_{i=1}^{m}(X_i\partial x_i + Y_i\partial y_i) + Z\partial z\Big) = \sum_{i=1}^{m}(Y_i\partial x_i + X_i\partial y_i) + \sum_{i=1}^{m} Y_i y^i \partial z,$$

where (x^i, y^i, z) are the cartesian coordinates on $\mathbb{R}_q^{2m+1}$.

Example 1. Let $(\mathbb{R}_2^7, \overline{g}, \phi, \eta, V)$ be an indefinite para-Sasakian manifold, where $\overline{g}$ is of signature $(-,+,+,-,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial y_1, \partial y_2, \partial y_3, \partial z\}$. Suppose M is a submanifold of $\mathbb{R}_2^7$ given by

$$-x^1 = y^2 = u_1, \quad x^2 = y^1 = u_2, \quad x^3 = u_3, \quad y^3 = u_4 \quad \text{and} \quad z = u_5.$$

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5\}$, where

$$Z_1 = 2(-\partial x_1 + \partial y_2 - y^1\partial z), \qquad Z_2 = 2(\partial x_2 + \partial y_1 + y^2\partial z),$$
$$Z_3 = 2(\partial x_3 + y^3\partial z), \qquad Z_4 = 2\partial y_3 \qquad \text{and} \qquad Z_5 = V = 2\partial z.$$

Hence $RadTM = span\{Z_1, Z_2\}$, $S(TM) = span\{Z_3, Z_4, V\}$ and $ltr(TM)$ is spanned by $N_1 = \partial x_1 + \partial y_2 + y^1\partial z$, $N_2 = \partial x_2 - \partial y_1 + y^2\partial z$.

It follows that $\phi Z_1 = 2N_2$, $\phi Z_2 = 2N_1$, $\phi Z_3 = Z_4$, $\phi Z_4 = Z_3$. Thus $\phi RadTM = ltr(TM)$ and $\phi D = D$. Hence M is a radical transversal 2-lightlike submanifold of $\mathbb{R}_2^7$.

Example 2. Let $(\mathbb{R}_2^9, \overline{g}, \phi, \eta, V)$ be an indefinite para-Sasakian manifold, where $\overline{g}$ is of signature $(-,+,+,+,-,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}$. Suppose M is a submanifold of $\mathbb{R}_2^9$ given by $x^1 = y^2 = u_1$, $-x^2 = y^1 = u_2$, $x^3 = y^4 = u_3$, $x^4 = y^3 = u_4$ and $z = u_5$.
The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5\}$, where

$$Z_1 = 2(\partial x_1 + \partial y_2 + y^1\partial z), \qquad Z_2 = 2(-\partial x_2 + \partial y_1 - y^2\partial z)$$
$$Z_3 = 2(\partial x_3 + \partial y_4 + y^3\partial z), \qquad Z_4 = 2(\partial x_4 + \partial y_3 + y^4\partial z),$$
$$Z_5 = V = 2\partial z.$$

Hence $RadTM = span\{Z_1, Z_2\}$ and $S(TM) = span\{Z_3, Z_4, V\}$.

Now $ltr(TM)$ is spanned by $N_1 = \partial x_1 - \partial y_2 + y^1\partial z$, $N_2 = \partial x_2 + \partial y_1 + y^2\partial z$ and $S(TM^\perp)$ is spanned by $W_1 = 2(\partial x_3 - \partial y_4 + y^3\partial z), W_2 = 2(-\partial x_4 + \partial y_3 - y^4\partial z)$.

It follows that $\phi Z_1 = 2N_2$, $\phi Z_2 = 2N_1$, $\phi Z_3 = Z_4$, $\phi Z_4 = Z_3$, $\phi W_1 = W_2$ and $\phi W_2 = W_1$. Thus $\phi RadTM = ltr(TM)$, $\phi D = D$ and $\phi S(TM^{\perp}) = S(TM^{\perp})$. Hence M is a radical transversal 2-lightlike submanifold of $\mathbb{R}^9_2$.

Theorem 3.1. *Let M be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then the distribution $S(TM^{\perp})$ is invariant with respect to ϕ, i.e. $\phi(S(TM^{\perp})) \subseteq S(TM^{\perp})$.*

Proof. Let M be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then from (2.23), we have

$$\overline{g}(\phi W, \xi) = \overline{g}(W, \phi\xi) = 0, \quad \forall W \in \Gamma(S(TM^{\perp})) \text{ and } \forall \xi \in \Gamma(RadTM), \tag{3.3}$$

$$\overline{g}(\phi W, N) = \overline{g}(W, \phi N) = 0, \quad \forall W \in \Gamma(S(TM^{\perp})) \text{ and } \forall N \in ltr(TM). \tag{3.4}$$

From (3.3) and (3.4), we get

$$\phi(S(TM^{\perp})) \cap RadTM = \{0\} \quad \text{and} \quad \phi(S(TM^{\perp})) \cap ltr(TM) = \{0\}\,.$$

From (3.2), we have

$$\overline{g}(\phi W, X) = \overline{g}(W, \phi X) = 0, \qquad \forall X \in (S(TM)), \tag{3.5}$$

which shows that $\phi(S(TM^{\perp})) \cap S(TM) = \{0\}$. Therefore the distribution $S(TM^{\perp})$ is invariant with respect to ϕ. This completes the proof. ∎

Let M be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Let P_1 and P_2 be the projection morphisms on $RadTM$ and D, respectively. Then, for $X \in \Gamma(TM)$, we have

$$X = P_1X + P_2X + \eta(X)V, \tag{3.6}$$

where $P_1X \in \Gamma(RadTM)$ and $P_2X \in \Gamma(D)$.

