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HYBRID FIXED POINT THEOREMS IN SYMMETRIC SPACES VIA COMMON LIMIT RANGE PROPERTY

Abstract.   In this paper, we point out that some recent results of Vijaywar et al. (Coincidence and common fixed point theorems for hybrid contractions in symmetric spaces, Demonstratio Math. 45 (2012), 611–620) are not true in their present form. With a view to prove corrected and improved versions of such results, we introduce the notion of common limit range property for a hybrid pair of mappings and utilize the same to obtain some coincidence and fixed point results for mappings defined on an arbitrary set with values in symmetric (semi-metric) spaces. Our results improve, generalize and extend some results of the existing literature especially due to Imdad et al., Javid and Imdad, Vijaywar et al. and some others. Some illustrative examples to highlight the realized improvements are also furnished.

1. Introduction and preliminaries

The classical Banach Contraction Principle is indeed the most fundamental result of metric fixed point theory which is very effectively utilized to establish the existence of solutions of nonlinear Volterra integral equations, Fredholm integral equations, nonlinear integro-differential equations in Banach spaces besides supporting the convergence of algorithms in Computational Mathematics. However, sometimes one may come across situations wherein the full force of metric requirements are not used in the proofs of certain metrical fixed point theorems. Motivated by this fact, Hicks and Rhoades [10] proved some common fixed point theorems in symmetric spaces and showed that a general probabilistic structures admits a compatible symmetric or semi-metric. Mihet [23] pointed out that Hicks and Rhoades [10] have inadvertently used triangle inequality in their results.

Though the notion of weak commutativity in metric fixed point theory was introduced by Sessa [29] in 1982, yet the earliest use of a weak commutativity condition for a hybrid pair can be traced back to Itoh and Takahashi [16] wherein authors proved some coincidence point theorems in metric spaces. Kaneko and Sessa [21] extended the concept of compatibility (due to Jungck [18]) to a hybrid pair of mappings defined on metric spaces. Pathak [28] extended the concept of compatibility (due to Jungck [19]) by defining weak compatibility for hybrid pairs of mappings (including single valued case) and utilize the same to prove some coincidence and common fixed point theorems satisfying a suitable contraction condition. Naturally, compatible mappings are weakly compatible but not conversely.

This remains an established fact that the contractive conditions do not ensure the existence of fixed points unless the underlying space is assumed compact or the contractive conditions are replaced by relatively stronger conditions. Firstly, Pant [26, 27] studied metrical fixed point theorems for single-valued non-compatible mappings under strict contractions. In 2004, Kamran [20] extended the notion of the property (E.A) (due to Aamri and Moutawakil [1]) to hybrid pairs of mappings and proved some coincidence and common fixed point theorems. It is observed in Imdad and Ali [13] that the property (E.A) buys the suitable required containment of the range of one mapping into the range of another up to a pair of mappings. Sintunavarat and Kumam [31] coined the idea of 'common limit range property' for single-valued mappings whose use does not demand the completeness (or closedness) of the underlying subspaces.

In 2010, Ali and Imdad [3] noticed some errors in certain results of Singh and Hashim [30] and proved some fixed point results for two pairs of hybrid mappings in symmetric (or semi-metric) spaces. Recently, Vijaywar et al. [33] proved some fixed point theorems for a pair of hybrid mappings satisfying strict contractive conditions in symmetric (semi-metric) spaces under the property (E.A). Our main purpose in this paper is twofold. Firstly, we point out that some results of Vijaywar et al. [33] are not true in their present form. Secondly, we introduce the notion of common limit range property for a pair of hybrid mappings which are defined on an arbitrary nonempty set with values in a symmetric (semi-metric) space and utilize the same to prove corrected and improved versions of some results due to Vijaywar et al. [33] in symmetric spaces. We furnish some examples to support our main result besides deriving some related results.

The following definitions and results will be needed in the sequel.

A symmetric on a non-empty set X is a non-negative real valued function d on $X \times X$ such that

(1) $d(x, y) = 0$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$.

