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ON THE NEMYTSKII OPERATOR IN THE SPACE OF FUNCTIONS OF BOUNDED $(p, 2, \alpha)$-VARIATION WITH RESPECT TO THE WEIGHT FUNCTION

Abstract. In this paper, we consider the Nemytskii operator $\big(Hf\big)\ (t) = h(t, f(t))$, generated by a given function h. It is shown that if H is globally Lipschitzian and maps the space of functions of bounded $(p, 2, \alpha)$-variation (with respect to a weight function α) into the space of functions of bounded $(q, 2, \alpha)$-variation (with respect to α) $1 < q < p$, then H is of the form $\big(Hf\big)(t) = A(t)f(t) + B(t)$. On the other hand, if $1 < p < q$ then H is constant. It generalize several earlier results of this type due to Matkowski–Merentes and Merentes. Also, we will prove that if a uniformly continuous Nemytskii operator maps a space of bounded variation with weight function in the sense of Merentes into another space of the same type, its generator function is an affine function.

1. Introduction

Let $I, J \subset \mathbb{R}$ be intervals. By J^I, we denote the set of all functions $f : I \to J$. For a given function $h : I \times J \to \mathbb{R}$, the mapping $H : J^I \to \mathbb{R}^I$ defined by

$$(Hf)(t) := h(t, f(t)), \quad f \in J^I, \quad t \in I,$$

is called a superposition operator (sometimes also composition operator, substitution operator, or Nemytskii operator) generated by h. The superposition operators play an important role in the theory of differential equations, integral equations and functional equations. In 1982, J. Matkowski showed (cf. [9]) that a composition operator mapping the function space $\mathrm{Lip}(I, \mathbb{R})$ $(I = [0, 1])$ into itself is Lipschitzian with respect to the Lipschitzian norm if and only if its generator h has the form

$$h(t, y) = A(t)y + B(t), \quad t \in I, \quad y \in \mathbb{R}, \tag{1}$$

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for some $A, B \in \mathrm{Lip}(I, \mathbb{R})$. This result was extended to a lot of spaces by J. Matkowski and others.

In [12], N. Merentes and K. Nikodem showed that Nemystkii operator H, generated by a set-valued function h, mapping the space of functions of bounded p-variation $(1 < p < \infty)$ into the space of set-valued functions of bounded p-variation and globally Lipschitzian, has to be of the form (1), where $A(t)$ is a linear continuous set-valued function and B is a set-valued function of bounded p-variation. In 2000, V. V. Chistyakov in [4] proved that Lipschtzian Nemystkii operators H, which map between spaces of real valued functions of bounded generalized variation of Riesz–Orlicz type including weight, are of the form (1), where A and B are functions of bounded generalized variation of Riesz–Orlicz type including weight.

The aim of this paper is to prove an analogous result in the case when the Nemytskii operator H maps the space of set-valued functions of bounded p-variation in the sense of Riesz with respect to the weight α into the space of set-valued functions of bounded q-variation in the sense of Riesz with respect to the weight α, where $1 < q \leq p < \infty$ and H is globally Lipschitzian. The particular case $p = q$ has been already considered by authors in [10, 12, 13, 19, 20], but the present case of possibly different spaces requires a different proof technique and this extension may turn out to be useful in some applications.

In the present paper, we will extend the concept of bounded $(p, 2)$-variation, $1 < p < +\infty$, and will prove a characterization of the class $AC_p^2[a, b]$ in terms of this concept. $AC_{p,\alpha}^2[a, b]$ is the class of functions $f : [a, b] \to \mathbb{R}$, for which f' is absolutely continuous on $[a, b]$ and $f'' \in L_{p,\alpha}[a, b]$. Moreover, the $(p, 2, \alpha)$-variation of a function f on $[a, b]$ is given by $\|f''\|^p_{L_{p,\alpha}[a,b]}$, that is

$$V_{p,\alpha}^2(f, [a, b]) = \|f''\|^p_{L_{p,\alpha}[a,b]}.$$

So, the obtained characterization can be considered as a "natural" generalization of that the given by [17] for the class $AC_{p,\alpha}[a, b]$. This results will provide us with an alternative characterization for the Sobolev space $W_{p,\alpha}^2[a, b]$.