Applying ϕ to (3.6), we obtain

$$\phi X = \phi P_1X + \phi P_2X, \tag{3.7}$$

where $\phi P_1X \in \Gamma(ltr(TM))$ and $\phi P_2X \in \Gamma(D)$.

From (2.24), we have

$$\overline{\nabla}_X\phi Y - \phi\overline{\nabla}_XY = -g(\phi X, \phi Y)V - \eta(Y)\phi^2X. \tag{3.8}$$

In view of (2.7), (2.8), (3.7) and (3.8), we obtain

$$\begin{aligned} &- g(\phi X, \phi Y)V - \eta(Y)\phi^2X \\ &\quad = \nabla_X\phi P_2Y + h^l(X, \phi P_2Y) + h^s(X, \phi P_2Y) \\ &\qquad - A_{\phi P_1Y}X + \nabla^l_X(\phi P_1Y) + D^s(X, \phi P_1Y) \\ &\qquad - \phi P_2(\nabla_XY) - \phi P_1(\nabla_XY) - \phi h^l(X, Y) - \phi h^s(X, Y). \end{aligned} \tag{3.9}$$

Now equating tangential, screen transversal and lightlike transversal components in both sides in equation (3.9) respectively, we obtain

$$g(\phi X,\phi Y)V+\eta(Y)\phi^2X=\phi P_2(\nabla_XY)+\phi h^l(X,Y)+A_{\phi P_1Y}X-\nabla_X\phi P_2Y, \tag{3.10}$$

$$h^l(X,\phi P_2Y)+\nabla^l_X(\phi P_1Y)+\phi P_1(\nabla_XY)=0, \tag{3.11}$$

$$h^s(X,\phi P_2Y)+D^s(X,\phi P_1Y)-\phi h^s(X,Y)=0. \tag{3.12}$$

Lemma 3.2. *Let M be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then we have*

(i) $g(\nabla_XY,V)=\overline{g}(Y,\phi X),\forall X,Y\in\Gamma(TM)-\{V\}$,
(ii) $g([X,Y],V)=0$, $\forall X,Y\in\Gamma(TM)-\{V\}$.

Proof. Let M be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then (2.7), we have

$$g(\nabla_XY,V)=\overline{g}(\overline{\nabla}_XY,V),\qquad \forall X,Y\in\Gamma(TM)-\{V\}. \tag{3.13}$$

Since $\overline{\nabla}$ is a metric connection, from (3.13) we get

$$g(\nabla_XY,V)=\nabla_Xg(Y,V)-\overline{g}(Y,\overline{\nabla}_XV), \tag{3.14}$$

which implies

$$g(\nabla_XY,V)=-\overline{g}(Y,\overline{\nabla}_XV),\qquad \forall X,Y\in\Gamma(TM)-\{V\}. \tag{3.15}$$

From (2.25) and (3.15), we obtain

$$g(\nabla_XY,V)=-\overline{g}(Y,\phi X),\qquad \forall X,Y\in\Gamma(TM)-\{V\}. \tag{3.16}$$

On interchanging X and Y in (3.16), we get

$$g(\nabla_YX,V)=-\overline{g}(X,\phi Y),\qquad \forall X,Y\in\Gamma(TM)-\{V\}. \tag{3.17}$$

From (2.23), (3.16) and (3.17), we have

$$g([X,Y],V)=0,\quad \forall X,Y\in\Gamma(TM)-\{V\}.\ \blacksquare$$

Theorem 3.3. *Let M be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then D is integrable if and only if $h^l(X,\phi Y)=h^l(Y,\phi X)$, $\forall X,Y\in\Gamma(D)$.*

Proof. Let M be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. On interchanging the role of X and Y in equation (3.11), we obtain

$$h^l(Y,\phi P_2X)+\nabla^l_Y(\phi P_1X)+\phi P_1(\nabla_YX)=0,\qquad \forall X,Y\in\Gamma(D). \tag{3.18}$$

Then from (3.11) and (3.18), we get

$$h^l(X,\phi Y) - h^l(Y,\phi X) = \phi P_1[X,Y], \qquad \forall X,Y \in \Gamma(D). \tag{3.19}$$

Since D is integrable if and only if $[X,Y] \in \Gamma(D)$, $\forall X,Y \in \Gamma(D)$.