Let d be a symmetric on a set X and for $r > 0$ and any $x \in X$, let $\mathcal{B}(x, r) = \{y \in X : d(x, y) < r\}$. A topology $\mathcal{T}(d)$ on X is given by $U \in \mathcal{T}(d)$ if and only if for each $x \in U$, $\mathcal{B}(x, r) \subset U$ for some $r > 0$. A symmetric d is a semi-metric if for each $x \in X$ and each $r > 0$, $\mathcal{B}(x, r)$ is a neighborhood of x in the topology $\mathcal{T}(d)$. Note that $\lim_{n \to \infty} d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology $\mathcal{T}(d)$.

Notice that symmetric spaces are not essentially Hausdorff and also the symmetric d is not continuous in general. Therefore, in the course of proving fixed point theorems, some additional axioms are required. The following axioms are available in the papers of Aliouche [4], Galvin and Shore [7], Hicks and Rhoades [10] and Wilson [34].

(W_3) [34] Given $\{x_n\}$; $x, y \in X$, $\lim_{n \to \infty} d(x_n, x) = 0$ and $\lim_{n \to \infty} d(x_n, y) = 0$ imply $x = y$.

(W_4) [34] Given $\{x_n\}, \{y_n\}$; $x \in X$, $\lim_{n \to \infty} d(x_n, x) = 0$ and $\lim_{n \to \infty} d(x_n, y_n) = 0$ imply $\lim_{n \to \infty} d(y_n, x) = 0$.

(HE) [4] Given $\{x_n\}, \{y_n\}$; $x \in X$, $\lim_{n \to \infty} d(x_n, x) = 0$ and $\lim_{n \to \infty} d(y_n, x) = 0$ imply $\lim_{n \to \infty} d(x_n, y_n) = 0$.

$(1C)$ [7] A symmetric d is said to be 1-continuous if $\lim_{n \to \infty} d(x_n, x) = 0$ implies $\lim_{n \to \infty} d(x_n, y) = d(x, y)$, where $\{x_n\}$ is a sequence in X and $x, y \in X$.

(CC) [7] A symmetric d is said to be continuous if $\lim_{n \to \infty} d(x_n, x) = 0$ and $\lim_{n \to \infty} d(y_n, y) = 0$ imply $\lim_{n \to \infty} d(x_n, y_n) = d(x, y)$, where $\{x_n\}$, $\{y_n\}$ are sequences in X and $x, y \in X$.

Here, it is observed that $(CC) \implies (1C)$, $(W_4) \implies (W_3)$, and $(1C) \implies (W_3)$ but the converse implications are not true. In general, all other possible implications amongst (W_3), $(1C)$, and (HE) are not true. For detailed description, we refer to Cho et al. [6] which also contains some illustrative examples. However, (CC) implies all the remaining four conditions namely: (W_3), (W_4), (HE) and $(1C)$. Employing these axioms, several authors proved common fixed point theorems in the framework of symmetric spaces (e.g. [5, 8, 9, 12, 13, 15, 17, 22, 32]). With a view to obtain our results under optimal conditions, we utilize condition (W_3) or $(1C)$ (along with (HE)) instead of (W_4).

DEFINITION 1. Let (X, d) be a semi-metric space. A subset A of X is said to be

(1) closed if $A = \overline{A}$ where $\overline{A} = \{x \in X : d(x, A) = 0\}$ and

(2) bounded if $\delta(A) < \infty$ where $\delta(A) = \sup\{d(a, b) : a, b \in A\}$.

Let (X, d) be a symmetric (or semi-metric) space. Then, on the lines of Nadler [25], we adopt

(1) $CL(X) = \{A : A \text{ is a non-empty closed subset of } X\}$ and
(2) $CB(X) = \{A : A \text{ is a non-empty closed and bounded subset of } X\}$,
(3) for non-empty closed and bounded subsets A, B of X and $x \in X$,

$$d(x, A) = \inf\{d(x, a) : a \in A\}$$

and

$$H(A, B) = \max\left\{\sup\{d(a, B) : a \in A\}, \sup\{d(A, b) : b \in B\}\right\}.$$

It is easy to see that $(CB(X), H)$ is a semi-metric space (see [24]). It is also well known that $CB(X)$ is a metric space under the metric H, which is known as the Hausdorff–Pompeiu metric on $CB(X)$ provided (X, d) is a metric space.

