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Alireza Kamel Mirmostafaee, Omid Pourbahari Rahpeyma, Mohsen Erfanian Omidvar

NUMERICAL RADIUS INEQUALITIES FOR FINITE SUMS OF OPERATORS

Abstract. In this paper, we obtain some sharp inequalities for numerical radius of finite sums of operators. Moreover, we give some applications of our result in estimation of spectral radius. We also compare our results with some known results.

1. Introduction

Let $(\mathcal{H}, \langle ., . \rangle)$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ denote the C^* -algebra of all bounded linear operators on $\mathcal{H}$. The numerical radius of $A \in \mathcal{B}(\mathcal{H})$ is defined by

$$\omega(A) = \sup\{|\langle Ax, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}.$$

It is well known that $\omega(.)$ defines a norm on $\mathcal{B}(\mathcal{H})$. In fact, for each $A \in \mathcal{B}(\mathcal{H})$, we have

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\| \tag{1}$$

(see [5, Theorem 1.3-1]). Thus the usual operator norm and the numerical radius are equivalent.

It is of interest to investigate interrelationship between numerical and spectral radius of operators (see e.g. [5, 13, 11, 12, 16]). In Section 2, we will show that if $X, Y \in \mathcal{B}(\mathcal{H})$, $0 \leq \alpha \leq 1$ and $r \geq 1, r' \geq 1$,

$$\begin{aligned}&|\langle Xx, x\rangle|^r |\langle Yx, x\rangle|^{r'}\\ &\qquad \leq \tfrac{1}{2}(\|\alpha|X|^{2r} + (1-\alpha)|X^*|^{2r} + \alpha|Y|^{2r'} + (1-\alpha)|Y^*|^{2r'}\|),\end{aligned}$$

where x is a unit vector in $\mathcal{H}$. We compare our result with the corresponding inequality obtained in [10]. Our results enable us to obtain a sharp inequality for spectral radius.

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In Section 3, we obtain an upper bound for numerical radius by means of Cartesian decomposition. More precisely, we will prove that

$$\omega^r\Big(\sum_{j=1}^{n} A_j\Big) \leq (2n)^{r-1}\sum_{j=1}^{n}\big\||B_j|^{2r}+|C_j|^{2r}\big\|^{\frac{1}{2}},$$

where $A_j \in \mathcal{B}(\mathcal{H})$ and $A_j = B_j + iC_j$, $1 \leq j \leq n$. In some special cases, we show that our result gives a sharper estimation for numerical radius than the corresponding result obtained in [6].

2. Numerical radius for sums of operator

In order to obtain the main result of this section, we need the following known results. The first, one due to McCarthy, is an operator variant of the Hölder inequality (see [4, 15]).

Lemma 2.1. (Hölder–McCarthy inequality) *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator on $\mathcal{H}$. Then the following inequalities hold:*

$$\langle A^r x, x\rangle \geq ||x||^{2(1-r)}\langle Ax, x\rangle^r, \qquad \text{for } x \in \mathcal{H} \text{ if } r \geq 1. \tag{2}$$

$$\langle A^r x, x\rangle \leq ||x||^{2(1-r)}\langle Ax, x\rangle^r, \qquad \text{for } x \in \mathcal{H} \text{ if } 0 < r < 1. \tag{3}$$

The next lemma is known as a generalized mixed Cauchy–Schwarz inequality.

Lemma 2.2. [9] (Cauchy–Schwarz inequality) *Let $A \in \mathcal{B}(\mathcal{H})$ and $0 \leq \alpha \leq 1$ then*

$$|\langle Ax, y\rangle|^2 \leq \big\langle |A|^{2\alpha}x, x\big\rangle\langle |A^*|^{2(1-\alpha)}y, y\rangle \quad (x, y \in \mathcal{H}). \tag{4}$$

The following lemma is a consequence of the convexity of the function $f(t) = t^r, r \geq 1$ (see [1, 2, 8, 14]).