The proof follows from (3.19) and Lemma 3.2. ∎

Theorem 3.4. *Let M be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then $RadTM$ is integrable if and only if $A_{\phi X}Y = A_{\phi Y}X$, $\forall X,Y \in \Gamma(RadTM)$.*

Proof. Let M be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then from (3.10), we have

$$A_{\phi Y}X + \phi P_2(\nabla_X Y) + \phi h^l(X,Y) = 0, \qquad \forall X,Y \in \Gamma(RadTM). \tag{3.20}$$

Interchanging the role of X and Y in (3.20), we obtain

$$A_{\phi X}Y + \phi P_2(\nabla_Y X) + \phi h^l(Y,X) = 0, \qquad \forall X,Y \in \Gamma(RadTM). \tag{3.21}$$

Now from (3.20) and (3.21), we get

$$\phi P_2(\nabla_X Y) - \phi P_2(\nabla_Y X) + \phi h^l(X,Y) - \phi h^l(Y,X) = A_{\phi X}Y - A_{\phi Y}X.$$

Since h^l is symmetric, from above equation, we obtain

$$\phi P_2[X,Y] = A_{\phi X}Y - A_{\phi Y}X, \qquad \forall X,Y \in \Gamma(RadTM). \tag{3.22}$$

Since $Rad(TM)$ is integrable if and only if $[X,Y] \in \Gamma(RadTM)$, $\forall X,Y \in \Gamma(RadTM)$.

The proof follows from (3.22) and Lemma 3.2. ∎

Theorem 3.5. *Let M be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then $RadTM \oplus \{V\}$ defines a totally geodesic foliation on M if and only if $\overline{g}(\phi Y,X)\eta(Z) = -\overline{g}(A_{\phi Y}X,\phi Z)$, $\forall X,Y \in \Gamma(RadTM) \oplus \{V\}$ and $Z \in \Gamma(D)$.*

Proof. Let M be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. By definition of radical transversal lightlike submanifold, $RadTM \oplus \{V\}$ defines a totally geodesic foliation if and only if $\overline{g}(\nabla_X Y,Z) = 0$, $\forall X,Y \in \Gamma(RadTM) \oplus \{V\}$ and $Z \in \Gamma(S(TM))$.

Since $\overline{\nabla}$ is a metric connection, using (2.7), we have

$$\overline{g}(\nabla_X Y,Z) = X\overline{g}(X,Z) - \overline{g}(Y,\overline{\nabla}_X Z) = -\overline{g}(Y,\overline{\nabla}_X Z), \quad \forall Z \in \Gamma(D) \text{ and } \forall X,Y \in \Gamma(RadTM) \oplus \{V\}. \tag{3.23}$$

Using (2.7), (2.20), (2.24) and (3.23), we get

$$\overline{g}(\nabla_X Y,Z) = -\overline{g}(\phi Y,X)\eta(Z) - \overline{g}(\phi Y,\nabla_X \phi Z), \quad \forall Z \in \Gamma(D) \text{ and } \forall X,Y \in \Gamma(RadTM) \oplus \{V\}. \tag{3.24}$$

From (2.13), (2.16) and (3.24), we have

$$\overline{g}(\nabla_X Y, Z) = -\overline{g}(\phi Y, X)\eta(Z) - \overline{g}(A_{\phi Y}X, \phi Z), \quad \forall Z \in \Gamma(D) \text{ and } \forall X, Y \in \Gamma(RadTM) \oplus \{V\}. \tag{3.25}$$

The proof follows from (3.25) and Lemma 3.2. ■

Theorem 3.6. *Let M be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then screen distribution defines a totally geodesic foliation if and only if $A^*_{\phi N}X$ has no components in D, $\forall N \in \Gamma(ltr(TM))$ and $\forall X \in \Gamma(S(TM))$.*

Proof. Let M be a radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. By definition of radical transversal lightlike submanifold, $S(TM)$ defines a totally geodesic foliation if and only if $\overline{g}(\nabla_X Y, N) = 0$, $\forall X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$. From (2.7), we have

$$\overline{g}(\nabla_X Y, N) = \overline{g}(\overline{\nabla}_X Y, N), \quad \forall X, Y \in \Gamma(S(TM)) \text{ and } N \in \Gamma(ltr(TM)). \tag{3.26}$$

From (2.20), (2.24) and (3.26), we get

$$\overline{g}(\nabla_X Y, N) = \overline{g}(\overline{\nabla}_X \phi Y, \phi N), \quad \forall X, Y \in \Gamma(S(TM)) \text{ and } N \in \Gamma(ltr(TM)). \tag{3.27}$$

In view of equations (2.7), (2.15) and (3.27), we obtain

$$\overline{g}(\nabla_X Y, N) = \overline{g}(A^*_{\phi N}X, \phi Y), \quad \forall X, Y \in \Gamma(S(TM)) \text{ and } N \in \Gamma(ltr(TM)). \tag{3.28}$$