DEFINITION 2. [26] Let (X, d) be a symmetric (semi-metric) space with $F : X \to CB(X)$ and $g : X \to X$. The pair of hybrid mappings (F, g) is said to be R-weakly commuting if, for every $x \in X$ and $gFx \in CB(X)$, there exists some positive real number R such that $H(Fgx, gFx) \leq Rd(Fx, gx)$.

DEFINITION 3. [21] Let (X, d) be a symmetric (semi-metric) space with $F : X \to CB(X)$ and $g : X \to X$. The pair of hybrid mappings (F, g) is said to be compatible if $gFx \in CB(X)$ for all $x \in X$ and $\lim_{n\to\infty} H(Fgx_n, gFx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} gx_n = t \in A = \lim_{n\to\infty} Fx_n$.

Here it may be noted that compatible mappings need not be R-weakly commuting (see [26]). Also, on coincidence points, R-weak commutativity is equivalent to commutativity and remains a necessary minimal condition for the existence of common fixed points for contractive type mappings.

DEFINITION 4. [12] Let (X, d) be a symmetric (semi-metric) space wherein d satisfies condition (W_3) (Hausdorffness of $\tau(d)$) with $F : X \to CB(X)$ and $g : X \to X$. The pair of hybrid mappings (F, g) is said to be non-compatible if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} gx_n = t \in A = \lim_{n\to\infty} Fx_n$ but $\lim_{n\to\infty} H(Fgx_n, gFx_n)$ is either non-zero or nonexistent.

DEFINITION 5. [11] Let Y be a non-empty subset of X, $F : Y \to 2^X$ and $g : Y \to X$. The pair of hybrid mappings (F, g) is said to be quasi-coincidentally commuting if $gx \in Fx$ (for $x \in X$ with $Fx, gx \in Y$) implies gFx is contained in Fgx.

DEFINITION 6. [11] Let Y be a non-empty subset of X, $F : Y \to 2^X$ and $g : Y \to X$. The mapping g is said to be coincidentally idempotent with

respect to mapping F, if $gx \in Fx$ with $gx \in Y$ imply $ggx = gx$, that is, g is idempotent at coincidence points of the pair (F, g).

DEFINITION 7. [2] Let (X, d) be a symmetric (semi-metric space) wherein d satisfies condition (W_3) whereas Y be an arbitrary non-empty set with $F : Y \to CB(X)$ and $g : Y \to X$. Then the pair of hybrid mappings (F, g) is said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in Y, for some $t \in X$ and $A \in CB(X)$ such that

$$\lim_{n \to \infty} gx_n = t \in A = \lim_{n \to \infty} Fx_n.$$

2. Main result

Firstly, we introduce the notion of common limit range property with respect to mapping g (briefly, (CLRg) property) as follows:

DEFINITION 8. Let (X, d) be a semi-metric space wherein d satisfies condition (W_3) whereas Y be an arbitrary non-empty set with $F : Y \to CB(X)$ and $g : Y \to X$. Then the pair of hybrid mappings (F, g) is said to satisfy the (CLRg) property if there exists a sequence $\{x_n\}$ in Y, for some $u \in X$ and $A \in CB(X)$ such that

$$\lim_{n \to \infty} gx_n = gu \in A = \lim_{n \to \infty} Fx_n.$$

Now, we present some examples demonstrating the preceeding definition.

EXAMPLE 1. Let us consider $X = [0, 1]$ with the symmetric $d(x, y) = (x - y)^2$. Define $F : X \to CB(X)$ and $g : X \to X$ as follows:

$$Fx = \begin{cases} \left[\frac{1}{2}, \frac{3}{4}\right], & \text{if } 0 \le x \le \frac{1}{2}; \\ \left[\frac{1}{4}, \frac{1}{2}\right), & \text{if } \frac{1}{2} < x \le 1. \end{cases} \qquad gx = \begin{cases} 1 - x, & \text{if } 0 \le x \le \frac{1}{2}; \\ \frac{4}{5}, & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

If we consider the sequence $\{x_n\} = \left\{\frac{1}{2} - \frac{1}{n}\right\}_{n \in \mathbb{N}}$, then one can verify that the pair (F, g) enjoys the (CLRg) property as

$$\lim_{n \to \infty} g\left(\frac{1}{2} - \frac{1}{n}\right) = g\left(\frac{1}{2}\right) \in \left[\frac{1}{2}, \frac{3}{4}\right] = \lim_{n \to \infty} F\left(\frac{1}{2} - \frac{1}{n}\right).$$