Lemma 2.3. (Bohr's inequality) *Let a_i be a positive real number, $1 \leq i \leq n$. Then for each $r \geq 1$*

$$\Big(\sum_{i=1}^{n} a_i\Big)^r \leq n^{r-1}\sum_{i=1}^{n} a_i^r. \tag{5}$$

The following lemma is a consequence of the classical Jensen inequality concerning the convexity or the concavity of certain power functions.

Lemma 2.4. [8] *For $a, b \geq 0$ and $0 < \alpha < 1$ and $r \neq 0$,*

$$(\alpha a^r + (1-\alpha)b^r)^{\frac{1}{r}} \leq (\alpha a^s + (1-\alpha)b^s)^{\frac{1}{s}}, \qquad \text{for } r \leq s, \tag{6}$$

$$a^{\alpha}b^{(1-\alpha)} \leq (\alpha a^r + (1-\alpha)b^r)^{\frac{1}{r}}, \qquad \text{for } r > 0, \tag{7}$$

$$(a^s + b^s)^{\frac{1}{s}} \leq (a^r + b^r)^{\frac{1}{r}}, \qquad \text{for } 0 < r \leq s, \tag{8}$$

$$(ab)^{\frac{1}{2}} \leq \frac{1}{2}(a+b). \tag{9}$$

Now we are ready to state the main result of this section.

Theorem 2.5. *Let $X, Y \in \mathcal{B}(\mathcal{H})$ and $r \geq 1$, $r' \geq 1$ and $0 \leq \alpha \leq 1$. Then for each unit vector x in $\mathcal{H}$,*

$$|\langle Xx, x\rangle|^r |\langle Yx, x\rangle|^{r'} \leq \tfrac{1}{2}(\|\alpha|X|^{2r} + (1-\alpha)|X^*|^{2r} + \alpha|Y|^{2r'} + (1-\alpha)|Y^*|^{2r'}\|). \tag{10}$$

Proof. For arbitrary unit vectors x in $\mathcal{H}$, we have

$$\begin{aligned}
&\big|\langle Xx, x\rangle\big|^r \big|\langle Yx, x\rangle\big|^{r'}\\
&\leq \big(\langle |X|^{2\alpha}x, x\rangle\langle |X^*|^{2(1-\alpha)}x, x\rangle\big)^{\frac{r}{2}} \big(\langle |Y|^{2\alpha}x, x\rangle\langle |Y^*|^{2(1-\alpha)}x, x\rangle\big)^{\frac{r'}{2}} \quad \text{by (4)}\\
&\leq \tfrac{1}{2}\big((\langle |X|^{2\alpha}x, x\rangle\langle |X^*|^{2(1-\alpha)}x, x\rangle)^r + (\langle |Y|^{2\alpha}x, x\rangle\langle |Y^*|^{2(1-\alpha)}x, x\rangle)^{r'}\big) \quad \text{by (9)}\\
&\leq \tfrac{1}{2}\big(\langle |X|^{2\alpha r}x, x\rangle\langle |X^*|^{2(1-\alpha)r}x, x\rangle + \langle |Y|^{2\alpha r'}x, x\rangle\langle |Y^*|^{2(1-\alpha)r'}x, x\rangle\big) \quad \text{by (2)}\\
&\leq \tfrac{1}{2}\big(\langle |X|^{2r}x, x\rangle^{\alpha}\langle |X^*|^{2r}x, x\rangle^{(1-\alpha)} + \langle |Y|^{2r'}x, x\rangle^{\alpha}\langle |Y^*|^{2r'}x, x\rangle^{(1-\alpha)}\big) \quad \text{by (3)}\\
&\leq \tfrac{1}{2}\big(\langle (\alpha|X|^{2r} + (1-\alpha)|X^*|^{2r})x, x\rangle + \langle (\alpha|Y|^{2r'} + (1-\alpha)|Y^*|^{2r'})x, x\rangle\big) \quad \text{by (7)}\\
&\leq \tfrac{1}{2}\big(\big\|\alpha|X|^{2r} + (1-\alpha)|X^*|^{2r} + \alpha|Y|^{2r'} + (1-\alpha)|Y^*|^{2r'}\big\|\big). \ \blacksquare
\end{aligned}$$