2. Preliminary results

The section is devoted to present some auxiliary facts, which will be used later on.

Let $\big(X, \|\cdot\|\big)$ be a normed space and $p \geq 1$ be a fixed number. Given $\alpha : [a, b] \to \mathbb{R}$, a fixed continuous strictly increasing function called a weight, $f : [a, b] \to X$ and a partition $\pi : a = t_0 < t_1 < \cdots < t_n = b$ of the interval

$[a, b]$, we define:

$$\sigma_{p,\alpha}(f;\pi) := \sum_{i=1}^{n} \frac{\|f(t_i) - f(t_{i-1})\|^p}{|\alpha(t_i) - \alpha(t_{i-1})|^{p-1}}.$$

The number:

$$V_{p,\alpha}(f, [a, b]) := \sup_{\pi} \sigma_{p,\alpha}(f, \pi),$$

where the supremum is taken over all partitions π of $[a, b]$, is called the p-variation in the sense of Riesz of the function f with respect to the weight function α (cf. [4]). A function f is said to be of bounded p-variation if $V_{p,\alpha}(f, [a, b]) < +\infty$. Denote by $RV_{p,\alpha}([a, b]; X)$, the space of all functions $f : [a, b] \to X$ of bounded p-variation in the sense of Riesz with respect to the weight function α equipped with the norm

$$\|f\|_p := \|f(a)\| + \big(V_{p,\alpha}(f, [a, b])\big)^{1/p}.$$

F. Riesz [17] introduced the so-called Riesz class $A_p[a, b]$ $(1 < p < +\infty)$ in the following way:

Lemma 2.1. [17] *A real function f defined on the interval $[a, b]$ belongs to the class $A_p[a, b]$ $(1 < p < +\infty)$ if and only if $V_p(f) < +\infty$ and $f' \in L_p[a, b]$. Moreover:*

$$V_p(f) = \|f'\|^p_{L_p[a,b]}.$$

A. M. Russell and R. Castillo (cf. [18, 1]) generalized the definition introduced by De la Vallée Poussin [15] in the following way. Let $\alpha : [a, b] \to \mathbb{R}$ be a continuous and strictly monotonic weight function, and let π denote a partition of the interval $[a, b]$ of the form

$$\pi : \ a = t_0 < t_1 < \cdots < t_n = b. \tag{2}$$

For a function $f : [a, b] \to \mathbb{R}$ put

$$\sigma^2_\alpha(f;\pi) := \sum_{i=1}^{n-1} \left| \frac{f(t_{i+1}) - f(t_i)}{\alpha(t_{i+1}) - \alpha(t_i)} - \frac{f(t_i) - f(t_{i-1})}{\alpha(t_i) - \alpha(t_{i-1})} \right|$$

and

$$V^2_\alpha(f) := \sup_{\pi} \sigma^2_\alpha(f;\pi),$$

where the supremum is taken over all partitions π of the form (2).

The number $V^2_\alpha(f)$ is called De la Vallée Poussin second variation of the function f on $[a, b]$ with respect to the weight function α.

If $V^2_\alpha(f) < +\infty$, the function f is said to be of $(2, \alpha)$-bounded variation.

The set of all these functions will be denoted by $BC^2_\alpha[a, b]$. The class $BC^2_\alpha[a, b]$ is a Banach space equipped with the norm

$$\|f\|_{BC^2_\alpha[a,b]} = |f(a)| + |f'_\alpha(a)| + V^2_\alpha(f).$$

With the implementation of the weight function ($\alpha \in \mathbb{R}^{[a,b]}$) in the definition given by De la Vallée Poussin for functions of bounded variation [15], we considerably generalize the results on [6, 11, 14].