EXAMPLE 2. Consider $X = [0, 1]$ with the symmetric $d(x, y) = (x - y)^2$. Define $F : X \to CB(X)$ and $g : X \to X$ by

$$Fx = \begin{cases} \left[\frac{1}{2}, \frac{3}{4}\right], & \text{if } 0 \le x < \frac{1}{2}; \\ \left[\frac{1}{4}, \frac{1}{2}\right), & \text{if } \frac{1}{2} \le x \le 1. \end{cases} \qquad gx = \begin{cases} 1 - x, & \text{if } 0 \le x < \frac{1}{2}; \\ \frac{4}{5}, & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Consider the sequence $\{x_n\} = \left\{\frac{1}{2} - \frac{1}{n}\right\}_{n \in \mathbb{N}}$, then the pair (F, g) enjoys the property (E.A) as

$$\lim_{n \to \infty} g\left(\frac{1}{2} - \frac{1}{n}\right) = \lim_{n \to \infty} g\left(1 - \frac{1}{2} + \frac{1}{n}\right) = \frac{1}{2} \in \left[\frac{1}{2}, \frac{3}{4}\right] = \lim_{n \to \infty} F\left(\frac{1}{2} - \frac{1}{n}\right).$$

EXAMPLE 3. Consider $X = [0,1]$ with the symmetric $d(x,y) = (x - y)^2$. Define $F : X \to CB(X)$ and $g : X \to X$ by

$$Fx = \begin{cases} \left[\frac{1}{8}, \frac{1}{4}\right], & \text{if } 0 \le x \le \frac{1}{2}; \\ \left(\frac{1}{4}, \frac{3}{8}\right], & \text{if } \frac{1}{2} < x \le 1. \end{cases} \qquad gx = \begin{cases} 1 - x, & \text{if } 0 \le x \le \frac{1}{2}; \\ \frac{4}{5}, & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

If we consider $\{x_n\} = \left\{\frac{1}{2} - \frac{1}{n}\right\}_{n \in \mathbb{N}}$, then we find

$$\lim_{n \to \infty} g\left(\frac{1}{2} - \frac{1}{n}\right) = \lim_{n \to \infty} g\left(1 - \frac{1}{2} + \frac{1}{n}\right)$$

$$= g\left(\frac{1}{2}\right) = \frac{1}{2} \notin \left[\frac{1}{8}, \frac{1}{4}\right] = \lim_{n \to \infty} F\left(\frac{1}{2} - \frac{1}{n}\right).$$

Similar verification can be carried in respect of the other possible sequences. Thus in all, the pair (F, g) doesn't satisfy the property (E.A) as well as (CLRg) property.

REMARK 1. If the pair (F, g) satisfies the property (E.A) along with the closedness of $g(X)$, then the pair also satisfies the (CLRg) property.

For the sake of completeness, we state the following theorem due to Vijaywar et al. [33] proved for a pair of hybrid mappings defined on a symmetric (semi-metric) space (X, d).

THEOREM 1. [33, Theorem 1] *Let (X, d) be a symmetric (semi-metric) space wherein d enjoys (W_3) (the Hausdorffness of $\tau(d)$). Suppose that the mappings $F : X \to CB(X)$ and $g : X \to X$ are such that*

(1) *the mappings F and g satisfy the property* (E.A) *and*
(2) *for all $x \neq y \in X$,*

$$(2.1) \quad H(Fx, Fy) < \max\Big\{d(gx, gy), \frac{1}{2}[d(gx, Fx) + d(gy, Fy)],$$

$$\frac{1}{2}[d(gy, Fx) + d(gx, Fy)]\Big\}.$$

If $g(X)$ is a d-closed ($\tau(d)$-closed) subset of X, then F and g have a coincidence point.

Unfortunately, the preceeding theorem is not true in it's present form as authors use the continuity of the symmetric d but fail to mention the same. To substantiate the claim, we furnish an example of a discontinuous symmetric d which demonstrates that Theorem 1 is not valid in it's present form even for single valued pair of mappings.