The proof follows from (3.28) and Lemma 3.2. ■

4. Totally contact umbilical radical transversal lightlike submanifolds

Definition 4.1. A lightlike submanifold M, tangent to the structure vector field V, of an indefinite para-Sasakian manifold $\overline{M}$ is said to be totally contact umbilical radical transversal lightlike submanifold if the second fundamental form h of M satisfies:

$$h^l(X, Y) = [g(X, Y) - \eta(X)\eta(Y)]\alpha_L + \eta(X)h^l(Y, V) + \eta(Y)h^l(X, V), \quad \forall X, Y \in \Gamma(TM) \text{ and } \alpha_L \in \Gamma(ltr(TM)). \tag{4.1}$$

$$h^s(X, Y) = [g(X, Y) - \eta(X)\eta(Y)]\alpha_S + \eta(X)h^s(Y, V) + \eta(Y)h^s(X, V), \quad \forall X, Y \in \Gamma(TM) \text{ and } \alpha_S \in \Gamma(S(TM^\perp)). \tag{4.2}$$

Theorem 4.1. *Let M be a totally contact umbilical radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then distribution D is integrable.*

Proof. Let M be a totally contact umbilical radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then for any $X, Y \in \Gamma(D)$ and $N \in \Gamma(ltr(TM))$, we have

$$\overline{g}([X,Y],N) = \overline{g}(\nabla_X Y, N) - \overline{g}(\nabla_Y X, N). \tag{4.3}$$

Now from (2.7), (2.20) and (4.3), we have

$$\overline{g}([X,Y],N) = \overline{g}(h^l(X,\phi Y),\phi N) - \overline{g}(h^l(Y,\phi X),\phi N). \tag{4.4}$$

Replacing Y by ϕY in (4.1), we get

$$h^l(X,\phi Y) = [g(X,\phi Y)]\alpha_L + \eta(X)h^l(\phi Y, V), \qquad \forall X,Y \in \Gamma(D). \tag{4.5}$$

Similarly, we have

$$h^l(Y,\phi X) = [g(Y,\phi X)]\alpha_L, \qquad \forall X,Y \in \Gamma(D). \tag{4.6}$$

Now, from (4.4), (4.5) and (4.6), we get

$$\overline{g}([X,Y],N) = \overline{g}(g(X,\phi Y)\alpha_L,\phi N) - \overline{g}(g(Y,\phi X)\alpha_L,\phi N), \tag{4.7}$$

which implies

$$\overline{g}([X,Y],N) = g(Y,\phi X)(\overline{g}(\alpha_L,\phi N) - \overline{g}(\alpha_L,\phi N)) = 0, \quad \forall X,Y \in \Gamma(D) \text{ and } N \in \Gamma(ltr(TM)). \tag{4.8}$$

The proof follows from (4.8) and Lemma 3.2. ∎

Theorem 4.2. *Let M be a totally contact umbilical radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then $\alpha_L = 0$ if and only if $h^*(X,\phi Y) = 0$, $\forall X,Y \in \Gamma(D)$.*

Proof. Let M be a totally contact umbilical radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then from (3.9), we have

$$\begin{aligned} -g(\phi X,\phi Y)V - \eta(Y)\phi^2 X = {} & \nabla_X \phi Y + h^l(X,\phi Y) + h^s(X,\phi Y) \\ & - \phi P_2(\nabla_X Y) - \phi P_1(\nabla_X Y) - \phi h^l(X,Y) \\ & - \phi h^s(X,Y), \ \forall X,Y \in \Gamma(D). \end{aligned} \tag{4.9}$$

Now, from (4.9), we get

$$\overline{g}(\nabla_X \phi Y, \phi Z) - \overline{g}(\phi h^l(X,Y),\phi Z) = 0, \quad \forall X,Y \in \Gamma(D) \text{ and } \forall Z \in \Gamma(Rad(TM)). \tag{4.10}$$

From (2.13), (2.20) and (4.10), we have

$$\overline{g}(h^*(X,\phi Y),\phi Z)-\overline{g}(h^l(X,Y),Z)=0, \quad \forall X,Y\in\Gamma(D) \text{ and } \forall Z\in\Gamma(Rad(TM)). \tag{4.11}$$

Now using (4.1) in (4.11), we get

$$\overline{g}(h^*(X,\phi Y),\phi Z)-\overline{g}(g(X,Y)\alpha_L,Z)=0, \quad \forall X,Y\in\Gamma(D) \text{ and } \forall Z\in\Gamma(Rad(TM)). \tag{4.12}$$

This completes the proof. ■

Theorem 4.3. *Let M be a totally contact umbilical radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then the induced connection ∇ on M is a metric connection if and only if $A_{\phi Y}X=-\eta(X)Y$, for $X\in\Gamma(TM)$ and $Y\in\Gamma(RadTM)$.*

Proof. Let M be a totally contact umbilical radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. It is known that the induced connection is metric connection if and only if $\nabla_X Y\in\Gamma(RadTM)$, for $X\in\Gamma(TM)$ and $Y\in\Gamma(RadTM)$.