Definition 2.2. Let $\alpha : [a,b] \to \mathbb{R}$ be a continuous and strictly monotonic function. A function $f : [a,b] \to \mathbb{R}$ is called α-Lipschitz if there is $M > 0$ such that

$$|f(x) - f(y)| \leq M|\alpha(x) - \alpha(y)| \quad (x, y \in [a,b],\ x \neq y).$$

By α-$Lip[a,b]$, we will denote the space of functions which are α-Lipschitz. If $f \in \alpha$-$Lip[a,b]$, we define

$$Lip_\alpha(f) = \inf\{M > 0 : |f(x) - f(y)| \leq M|\alpha(x) - \alpha(y)|,\ x \neq y \in [a,b]\}$$

and α-$Lip[a,b]$ equipped with the norm

$$\|f\|_{\alpha\text{-}Lip[a,b]} = |f(a)| + Lip_\alpha(f)$$

is a Banach space.

Definition 2.3. [18, Definition 4] Let $\alpha : [a,b] \to \mathbb{R}$ be a continuous and strictly monotonic function. A function $f : [a,b] \to \mathbb{R}$ is called α-convex if for every $x, y \in [a,b]$ such that $a \leq x < z < y \leq b$, it satisfies

$$f(y) \leq \frac{\alpha(y) - \alpha(x)}{\alpha(z) - \alpha(x)} f(z) + \frac{\alpha(z) - \alpha(y)}{\alpha(z) - \alpha(x)} f(x).$$

Lemma 2.4. [18, Theorem 1.1] *If $V_\alpha^2(f) < +\infty$, then there exists a non-negative constant M such that*

$$|f(x) - f(y)| \leq M|\alpha(x) - \alpha(y)|; \quad (x, y \in [a,b],\ x \neq y)$$

and the function f can be expressed as a difference of two α-convex functions.

Theorem 2.5. [18, Lemma 1.6] *Let $f : [a,b] \to \mathbb{R}$ be an α-convex function and $a \leq x < y < z \leq b$. Then*

$$f_\alpha[x,y] \leq f_\alpha[x,z] \leq f_\alpha[y,z],$$

where $f_\alpha[x,y] = \dfrac{f(y) - f(x)}{\alpha(y) - \alpha(x)}$.

The α-convexity of f implies the function

$$f_\alpha^\rho : \{[a,b] \backslash \{\alpha, \rho\}\} \to \mathbb{R}; \quad f_\alpha^\rho(t) = f_\alpha[t,\rho] = \frac{f(t) - f(\rho)}{\alpha(t) - \alpha(\rho)},$$
$$t, \rho \in [a,b],\ t \neq \rho,$$

increases with respect to t and ρ. It follows that there exist the lateral limits of function $f_\alpha^{t_0}$, for every t_0 in $[a,b]$:

$$(3) \qquad f_\alpha^\rho(\rho^+) = \lim_{t \to \rho^+} f_\alpha^\rho(t) = \lim_{t \to \rho^+} f_\alpha[t,\rho] = \lim_{t \to \rho^+} \frac{f(t) - f(\rho)}{\alpha(t) - \alpha(\rho)} = f'_{\alpha^+}(\rho),$$

and

$$f_\alpha^\rho(\rho^-) = \lim_{t\to\rho^-} f_\alpha^\rho(t) = \lim_{t\to\rho^-} f_\alpha[t,\rho] = \lim_{t\to\rho^-} \frac{f(t)-f(\rho)}{\alpha(t)-\alpha(\rho)} = f'_{\alpha^-}(\rho). \tag{4}$$

Moreover

$$f'_{\alpha^+}(a) \le f_\alpha[x,y] \le f'_{\alpha^-}(b).$$

Definition 2.6. Let $\big([a,b],\sum,\mu_\alpha\big)$ be a measure space equipped with the Lebesgue–Stieltjes measure. A measurable function $f:[a,b]\to\mathbb{R}$ is said to be in $L_{p,\alpha}[a,b]$ for $1\le p<+\infty$ if

$$\int_a^b |f|^p d\alpha < +\infty.$$

Moreover, let α be a function strictly increasing and continuous in $[a,b]$. A set $E\subset[a,b]$ of α-measure (μ_α) zero is a set of values $x\in[a,b]$, which can be covered by a finite number or by a denumerable sequence of intervals whose total length (i.e. the sum of the individual lengths respect to α) is arbitrarily small (cf. [16], §25).