EXAMPLE 4. Consider $X = [0, 1]$ equipped with the symmetric

$$d(x, y) = \begin{cases} \frac{e^{|x-y|}-1}{xy}, & \text{if } x \neq 0, y \neq 0, \\ 0, & \text{if } x = y, \\ 1, & x = 0, \ y \neq 0 \ (\text{or } x \neq 0, \ y = 0). \end{cases}$$

Define $f, g : X \to X$ as follows:

$$f(x) = \begin{cases} 1 - x, & \text{if } 0 \leq x < \frac{1}{3}, \\ 0, & \text{if } \frac{1}{3} \leq x \leq 1, \end{cases} \qquad g(x) = \begin{cases} \frac{2}{3}, & \text{if } 0 \leq x < \frac{1}{3}, \\ \frac{3}{4}, & \text{if } \frac{1}{3} \leq x \leq 1. \end{cases}$$

Clearly $f(X) = \{0\} \cup [\frac{1}{3}, 1]$ is d-closed in X. The pair (f, g) enjoys the property (E.A) as in respect of the sequence $x_n = \frac{1}{3} - \frac{1}{3n}, n = 1, 2, 3, \ldots$, we have

$$\lim_{n \to \infty} d\left(f\left(\frac{1}{3} - \frac{1}{3n} \right), \frac{2}{3} \right) = \lim_{n \to \infty} d\left(g\left(\frac{1}{3} - \frac{1}{3n} \right), \frac{2}{3} \right) = 0,$$

where $\frac{2}{3} \in X$. By a routine calculation one can easily show that the following contractive condition

$$d(gx, gy)$$
$$< \max\left\{ d(fx, fy), \frac{k}{2}[d(gx, fx) + d(gy, fy)], \frac{k}{2}[d(gy, fx) + d(gx, fy)] \right\}$$

holds for all $x \neq y$, $1 \leq k < 2$.

Notice that, in the foregoing example, all the conditions of Theorem 1 are satisfied but f and g have no coincidence point.

However a more general result can be obtained by using common limit range property under additional conditions $(1C)$ and (HE).

THEOREM 2. *Let (X, d) be a symmetric (semi-metric) space wherein d satisfies conditions $(1C)$ and (HE) while Y is an arbitrary non-empty set with $F : Y \to CL(X)$ and $g : Y \to X$. Suppose that*

(1) *the hybrid pair (F, g) enjoys the $(CLRg)$ property and*
(2) *for all $x \neq y \in Y$ and $0 < k < 2$,*

$$(2.2) \quad H(Fx, Fy) < \max\left\{ d(gx, gy), \frac{k}{2}[d(gx, Fx) + d(gy, Fy)], \right.$$
$$\left. \frac{k}{2}[d(gy, Fx) + d(gx, Fy)] \right\}.$$

Then F and g have a coincidence point.
In particular, if $Y \subset X$ and the pair of mappings (F, g) is quasi-coincidentally commuting and coincidentally idempotent, then the pair (F, g) has a common fixed point.

Proof. Firstly, one needs to note that a sequence $\{x_n\}$ in a potent semi-metric space (X, d) converges to a point x in $\tau(d)$ iff $d(x_n, x) \to 0$. To substantiate this, suppose $x_n \to x$ and let $\epsilon > 0$. Since $S(x, \epsilon)$ is a neighbourhood of x, there exists $U \in \tau(d)$ such that $x \in U \subset S(x, \epsilon)$. Since $x_n \to x$, there is a $m \in \mathbb{N}$ (the natural number) such that $x_n \in U \subset S(x, \epsilon)$ for $n \geq m$ so that $d(x_n, x) < \epsilon$ for $n \geq m$, that is, $d(x_n, x) \to 0$. The converse part is obvious in view of the definition of $\tau(d)$.