From (3.9), we have

$$\phi P_2(\nabla_X Y)+\phi P_1(\nabla_X Y)+\phi h^l(X,Y)+\phi h^s(X,Y)=\nabla^l_X(\phi P_1 Y) - A_{\phi P_1 Y}X+D^s(X,\phi P_1 Y), \ \forall X\in\Gamma(TM) \text{ and } Y\in\Gamma(RadTM). \tag{4.13}$$

Now, using (4.1) and (4.2) in (4.13), we obtain

$$\phi P_2(\nabla_X Y)+\phi P_1(\nabla_X Y)+\eta(X)\phi h^l(Y,V)+\eta(X)\phi h^S(Y,V) = \nabla^l_X(\phi P_1 Y)-A_{\phi P_1 Y}X+D^s(X,\phi P_1 Y), \ \forall X\in\Gamma(TM) \text{ and } Y\in\Gamma(RadTM). \tag{4.14}$$

Taking tangential component of above equation, we get

$$\phi P_2(\nabla_X Y)+\eta(X)\phi h^l(Y,V)=-A_{\phi Y}X, \quad \forall X\in\Gamma(TM) \text{ and } Y\in\Gamma(RadTM). \tag{4.15}$$

Also from (2.7) and (2.25), we have

$$\phi Y=h^l(Y,V), \ \forall Y\in\Gamma(RadTM). \tag{4.16}$$

Now, from (4.15) and (4.16), we get

$$\phi P_2(\nabla_X Y)=-\eta(X)Y-A_{\phi Y}X, \quad \forall X\in\Gamma(TM) \textit{ and } Y\in\Gamma(RadTM). \tag{4.17}$$

The proof follows from (4.17) and Lemma 3.2. ■

5. Screen slant radical transversal lightlike submanifolds

At first, we state the following Lemma for later use:

Lemma 5.1. *Let M be a $2q$-lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$, of index $2q$ such that $2q < \dim(M)$ with structure vector field tangent to M. Then the screen distribution $S(TM)$ of lightlike submanifold M is Riemannian.*

The proof of above Lemma follows as in Lemma 4.1 of [11], so we omit it.

Definition 5.1. Let M be a $2q$-lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ of index $2q$ such that $2q < dim(M)$ with structure vector field tangent to M. Then we say that M is screen slant radical transversal lightlike submanifold of $\overline{M}$ if following conditions are satisfied:

- (i) $\phi(RadTM) = ltr(TM)$,
- (ii) For each non-zero vector field X tangent to D at $x \in U \subset M$, the angle $\theta(X)$ between ϕX and the vector space D_x is constant, i.e. it is independent of the choice of $x \in U \subset M$ and $X \in D_x$, where D is complementary non-degenerate distribution to $\{V\}$ in $S(TM)$ such that $S(TM) = D \perp \{V\}$.

This constant angle $\theta(X)$ is called slant angle of distribution D. A screen slant lightlike submanifold is said to be proper if $D \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.

From the above definition, we have the following decomposition

$$TM = RadTM \perp D \perp \{V\}. \tag{5.1}$$

Theorem 5.2. *Let M be a screen slant radical transversal lightlike submanifold of $\overline{M}$. Then M is radical transversal lightlike submanifold (resp. transversal lightlike submanifold) if and only if $\theta = 0$ (resp. $\theta = \frac{\pi}{2}$).*

The proof of above theorem follows from definitions of radical transversal lightlike submanifolds and transversal lightlike submanifolds.

Example 3. Let $(\mathbb{R}^9_2, \overline{g}, \phi, \eta, V)$ be an indefinite para-Sasakian manifold, where $\overline{g}$ is of signature $(-,+,+,+,-,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}$. Suppose M is a submanifold of $\mathbb{R}^9_2$ given by

$$\begin{aligned} &x^1 = y^2 = u_1,\ x^2 = -y^1 = u_2,\ x^3 = u_3\cos\theta,\ x^4 = -u_3\sin\theta, \\ &y^3 = u_4\cos\theta,\ y^4 = u_4\sin\theta,\ z = u_5. \end{aligned}$$

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5\}$, where

$$Z_1 = 2(\partial x_1 + \partial y_2 + y^1 \partial z),$$
$$Z_2 = 2(\partial x_2 - \partial y_1 + y^2 \partial z),$$
$$Z_3 = 2(\cos\theta \partial x_3 - \sin\theta \partial x_4 + y^3 \cos\theta \partial z - y^4 \sin\theta \partial z),$$
$$Z_4 = 2(\cos\theta \partial y_3 + \sin\theta \partial y_4),$$
$$Z_5 = V = 2\partial z.$$

Hence $RadTM = span\,\{Z_1, Z_2\}$ and $S(TM) = span\,\{Z_3, Z_4, V\}$.

Now $ltr(TM)$ is spanned by $N_1 = \partial x_1 - \partial y_2 + y^1 \partial z$, $N_2 = \partial x_2 + \partial y_1 + y^2 \partial z$ and $S(TM^\perp)$ is spanned by

$$W_1 = 2(\sin\theta \partial x_3 + \cos\theta \partial x_4 + y^3 \sin\theta \partial z + y^4 \cos\theta \partial z),$$
$$W_2 = 2(\sin\theta \partial y_3 - \cos\theta \partial y_4).$$

It follows that $\phi Z_1 = N_2$, $\phi Z_2 = -N_1$, which implies that $\phi RadTM = ltr(TM)$. On otherhand, we can see that $D = span\,\{Z_3, Z_4\}$ is a slant distribution with slant angle 2θ. Hence M is screen slant radical transversal 2-lightlike submanifold of $\mathbb{R}^9_2$.