The above theorem enables us to obtain a sharp estimation for spectral radius of some operators:

Corollary 2.6. *Let $A \in \mathcal{B}(\mathcal{H})$ and $r(A)$ denote the spectral radius of A. Then*

$$r(A) \leq \frac{1}{\sqrt[4]{2}}\Big\||A^2|^2 + |(A^*)^2|^2\Big\|^{\frac{1}{4}} \leq \|A\|.$$

Proof. Put $\alpha = \frac{1}{2}$, $X = Y = A^2$ and $r = r' = 1$ in Theorem (2.5), to obtain

$$\omega(A^2) \leq \frac{\sqrt{2}}{2}\Big\||A^2|^2 + |(A^*)^2|^2\Big\|^{\frac{1}{2}}.$$

It follows that

$$r^2(A) = r(A^2) \leq \omega(A^2) \leq \frac{\sqrt{2}}{2}\Big\||A^2|^2 + |(A^*)^2|^2\Big\|^{\frac{1}{2}}.$$

Therefore

$$r(A) \leq \frac{1}{\sqrt[4]{2}}\Big\||A^2|^2 + |(A^*)^2|^2\Big\|^{\frac{1}{4}} \leq \|A\|. \ \blacksquare \tag{11}$$

Corollary 2.7. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\omega(AB + BA) \leq \frac{\sqrt{2}}{2}\Big\||AB + BA|^2 + |A^*B^* + B^*A^*|^2\Big\|^{\frac{1}{2}}. \tag{12}$$

Proof. Put $X = Y = AB + BA$, $\alpha = \frac{1}{2}$ and $r = r' = 1$ in Theorem 2.5. $\blacksquare$

Remark 2.8. In [11, Theorem 1] Kittaneh proved that for each $A_1, A_2, B_1, B_2 \in \mathcal{B}(\mathcal{H})$,

$$\begin{aligned}&r(A_1B_1 + A_2B_2)\\ &\quad\leq \frac{1}{2}\Big(\|B_1A_1\| + \|B_2A_2\| + \sqrt[2]{(\|B_1A_1\| - \|B_2A_2\|)^2 + 4\,\|B_1A_2\|\,\|B_2A_1\|}\Big).\end{aligned}$$

In particular, when $A_1 = B_2 = A$ and $A_2 = B_1 = B$, it follows that

$$r(AB+BA) \leq \frac{1}{2}\Big(\|AB\| + \|BA\| + \sqrt[2]{(\|AB\| - \|BA\|)^2 + 4\,\|A^2\|\,\|B^2\|}\Big) \tag{13}$$

(see [11, Corollary 2]). If A and B are normal operators in $\mathcal{B}(\mathcal{H})$, we have $\|AB\| = \|BA\|$ and $\|A^2\| = \|A\|^2$. Hence

$$\begin{aligned}r(AB+BA) &\leq \frac{1}{2}\Big(\|AB\| + \|BA\| + \sqrt[2]{(\|AB\| - \|BA\|)^2 + 4\,\|A^2\|\,\|B^2\|}\Big)\\ &= \|AB\| + \|A\|\|B\|.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\frac{\sqrt{2}}{2}\Big\||AB+BA|^2 + |A^*B^* + B^*A^*|^2\Big\|^{\frac{1}{2}}\\ &\qquad\leq \frac{\sqrt{2}}{2}\Big(\||AB+BA|^2\| + \||A^*B^* + B^*A^*|^2\|\Big)^{\frac{1}{2}}\\ &\qquad= \frac{\sqrt{2}}{2}\big(\|AB+BA\|^2 + \|A^*B^* + B^*A^*\|^2\big)^{\frac{1}{2}}\\ &\qquad= \big(\|AB+BA\|^2\big)^{\frac{1}{2}} = \|AB+BA\|.\end{aligned}$$