Suppose that the pair (F, g) enjoys the (CLRg) property, there exists a sequence $\{x_n\}$ in Y, for some $u \in X$ and $A \in CL(X)$ such that

$$\lim_{n\to\infty} gx_n = gu \in A = \lim_{n\to\infty} Fx_n.$$

Now we show that $gu \in Fu$. If not, then using inequality (2.2), one obtains

$$H(Fx_n, Fu) < \max\left\{ d(gx_n, gu), \frac{k}{2}\left[d(gx_n, Fx_n) + d(gu, Fu)\right], \right.$$
$$\left. \frac{k}{2}\left[d(gu, Fx_n) + d(gx_n, Fu)\right] \right\}.$$

On letting $n \to \infty$ and using conditions $(1C)$ and (HE), we have

$$H(A, Fu) < \max\left\{ 0, \frac{k}{2}\left[d(gu, A) + d(gu, Fu)\right], \frac{k}{2}\left[d(gu, A) + d(gu, Fu)\right] \right\}.$$

Since $gu \in A$, the above inequality implies

$$d(gu, Fu) \leq H(A, Fu)$$
$$< \max\left\{ \frac{k}{2}d(gu, Fu), \frac{k}{2}d(gu, Fu) \right\} = \frac{k}{2}d(gu, Fu) < d(Fu, gu),$$

which is a contradiction. Hence $gu \in Fu$ which shows that the pair (F, g) has a point of coincidence.

Since $Y \subset X$ and u is a point of coincidence of the pair (F, g), using the quasi-coincidentally commuting property of (F, g) and the coincidentally idempotent property of g with respect to F, one can have $gu \in Fu$ and $ggu = gu$. Therefore $gu = ggu \in g(Fu) \subset F(gu)$ which shows that gu is a common fixed point of the pair (F, g). ∎

EXAMPLE 5. Consider $X = Y = [0, 1]$ equipped with the symmetric defined by $d(x, y) = (x - y)^2$ for all $x, y \in X$ which satisfies conditions $(1C)$ and (HE). Define the mappings $F : X \to CL(X)$ and $g : X \to X$ as follows:

$$Fx = \begin{cases} \left[\frac{1}{3}, \frac{3}{4}\right], & \text{if } 0 \leq x \leq \frac{1}{2}; \\ \left[\frac{1}{4}, \frac{1}{3}\right], & \text{if } \frac{1}{2} < x \leq 1. \end{cases} \qquad gx = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq x \leq \frac{1}{2}; \\ \frac{2x}{3}, & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Consider a sequence $\{x_n\} = \left\{\frac{1}{2} - \frac{1}{n}\right\}_{n \in \mathbb{N}}$, one can see that the pair (F, g) enjoys the (CLRg) property,

$$\lim_{n \to \infty} g\left(\frac{1}{2} - \frac{1}{n}\right) = g\left(\frac{1}{2}\right) \in \left[\frac{1}{3}, \frac{3}{4}\right] = \lim_{n \to \infty} F\left(\frac{1}{2} - \frac{1}{n}\right).$$

By a routine calculation one can show that the contractive condition (2.2) holds for every $x \neq y \in X$. It is pointed out that $g(X) = \left(\frac{1}{3}, \frac{2}{3}\right]$ is not a closed ($\tau(d)$-closed) subset of X. Also the pair (F, g) is quasi-coincidentally commuting at $x = \frac{1}{2}$, that is, $g\left(\frac{1}{2}\right) \in F\left(\frac{1}{2}\right)$ and $gF\left(\frac{1}{2}\right) = \left(\frac{1}{3}, \frac{1}{2}\right] \subset \left[\frac{1}{3}, \frac{3}{4}\right] = Fg\left(\frac{1}{2}\right)$. Thus, all conditions of Theorem 2 are satisfied and $\frac{1}{2} = g\left(\frac{1}{2}\right) \in F\left(\frac{1}{2}\right)$.

COROLLARY 1. *Let (X, d) be a symmetric (semi-metric) space wherein d satisfies conditions $(1C)$ and (HE) while Y is an arbitrary non-empty set with $F : Y \to CL(X)$ and $g : Y \to X$. Suppose that the hybrid pair (F, g) enjoys the property $(E.A)$ and satisfies inequality (2.2). If $f(Y)$ is a closed $(\tau(d)$-closed) subset of X, then the pair (F, g) has a coincidence point.*

In particular, if $Y \subset X$ and the pair of mappings (F, g) is quasi-coincidentally commuting and coincidentally idempotent, then the pair (F, g) has a common fixed point.

Proof. The proof of this corollary easily follows in view of Remark 1. ∎

Since the class of compatible as well as non-compatible mappings are contained in the class of mappings pairs satisfying the property (E.A), therefore we have the following.