Now, we denote the projections on $RadTM$ and D in TM by P_1 and P_2 respectively. Similarly, we denote the projections on $ltr(TM)$ and $S(TM^\perp)$ in $tr(TM)$ by Q_1 and Q_2, respectively. Then, we get

$$X = P_1X + P_2X + \eta(X)V, \qquad \forall X \in \Gamma(TM). \tag{5.2}$$

On applying ϕ to (5.2), we have

$$\phi X = \phi P_1 X + \phi P_2 X, \tag{5.3}$$

which gives

$$\phi X = \phi P_1 X + f P_2 X + F P_2 X, \qquad \forall X \in \Gamma(TM), \tag{5.4}$$

where fP_2X (resp. FP_2X) denotes the tangential (resp. transversal) component of $\phi P_2 X$. Thus we get $\phi P_1 X \in ltr(TM), fP_2X \in \Gamma(D)$ and $FP_2X \in \Gamma(S(TM^\perp))$. Also, we have

$$W = Q_1W + Q_2W, \qquad \forall W \in \Gamma(tr(TM)). \tag{5.5}$$

Applying ϕ to (5.5), we obtain

$$\phi W = \phi Q_1 W + \phi Q_2 W, \tag{5.6}$$

which gives

$$\phi W = \phi Q_1 W + BQ_2W + CQ_2W, \tag{5.7}$$

where BQ_2W (resp. CQ_2W)) denote the tangential (resp. transversal) component of $\phi Q_2 W$. Thus we get $\phi Q_1 W \in RadTM, BQ_2W \in \Gamma(D)$ and $CQ_2W \in \Gamma(S(TM^\perp))$.

Now, by using (2.7), (2.8), (2.9), (2.24), (5.4) and (5.7) and equating tangential, lightlike transversal and screen transversal components, we obtain

$$(5.8)\quad -\overline{g}(\phi X,\phi Y)V-\eta(Y)\phi^2X=\nabla_X fP_2Y-A_{FP_2Y}X-A_{\phi P_1Y}X$$
$$-fP_2\nabla_XY+Bh^s(X,Y)+\phi h^l(X,Y),$$

$$(5.9)\quad h^l(X,fP_2Y)+D^l(X,FP_2Y)+\nabla^l_X\phi P_1Y=\phi P_1\nabla_XY,$$

(5.10)
$$D^s(X,\phi P_1Y)+h^s(X,fP_2Y)=Ch^s(X,Y)-\nabla^s_XFP_2Y+FP_2\nabla_XY.$$

Theorem 5.3. *Let M be a $2q$-lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ with structure vector field tangent to M such that $\phi RadTM = ltr(TM)$. Then M is screen slant radical transversal lightlike submanifold if and only if there exists a constant $\lambda\in[0,1]$ such that $P^2X=\lambda(X-\eta(X)V)$, $\forall X\in\Gamma(D)$.*

Proof. Let M be a $2q$-lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ with structure vector field tangent to M such that $\phi RadTM = ltr(TM)$. Suppose there exists a constant λ, such that $P^2X=\lambda(X-\eta(X)V)=\lambda\phi^2X$, $\forall X\in\Gamma(D)$.

Now

$$\cos\theta(X)=\frac{g(\phi X,PX)}{|\phi X||PX|}=\frac{g(X,\phi PX)}{|\phi X||PX|}=\frac{g(X,P^2X)}{|\phi X||PX|}$$
$$=\lambda\frac{g(X,\phi^2X)}{|\phi X||PX|}=\lambda\frac{g(\phi X,\phi X)}{|\phi X||PX|}.$$

From above equation, we get

$$(5.11)\qquad \cos\theta(X)=\lambda\frac{|\phi X|}{|PX|}.$$

Also $|PX|=|\phi X|\cos\theta(X)$, which implies

$$(5.12)\qquad \cos\theta(X)=\frac{|PX|}{|\phi X|}.$$

From (5.11) and (5.12), we get $\cos^2\theta(X)=\lambda(constant)$.

Hence M is a screen slant radical transversal lightlike submanifold.

Conversely, suppose that M is a screen slant radical transversal lightlike submanifold. Then $\cos^2\theta(X)=\lambda$, where λ is a constant. From (5.12), we have

$$(5.13)\qquad \frac{|PX|^2}{|\phi X|^2}=\lambda.$$

Now $g(PX,PX)=\lambda g(\phi X,\phi X)$, which gives $g(X,P^2X)=\lambda g(X,\phi^2X)$. Thus $g(X,(P^2-\lambda\phi^2)X)=0$. Since X is non-null vector, we have

$(P^2 - \lambda\phi^2)X = 0$. Hence

$$P^2X = \lambda\phi^2X = \lambda(X - \eta(X)V), \qquad \forall X \in \Gamma(D). \blacksquare$$

Corollary 5.1. *Let M be a screen slant radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ with slant angle θ, then*

$$g(PX, PY) = \cos^2\theta(g(X,Y) - \eta(X)\eta(Y)), \qquad \forall X, Y \in \Gamma(D), \tag{5.14}$$

$$g(FX, FY) = \sin^2\theta(g(X,Y) - \eta(X)\eta(Y)), \qquad \forall X, Y \in \Gamma(D). \tag{5.15}$$

The proof of above corollary follows using the steps as in the proof of Corollary 3.2 of [10].