Definition 2.7. [2, 3] A function $f:[a,b]\to\mathbb{R}$ is said to be absolutely continuous with respect to α if for every $\epsilon>0$, there exists some $\delta>0$ such that if $\big\{(a_j,b_j)\big\}_{j=1}^n$ is a class of disjoint open subintervals of $[a,b]$ then

$$\sum_{j=1}^n |\alpha(b_j)-\alpha(a_j)|^p \le \delta \text{ implies } \sum_{j=1}^n |f(b_j)-f(a_j)|^p \le \epsilon.$$

Thus, the collection α-$AC_p[a,b]$ of all α-absolutely continuous functions on $[a,b]$ is a function space and an algebra of functions.

3. Some results

In this section, we introduce the extension of the notion of Riesz $(p,2,\alpha)$-variation and we give a result similar to Lemma 2.1, for the class α-$AC_p^2[a,b]$ $(1<p<\infty)$ in terms of this concept. α-$AC_p^2[a,b]$ is the class of functions $f:[a,b]\to\mathbb{R}$, for which f' is absolutely continuous on $[a,b]$ with respect to a function α strictly increasing, and $f''\in L_{p,\alpha}[a,b]$. Also the $(p,\alpha,2)$-variation of a function f on $[a,b]$ is given by $\|f''\|^p_{L_{p,\alpha}[a,b]}$, that is

$$V^2_{(p,\alpha)}(f;[a,b]) = \|f''\|^p_{L_{p,\alpha}[a,b]}.$$

Let $f:[a,b]\to\mathbb{R}$ and $1<p<+\infty$. For a given partition π of the form $a=t_0<t_1<\cdots<t_n=b$, let

$$\sigma^2_{(p,\alpha)}(f;\pi) := \sum_{i=1}^{n-1} \left| \frac{f(t_{i+1})-f(t_i)}{\alpha(t_{i+1})-\alpha(t_i)} - \frac{f(t_i)-f(t_{i-1})}{\alpha(t_i)-\alpha(t_{i-1})} \right|^p \frac{1}{\big|\alpha(t_{i+1})-\alpha(t_{i-1})\big|^{p-1}}$$

and

$$V^2_{(p,\alpha)}(f;[a,b]) := \sup_{\pi} \sigma^2_{(p,\alpha)}(f;\pi),$$

where the supremum is taken over all partitions π of the interval $[a,b]$, is called the $(p,2,\alpha)$-variation of the function f on $[a,b]$ with respect the weight function α. If $V^2_{(p,\alpha)}(f;[a,b]) < +\infty$, the function is said to have bounded (or finite) Merentes $(p,2,\alpha)$-variation and the set $BV^2_{(p,\alpha)}[a,b]$ shall denote the Banach space of all functions f for which the norm

$$\|f\|_{(p,2,\alpha)} := |f(a)| + |f'(a)| + \left[V^2_{(p,\alpha)}(f;[a,b])\right]^{1/p}$$

is finite.

Importantly, this variation, which is a combination of the notions of bounded variation in the sense of Riesz and bounded variation in the sense of De la Vallée Poussin, was introduced by Merentes [14] in 1992. We include a weight function $\alpha \in [0,+\infty)^{[0,+\infty)}$, which is strictly increasing in the definition to prove some analogous results to those obtained in [11, 14].

The motivation for our work is due to the results of N. Merentes [14], J. Matkowski and N. Merentes [11], T. Kostrzewski [6, 7] and V. V. Chistyakov [4].