COROLLARY 2. *Let (X, d) be a symmetric (semi-metric) space wherein d satisfies conditions $(1C)$ and (HE) while Y is an arbitrary non-empty set with $F : Y \to CL(X)$ and $g : Y \to X$. Suppose that the hybrid pair (F, g) is compatible or non-compatible and satisfies inequality (2.2). If $f(Y)$ is a closed $(\tau(d)$-closed) subset of X, then the pair (F, g) has a coincidence point.*

In particular, if $Y \subset X$ and the pair of mappings (F, g) is quasi-coincidentally commuting and coincidentally idempotent, then the pair (F, g) has a common fixed point.

Our next theorem involves a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies the following properties:

(1) ϕ is upper semi-continuous on $\mathbb{R}^+$ and
(2) $0 < \phi(t) < t$ for each $t \in \mathbb{R}^+$.

THEOREM 3. *Let (X, d) be a symmetric (semi-metric) space wherein d satisfies conditions $(1C)$ and (HE) while Y is an arbitrary non-empty set with $F : Y \to CL(X)$ and $g : Y \to X$. Suppose that*

(1) *the hybrid pair (F, g) enjoys the (CLRg) property and*

(2) *for all $x \neq y \in Y$ and $0 < k \leq 2$,*

(2.3) $$H(Fx, Fy) \leq \phi(m(x, y)),$$

where

(2.4) $$m(x, y) = \max\left\{ d(gx, gy), \frac{k}{2}\left[d(gx, Fx) + d(gy, Fy)\right], \right.$$
$$\left. \frac{k}{2}\left[d(gy, Fx) + d(gx, Fy)\right] \right\}.$$

Then F and g have coincidence point.

In particular, if $Y \subset X$ and the pair of mappings (F, g) is quasi-coincidentally commuting and coincidentally idempotent, then the pair (F, g) has a common fixed point.

Proof. If the pair (F, g) satisfies the (CLRg) property, then there exists a sequence $\{x_n\}$ in Y, for some $u \in X$ and $A \in CL(X)$ such that

$$\lim_{n \to \infty} gx_n = gu \in A = \lim_{n \to \infty} Fx_n.$$

Now we assert that $gu \in Fu$. Suppose that $gu \notin Fu$, then using inequalities (2.3) and (2.4), one obtains

(2.5) $$H(Fx_n, Fu) \leq \phi(m(x_n, u)),$$

where

$$m(x_n, u) = \max\left\{ d(gx_n, gu), \frac{k}{2}\left[d(gx_n, Fx_n) + d(gu, Fu)\right], \right.$$
$$\left. \frac{k}{2}\left[d(gu, Fx_n) + d(gx_n, Fu)\right] \right\}.$$

Taking limit as $n \to \infty$ in (2.5) and using conditions $(1C)$ and (HE), we have

$$\lim_{n \to \infty} H(Fx_n, Fu)$$
$$\leq \phi\left(\lim_{n \to \infty} \max \left\{ \begin{array}{c} d(gx_n, gu), \frac{k}{2}\left[d(gx_n, Fx_n) + d(gu, Fu)\right], \\ \frac{k}{2}\left[d(gu, Fx_n) + d(gx_n, Fu)\right] \end{array} \right\} \right)$$

$$H(A, Fu) \leq \phi\left(\max\left\{ 0, \frac{k}{2}[d(gu, A) + d(gu, Fu)], \frac{k}{2}[d(gu, A) + d(gu, Fu)] \right\} \right).$$

Since $gu \in A$, we get

$$d(gu, Fu) \leq H(A, Fu) \leq \phi\left(\frac{k}{2}d(gu, Fu) \right) < d(gu, Fu),$$

which is a contradiction. Hence $gu \in Fu$, which shows that the pair (F, g) has a point of coincidence.