Theorem 5.4. *Let M be a screen slant radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then*

(i) *the radical distribution $RadTM$ is integrable if and only if $D^s(Y, \phi X) = D^s(X, \phi Y)$ and $A_{\phi X}Y = A_{\phi Y}X$, $\forall X, Y \in \Gamma(RadTM)$,*
(ii) *the distribution D is integrable if and only if $h^l(X, fY) + D^l(X, FY) = h^l(Y, fX) + D^l(Y, FX)$, $\forall X, Y \in \Gamma(D)$.*

Proof. Let M be a screen slant radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. From (5.10), we get

$$D^s(X, \phi Y) = Ch^s(X,Y) + FP_2\nabla_X Y, \qquad \forall X, Y \in \Gamma(RadTM). \tag{5.16}$$

Interchanging X and Y in (5.16), we get

$$D^s(Y, \phi X) = Ch^s(Y,X) + FP_2\nabla_Y X, \qquad \forall X, Y \in \Gamma(RadTM). \tag{5.17}$$

From (5.16) and (5.17), we get

$$D^s(X, \phi Y) - D^s(Y, \phi X) = FP_2(\nabla_X Y - \nabla_Y X) = FP_2[X,Y]. \tag{5.18}$$

From (5.8), we have

$$A_{\phi Y}X + fP_2\nabla_X Y = Bh^s(X,Y) + \phi h^l(X,Y), \qquad \forall X, Y \in \Gamma(RadTM). \tag{5.19}$$

Interchanging X and Y in (5.19), we get

$$A_{\phi X}Y + fP_2\nabla_Y X = Bh^s(Y,X) + \phi h^l(Y,X), \qquad \forall X, Y \in \Gamma(RadTM). \tag{5.20}$$

From (5.19) and (5.20), we get

$$A_{\phi X}Y - A_{\phi Y}X = fP_2[X,Y], \qquad \forall X, Y \in \Gamma(RadTM). \tag{5.21}$$

The proof of (i) follows from (5.18) and (5.21).

From (5.9), we have

$$h^l(X, fY) + D^l(X, FY) = \phi P_1\nabla_X Y, \qquad \forall X, Y \in \Gamma(D). \tag{5.22}$$

Interchanging X and Y in (5.20), we have

$$h^l(Y, fX) + D^l(Y, FX) = \phi P_1 \nabla_Y X, \quad \forall X, Y \in \Gamma(D). \tag{5.23}$$

From (5.22) and (5.23), we get

$$h^l(X, fY) - h^l(Y, fX) + D^l(X, FY) - D^l(Y, FX) = \phi P_1[X, Y], \quad \forall X, Y \in \Gamma(D). \tag{5.24}$$

Now the proof of (ii) follows from (5.24) and Lemma 3.2. ∎

Theorem 5.5. *Let M be a screen slant radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then the screen distribution $S(TM)$ defines a totally geodesic foliation if and only if $\overline{g}(A^*_{\phi N}X, fY) = -\overline{g}(D^l(X, FY), \phi N)$, $\forall X, Y \in \Gamma(S(TM))$ and $N \in ltr(TM)$.*

Proof. Let M be a screen slant radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. By definition of radical transversal lightlike submanifold, $S(TM)$ defines a totally geodesic foliation if and only if $\overline{g}(\nabla_X Y, N) = 0$, $\forall X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$. From (2.7), we have

$$\overline{g}(\nabla_X Y, N) = \overline{g}(\overline{\nabla}_X Y, N), \quad \forall X, Y \in \Gamma(S(TM)) \text{ and } N \in ltr(TM). \tag{5.25}$$

From (2.20), (2.24) and (5.25), we obtain

$$\overline{g}(\nabla_X Y, N) = \overline{g}(\overline{\nabla}_X \phi Y, \phi N), \quad \forall X, Y \in \Gamma(S(TM)) \text{ and } N \in ltr(TM). \tag{5.26}$$

In view of equations (5.4), (2.7), (2.9) and (5.26), we get

$$\overline{g}(\nabla_X Y, N) = \overline{g}(h^l(X, fY) + D^l(X, FY), \phi N), \quad \forall X, Y \in \Gamma(S(TM)) \text{ and } N \in ltr(TM). \tag{5.27}$$

From (2.15) and (5.27), we have

$$\overline{g}(\nabla_X Y, N) = \overline{g}(A^*_{\phi N}X, fY) + \overline{g}(D^l(X, FY), \phi N), \quad \forall X, Y \in \Gamma(S(TM)) \text{ and } N \in ltr(TM). \tag{5.28}$$