Hence, in this case, by Corollary 2.7

$$\begin{aligned}r(AB+BA) &\leq \omega(AB+BA)\\ &\leq \frac{\sqrt{2}}{2}\Big\||AB+BA|^2 + |A^*B^* + B^*A^*|^2\Big\|^{\frac{1}{2}}\\ &\leq \|AB+BA\| \leq \|AB\| + \|A\|\|B\|\\ &= \frac{1}{2}\Big(\|AB\| + \|BA\| + \sqrt[2]{(\|AB\| - \|BA\|)^2 + 4\,\|A^2\|\,\|B^2\|}\Big).\end{aligned}$$

Therefore Corollary 2.7 gives a sharper inequity than (13) in this case.

Remark 2.9. Kittaneh in [10] proved that

$$\omega(AB+BA) \leq \frac{1}{2}\|AA^* + A^*A + BB^* + B^*B\| \quad (A, B \in \mathcal{B}(\mathcal{H})). \tag{14}$$

Let A, B be normal operators then

$$\frac{1}{2}\|AA^* + A^*A + BB^* + B^*B\| = \|AA^* + BB^*\|.$$

On the other hand, according to the above remark

$$\begin{aligned}\frac{\sqrt{2}}{2}\big\||AB+BA|^2+|A^*B^*+B^*A^*|^2\big\|^{\frac{1}{2}}&\\ &\leq \|AB+BA\|=\|A^*B^*+B^*A^*\|\leq\|A^*B^*\|+\|B^*A^*\|\\ &\leq\frac{1}{2}\big(\|AA^*+B^*B\|+\|BB^*+A^*A\|\big)\quad [1, \text{Corollary4.3}]\\ &=\|AA^*+BB^*\|.\end{aligned}$$

Therefore, Corollary 2.7 is sharper than (14) in this case.

Corollary 2.10. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $r \geq 1$ then*

$$\omega^r(B^*A) \leq \frac{\sqrt{2}}{2}\big\||B^*A|^{2r}+|A^*B|^{2r}\big\|^{\frac{1}{2}}. \tag{15}$$

Proof. Put $X = Y = B^*A$, $r = r'$ and $\alpha = \frac{1}{2}$ in Theorem 2.5. ∎

Remark 2.11. Dragomir has shown that in [3], for $A, B \in \mathcal{B}(\mathcal{H})$ and $r \geq 1$

$$\omega^r(B^*A) \leq \frac{1}{2}\big\|(B^*B)^r+(A^*A)^r\big\|. \tag{16}$$

Let $A, B \geq 0$ and $AB = BA$. Then

$$\omega^r(B^*A) \leq \frac{1}{2}\big\|(B^*B)^r+(A^*A)^r\big\| = \frac{1}{2}\big\|B^{2r}+A^{2r}\big\|.$$

By Corollary 2.10, we have

$$\omega^r(B^*A) \leq \frac{\sqrt{2}}{2}\big\||B^*A|^{2r}+|A^*B|^{2r}\big\|^{\frac{1}{2}} = \big\||AB|^r\big\|.$$

On the other hand

$$0 \leq (AB)^r \leq \frac{1}{2}(B^{2r}+A^{2r}).$$

Hence, in this case, Corollary 2.10 gives a sharper inequality than (16).

3. Numerical radius and Cartesian decomposition

The main aim of this section is to obtain an upper bound for numerical radius by means of the Cartesian decomposition of operators.