Since $Y \subset X$ and u is a point of coincidence of the pair (F, g), using the quasi-coincidentally commuting property of (F, g) and the coincidentally

idempotent property of g with respect to F, one can have $gu \in Fu$ and $ggu = gu$. Therefore $gu = ggu \in g(Fu) \subset F(gu)$, which shows that gu is a common fixed point of the pair (F, g). ∎

COROLLARY 3. *Let (X, d) be a symmetric (semi-metric) space wherein d satisfies conditions $(1C)$ and (HE) while Y is an arbitrary non-empty set with $F : Y \to CL(X)$ and $g : Y \to X$. Suppose that*

(1) the hybrid pair (F, g) enjoys the (CLRg) property and
(2) for all $x \neq y \in Y$,

$$(2.6) \quad H(Fx, Fy)$$
$$\leq \phi \left(\max \left\{ d(gx, gy), d(gx, Fx), d(gy, Fy), d(gy, Fx), d(gx, Fy) \right\} \right).$$

Then F and g have a coincidence point.

In particular, if $Y \subset X$ and the pair of mappings (F, g) is quasi-coincidentally commuting and coincidentally idempotent, then the pair (F, g) has a common fixed point.

Proof. In view of the observation

$$H(Fx, Fy) \leq \phi \left(\max \left\{ d(gx, gy), d(gx, Fx), d(gy, Fy), d(gy, Fx), d(gx, Fy) \right\} \right)$$
$$\leq \phi \left(\max \left\{ d(gx, gy), d(gx, Fx) + d(gy, Fy), d(gy, Fx) + d(gx, Fy) \right\} \right),$$

the proof of this corollary easily follows from Theorem 3 (with $k = 2$). ∎

Our next result remains true for a pair of hybrid mappings in metric spaces.

THEOREM 4. *Let (X, d) be a metric space while Y is an arbitrary non-empty set with $F : Y \to CL(X)$ and $g : Y \to X$. Suppose that*

(1) the hybrid pair (F, g) enjoys the (CLRg) property and
(2) for all $x \neq y \in Y$ and $0 < k < 2$,

$$(2.7) \quad H(Fx, Fy)$$
$$< \max \left\{ d(gx, gy), \frac{k}{2} \left[d(gx, Fx) + d(gy, Fy) \right], \frac{k}{2} \left[d(gy, Fx) + d(gx, Fy) \right] \right\}.$$

Then F and g have a coincidence point.

In particular, if $Y \subset X$ and the pair of mappings (F, g) is quasi-coincidentally commuting and coincidentally idempotent, then the pair (F, g) has a common fixed point.

Proof. The proof of this theorem can be completed on the lines of the proof of Theorem 2, hence the details are avoided. ∎

Now, we utilize a relatively weaker condition (W_3) instead of condition $(1C)$ to prove our next result.

THEOREM 5. *Let (X, d) be a semi-metric (symmetric) space wherein d satisfies conditions (W_3) and (HE) while Y is an arbitrary non-empty set with $F : Y \to CL(X)$ and $g : Y \to X$. Suppose that*

(1) *the hybrid pair (F, g) enjoys the (CLRg) property and*
(2) *for all $x \neq y \in Y$,*

$$(2.8) \quad H(Fx, Fy)$$
$$< \max \left\{ d(gx, gy), \min\{d(gx, Fx), d(gy, Fy)\}, \min\{d(gy, Fx) + d(gx, Fy)\} \right\}.$$

Then F and g have a coincidence point.

In particular, if $Y \subset X$ and the pair of mappings (F, g) is quasi-coincidentally commuting and coincidentally idempotent, then the pair (F, g) has a common fixed point.

Proof. In view of (1), there exists a sequence $\{x_n\}$ in Y, for some $u \in X$ and $A \in CL(X)$ such that

$$\lim_{n \to \infty} gx_n = gu \in A = \lim_{n \to \infty} Fx_n.$$

Now we show that $gu \in Fu$. If not, then using inequality (2.8), one obtains

$$H(Fx_n, Fu) < \max\{d(gx_n, gu), \min\{d(gx_n, Fx_n), d(gu, Fu)\},$$
$$\min\{d(gu, Fx_n) + d(gx_n, Fu)\}\}.$$

On letting $n \to \infty$ and making use of conditions (W_3) and (HE), we get $\lim_{n \to \infty} H(Fx_n, Fu) = 0$ implying thereby $H(gu, Fu) = 0$, that is, $gu \in Fu$. Hence u is a coincidence point of the pair (F, g).

The rest of the proof run on the lines of the proof of Theorem 2. This concludes the proof. ∎

REMARK 2. The results similar to Theorem 3 can be proved under the contractive condition (2.8). Here, we avoid the detailed description.

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