The proof follows from (5.28) and Lemma 3.2. ∎

Theorem 5.6. *Let M be a screen slant radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then $RadTM \oplus \{V\}$ defines a totally geodesic foliation on M if and only if $A_{FZ}X = h^*(X, fZ) + \eta(Z)X$, $\forall X \in \Gamma(RadTM) \oplus \{V\}$ and $Z \in \Gamma(D)$.*

Proof. Let M be a screen slant radical transversal lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. By definition of radical transversal lightlike submanifold, $RadTM \oplus \{V\}$ defines a totally geodesic foliation if and only if $\overline{g}(\nabla_X Y, Z) = 0$, $\forall X, Y \in \Gamma(RadTM) \oplus \{V\}$ and $Z \in \Gamma(S(TM))$.

Since $\overline{\nabla}$ is a metric connection, using (2.7), we have

$$\overline{g}(\nabla_X Y, Z) = -\overline{g}(Y, \overline{\nabla}_X Z),\ Z \in \Gamma(D) \text{ and } \forall X, Y \in \Gamma(RadTM) \oplus \{V\}. \tag{5.29}$$

In view of equations (2.7), (2.20), (2.24) and (5.29), we obtain

$$\overline{g}(\nabla_X Y, Z) = -\overline{g}(\phi Y, X)\eta(Z) - \overline{g}(\phi Y, \nabla_X \phi Z), \quad Z \in \Gamma(D) \text{ and } \forall X, Y \in \Gamma(RadTM) \oplus \{V\}. \tag{5.30}$$

From (2.7), (2.9), (2.13), (5.4) and (5.26), we have

$$\overline{g}(\nabla_X Y, Z) = -\overline{g}(\phi Y, X)\eta(Z) - \overline{g}(\phi Y, h^*(X, fZ) + \overline{g}(\phi Y, A_{FZ}X), \quad Z \in \Gamma(D) \text{ and } \forall X, Y \in \Gamma(RadTM) \oplus \{V\}. \tag{5.31}$$

$$\overline{g}(\nabla_X Y, Z) = \overline{g}(\phi Y, A_{FZ}X - h^*(X, fZ) - \eta(Z)X), \quad Z \in \Gamma(D) and \forall X, Y \in \Gamma(RadTM) \oplus \{V\}. \tag{5.32}$$

The proof follows from (5.32) and Lemma 3.2.

Acknowledgement: Akhilesh Yadav gratefully acknowledges the financial support provided by the Council of Scientific and Industrial Research (C.S.I.R.), India.

References

[1] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics, 203, Birkhauser Boston, Inc., Boston, MA, 2002.

[2] K. L. Duggal, A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Vol. 364 of Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[3] K. L. Duggal, D. H. Jin, *Totally umbilical lightlike submanifolds*, Kodai Math. J. 26 (2003), 49–68.

[4] K. L. Duggal, B. Sahin, *Differential Geomety of Lightlike Submanifolds*, Birkhauser Verlag AG, Basel, Boston, Berlin, 2010.

[5] K. L. Duggal, B. Sahin, *Lightlike submanifolds of indefinite Sasakian manifolds*, Int. J. Math. Math. Sci., Article ID 57585, 2007.

[6] D. L. Johnson, L. B. Whitt, *Totally geodesic foliations*, J. Differential Geom. 15 (1980), 225–235.

[7] S. H. Kon, T. Loo, *Totally contact umbilical semi-invariant submanifolds of a Sasakian manifold*, SUT J. Math. 38 (2002), 75–82.

[8] F. Massamba, *Totally contact umbilical lightlike hypersurfaces of indefinite Sasakian manifolds*, Kodai Math. J. 31 (2008), 338–358.

[9] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.

[10] B. Sahin, *Screen slant lightlike submanifolds*, Int. Electronic J. of Geometry 2 (2009), 41–54.
[11] B. Sahin, C. Yildirim, *Slant lightlike submanifolds of indefinite Sasakian manifolds*, Faculty of Sci. and Math., Univ. of Nis, Serbia, 2 (2012), 71–81.
[12] M. M. Tripathi, E. Kihc, Y. Perktas, S. Keles, *Indefinite almost paracontact metric manifolds*, Int. J. Math. Math. Sci., Article ID 846195, 2010.
[13] Y. Wang, X. Liu, *Generalized transversal lightlike submanifolds of indefinite Sasakian manifolds*, Int. J. Math. Math. Sci., Article ID 361794, 2012.
[14] K. Yano, M. Kon, *Structures on Manifolds*, Vol. 3, Series in Pure Mathematics, World Scientfic, Singapore, 1984.
[15] C. Yildrim, B. Sahin, *Transversal lightlike submanifolds of indefinite Sasakian manifolds*, Turkish J. Math. 34 (2010), 561–583.

S. S. Shukla, A. Yadav
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALLAHABAD
ALLAHABAD-211002, INDIA
E-mail: ssshukla_au@rediffmail.com
akhilesh_mathau@rediffmail.com

Received April 29, 2013.
Communicated by S. Izumiya.