Theorem 3.1. *Let $A_j \in B(\mathcal{H})$ have the Cartesian decomposition $A_j = B_j + iC_j$ for $j = 1, \cdots, n$ and $r \geq 1$. Then*

$$\omega^r\Big(\sum_{j=1}^{n} A_j\Big) \leq (\sqrt{2}\, n)^{r-1}\sum_{j=1}^{n}\big\||B_j|^{2r}+|C_j|^{2r}\big\|^{\frac{1}{2}}. \tag{17}$$

Proof. For any unit vector $x \in \mathcal{H}$, we have

$$\begin{aligned}
\Big|\Big\langle \sum_{j=1}^{n} A_j x, x\Big\rangle\Big|^r &\leq \Big(\sum_{j=1}^{n}\Big(\langle B_j x, x\rangle^2 + \langle C_j x, x\rangle^2\Big)^{\frac{1}{2}}\Big)^r \\
&\leq \Big(\sum_{j=1}^{n}\big(\big\langle |B_j|^2 x, x\big\rangle + \big\langle |C_j|^2 x, x\big\rangle\big)^{\frac{1}{2}}\Big)^r \qquad \text{by Lemma 2.2 for } \alpha = 1 \\
&\leq n^{r-1}\sum_{j=1}^{n}\Big(\big\langle |B_j|^2 x, x\big\rangle + \big\langle |C_j|^2 x, x\big\rangle\Big)^{\frac{r}{2}} \qquad \text{by Lemma 2.3} \\
&\leq (\sqrt{2}\, n)^{r-1}\sum_{j=1}^{n}\Big(\big\langle |B_j|^2 x, x\big\rangle^r + \big\langle |C_j|^2 x, x\big\rangle^r\Big)^{\frac{1}{2}} \qquad \text{by Lemma 2.3} \\
&\leq (\sqrt{2}\, n)^{r-1}\sum_{j=1}^{n}\Big(\big\langle |B_j|^{2r} x, x\big\rangle + \big\langle |C_j|^{2r} x, x\big\rangle\Big)^{\frac{1}{2}} \qquad \text{by Lemma 2.1.}
\end{aligned}$$

Therefore

$$\begin{aligned}
\omega^r\Big(\sum_{j=1}^{n} A_j\Big) &= \sup\Big\{\Big|\Big\langle \sum_{j=1}^{n} A_j x, x\Big\rangle\Big|^r : x \in \mathcal{H}, \|x\| = 1\Big\} \\
&\leq (\sqrt{2}\, n)^{r-1}\sum_{j=1}^{n}\big\||B_j|^{2r} + |C_j|^{2r}\big\|^{\frac{1}{2}}. \blacksquare
\end{aligned}$$

In particular, when $n = 1$, we get to the following result.

Corollary 3.2. *Let $A \in \mathcal{B}(\mathcal{H})$ have the Cartesian decomposition $A = B + iC$. Then*

$$\omega^r(A) \leq 2^{\frac{r-1}{2}}\big\||B|^{2r} + |C|^{2r}\big\|^{\frac{1}{2}} \quad r \geq 1. \tag{18}$$

In [6], the authors proved that

$$\omega^r(A) \leq \||B|^r + |C|^r\| \quad 0 \leq r \leq 2. \tag{19}$$

Therefore, if $1 \leq r \leq 2$ and $|C|^r|B|^r = |B|^r|C|^r$, then

$$(|B|^r + |C|^r)^2 = B^{2r} + C^{2r} + |B|^r|C|^r + |C|^r|B|^r \geq B^{2r} + C^{2r} \geq 0.$$

So that in this case, inequality (18) is sharper than (19).

Theorem 3.3. *Let $A_j \in \mathcal{B}(\mathcal{H})$ have the Cartesian decomposition $A_j = B_j + iC_j$ for $j = 1, \ldots, n$ and $r \geq 1$. Then*

$$\omega^r\Big(\sum_{j=1}^{n} A_j\Big) \leq n^{r-1}2^{\frac{r}{2}-1}\sum_{j=1}^{n}\Big\||B_j + C_j|^{2r} + |B_j - C_j|^{2r}\Big\|^{\frac{1}{2}}.$$