Lemma 3.1. *If there is $p \in (1,+\infty)$ such that $V^2_{(p,\alpha)}(f;[a,b]) < +\infty$ then f has second variation and*

$$V^2_{\alpha}(f;[a,b]) \le \left[V^2_{(p,\alpha)}(f;[a,b])\right]^{\frac{1}{p}} |\alpha(b)-\alpha(a)|^{1-\frac{1}{p}}.$$

Proof. Let $\pi : a = t_0 < t_1 < \cdots < t_n = b$ be a partition of $[a,b]$. Then by Hölder's inequality, we obtain

$$\begin{aligned}\sum_{i=1}^{n-1} &\left|\frac{f(t_{i+1})-f(t_i)}{\alpha(t_{i+1})-\alpha(t_i)} - \frac{f(t_i)-f(t_{i-1})}{\alpha(t_i)-\alpha(t_{i-1})}\right| \frac{|\alpha(t_{i+1})-\alpha(t_{i-1})|^{1-\frac{1}{p}}}{|\alpha(t_{i+1})-\alpha(t_{i-1})|^{1-\frac{1}{p}}} \\ &\le \left[\sum_{i=1}^{n-1} \left|\frac{f(t_{i+1})-f(t_i)}{\alpha(t_{i+1})-\alpha(t_i)} - \frac{f(t_i)-f(t_{i-1})}{\alpha(t_i)-\alpha(t_{i-1})}\right|^p \frac{1}{|\alpha(t_{i+1})-\alpha(t_{i-1})|^{p-1}}\right]^{\frac{1}{p}} \\ &\qquad \times \left[\sum_{i=1}^{n-1} |\alpha(t_{i+1})-\alpha(t_{i-1})|\right]^{1-\frac{1}{p}}.\end{aligned}$$

Thus

$$V^2_{\alpha}(f;[a,b]) \le \left[V^2_{(p,\alpha)}(f;[a,b])\right]^{\frac{1}{p}} |\alpha(b)-\alpha(a)|^{1-\frac{1}{p}}. \blacksquare$$

By Lemmas 2.1, 2.4, and 3.1, we have

Corollary 3.2. $\|f\|_{BC^2_\alpha[a,b]} \leq \|f\|_{(p,2,\alpha)}$, $f \in BV^2_{(p,\alpha)}([a,b])$ *and*

$$BV^2_{(p,\alpha)}([a,b]) \hookrightarrow BC^2_\alpha[a,b] \hookrightarrow \alpha\text{-}Lip[a,b].$$

Corollary 3.3. *If* $V^2_{(p,\alpha)}(f;[a,b]) < +\infty$ $(1 < p < +\infty)$, *then* f *is* α-*absolutely continuous on* $[a,b]$ *and* f *can be expressed as a difference of two* α-*convex functions.*

Lemma 3.4. *If* $a < c < b$ *then*

$$V^2_{(p,\alpha)}(f;[a,b]) \geq V^2_{(p,\alpha)}(f;[a,c]) + V^2_{(p,\alpha)}(f;[c,b]).$$

Proof. This follows readily from definition of $V^2_{(p,\alpha)}(f;[a,b])$. ∎

Lemma 3.5. *If* $V^2_{(p,\alpha)}(f;[a,b]) < +\infty$ $(1 < p < +\infty)$, *then there exists the derivative* $f'_\alpha(\rho)$ *for all* $\rho \in (a,b)$.

Proof. By Corollary 3.3 and the consequences of Theorem 2.5, we obtain the existence of a right hand derivative $f'_{\alpha^+}(\rho)$ (see (3)) for all $\rho \in [a,b)$ and the left hand derivative $f'_{\alpha^-}(\rho)$ (see (4)) for all $\rho \in (a,b]$. Suppose that there exists $\rho \in (a,b)$ such that

$$\delta_\rho := \left|f'_{\alpha^+}(\rho) - f'_{\alpha^-}(\rho)\right| > 0.$$

By the definition of $(p,2,\alpha)$-variation, we have

$$\begin{aligned} V^2_{(p,\alpha)}(f;[a,b]) &\geq \lim_{h\to 0}\left|\frac{f(\rho+h)-f(\rho)}{\alpha(\rho+h)-\alpha(\rho)} - \frac{f(\rho)-f(\rho-h)}{\alpha(\rho)-\alpha(\rho-h)}\right|^p \frac{1}{2^p|\Delta\alpha|^{p-1}} \\ &= \frac{|\delta_\rho|^p}{2^{p-1}} \cdot \lim_{h\to 0}\frac{1}{|\Delta\alpha|^{p-1}} = +\infty, \end{aligned}$$

where $\Delta\alpha = \alpha(\rho+h)-\alpha(\rho-h)$. Consequently, the function f has a derivative $f'_\alpha(\rho)$ for all $\rho \in (a,b)$. ∎

Remark 3.6. We follow the same notation presented by De la Vallée Poisson and we put $BV^2_{p,\alpha}[a,b] = RV_{p,\alpha}[a,b]$.