Proof. For every unit vector x in $\mathcal{H}$, we have

$$\left|\left\langle \sum_{j=1}^{n} A_j x, x\right\rangle\right|^r \leq \Big(\sum_{j=1}^{n}(\langle B_j x, x\rangle^2 + \langle C_j x, x\rangle^2)^{\frac{1}{2}}\Big)^r$$

$$\leq \left(\sum_{j=1}^{n}\Big(\frac{1}{2}\Big(\langle (B_j + C_j)x, x\rangle^2 + \langle (B_j - C_j)x, x\rangle^2\Big)\Big)^{\frac{1}{2}}\right)^r$$

$$\leq n^{r-1} 2^{\frac{-r}{2}} \sum_{j=1}^{n}\Big(\langle (B_j + C_j)\, x, x\rangle^2 + \langle (B_j - C_j)\, x, x\rangle^2\Big)^{\frac{r}{2}} \quad \text{by Lemma 2.3}$$

$$\leq n^{r-1} 2^{\frac{-r}{2}} \sum_{j=1}^{n}\Big(\big\langle |B_j + C_j|^2 x, x\big\rangle + \big\langle |B_j - C_j|^2 x, x\big\rangle\Big)^{\frac{r}{2}} \quad \text{by Lemma 2.2}$$

$$\leq n^{r-1} 2^{\frac{r}{2}-1} \sum_{j=1}^{n}\Big(\big\langle |B_j + C_j|^2 x, x\big\rangle^r + \big\langle |B_j - C_j|^2 x, x\big\rangle^r\Big)^{\frac{1}{2}} \quad \text{by Lemma 2.3}$$

$$\leq n^{r-1} 2^{\frac{r}{2}-1} \sum_{j=1}^{n}\Big(\big\langle |B_j + C_j|^{2r} x, x\big\rangle + \big\langle |B_j - C_j|^{2r} x, x\big\rangle\Big)^{\frac{1}{2}} \quad \text{by Lemma 2.1.}$$

If follows from the above inequality that

$$\omega^r\Big(\sum_{j=1}^{n} A_j\Big) \leq n^{r-1} 2^{\frac{r}{2}-1} \sum_{j=1}^{n}\left\| |B_j + C_j|^{2r} + |B_j - C_j|^{2r}\right\|^{\frac{1}{2}}. \ \blacksquare$$

In particular for $n = 1$, we have the following result.

Corollary 3.4. *Let $A = B + iC$ be the Cartesian decomposition of A and $r \geq 1$. Then*

$$\omega^r(A) \leq 2^{\frac{r}{2}-1} \left\|(B + C)^{2r} + (B - C)^{2r}\right\|^{\frac{1}{2}}. \tag{20}$$

Remark 3.5. Kittaneh has shown in [6] that

$$\omega^r(A) \leq \frac{1}{2}\, \||B + C|^r + |B - C|^r\| \ \text{ for } r \geq 2. \tag{21}$$

Note that the inequality (20) holds for each $r \geq 1$, while (21) gives an upper bound for $\omega^r(A)$ only when $r \geq 2$.

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A. K. Mirmostafaee
CENTER OF EXCELLENCE IN ANALYSIS ON ALGEBRAIC STRUCTURES
DEPARTMENT OF PURE MATHEMATICS
FERDOWSI UNIVERSITY OF MASHHAD
P. O. BOX 1159, MASHHAD 91775, MASHHAD, IRAN
E-mail: mirmostafaei@ferdowsi.um.ac.ir

O. P. Rahpeyma
CENTER OF EXCELLENCE IN ANALYSIS ON ALGEBRAIC STRUCTURES
DEPARTMENT OF PURE MATHEMATICS
FERDOWSI UNIVERSITY OF MASHHAD
P. O. BOX 1159, MASHHAD 91775, MASHHAD, IRAN
E-mail: omidpourbahri@yahoo.com

M. E. Omidvar
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE, MASHHAD BRANCH
ISLAMIC AZAD UNIVERSITY, MASHHAD, IRAN
E-mail: mn-erfanian@yahoo.com

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