Lemma 3.7. *If* $V^2_{(p,\alpha)}(f;[a,b]) < +\infty$ $(1 < p < +\infty)$ *then* $f'_\alpha \in BV^2_{(p,\alpha)}[a,b]$. *Moreover*

$$V_{p,\alpha}(f'_\alpha;[a,b]) \leq V^2_{(p,\alpha)}(f;[a,b]).$$

Thus, $f'_\alpha \in \alpha\text{-}AC_p[a,b]$ *and* $f''_\alpha \in L_{p,\alpha}[a,b]$, *i.e.,* $f \in \alpha\text{-}AC^2_p[a,b]$.

Proof. Let $\pi : a = t_0 < t_1 < \cdots < t_n = b$ be a partition of $[a,b]$. Let $h > 0$ be such that

$$0 < h \leq \min\left\{\frac{\alpha(t_i)-\alpha(t_{i-1})}{2}\right\}_{i=1}^n.$$

We have

$$\sum_{i=1}^{n-1}\left|\frac{f(t_i)-f(t_i-h)}{\alpha(t_i)-\alpha(t_i-h)}-\frac{f(t_i+h)-f(t_i)}{\alpha(t_i+h)-\alpha(t_i)}\right|^p\frac{1}{|\alpha(t_i)-\alpha(t_{i-1})|^{p-1}} \le V^2_{(p,\alpha)}(f;[a,b]).$$

Hence, letting $h\to 0$, and by Lemma 3.5, we obtain

$$\sum_{i=1}^{n-1}\frac{|f'_\alpha(t_{i+1})-f'_\alpha(t_i)|^p}{|\alpha(t_{i+1})-\alpha(t_i)|^{p-1}}\le V^2_{(p,\alpha)}(f;[a,b]).$$

Now, by Lemma 2.1, we have $f'_\alpha\in RV_{p,\alpha}[a,b]$ and thus,

$$V_{p,\alpha}(f'_\alpha;[a,b])=\|f''\|^p_{L_{p,\alpha}[a,b]}\le V^2_{(p,\alpha)}(f;[a,b]).\ \blacksquare$$

Corollary 3.8. *If $f\in BV^2_{(p,\alpha)}[a,b]$ then $f'_\alpha\in\alpha\text{-}AC_p[a,b]$. Moreover*

$$\|f''_\alpha\|^p_{L_{p,\alpha}[a,b]}\le V^2_{(p,\alpha)}(f).$$

Corollary 3.9. *If f'_α is α-absolutely continuous on $[a,b]$, $f'_\alpha\in\alpha\text{-}AC_p[a,b]$, $(1<p<+\infty)$ then $f\in BV^2_{(p,\alpha)}[a,b]$. Moreover*

$$V^2_{(p,\alpha)}(f;[a,b])\le\|f''_\alpha\|^p_{L_{p,\alpha}[a,b]}.$$

Proof. Let $\pi: a=t_0<t_1<\cdots<t_n=b$ be a partition of $[a,b]$. Since we may assume that f'_α is continuous on $[a,b]$, we have

$$\left|\frac{f(t_{i+1})-f(t_i)}{\alpha(t_{i+1})-\alpha(t_i)}-\frac{f(t_i)-f(t_{i-1})}{\alpha(t_i)-\alpha(t_{i-1})}\right|^p=\left|f'_\alpha(\rho_i^+)-f'_\alpha(\rho_i^-)\right|^p$$

$$=\left|\int_{\rho_i^-}^{\rho_i^+}f''_\alpha(\tau)d(\tau)\right|^p\le\int_{t_{i-1}}^{t_i}\left|f''_\alpha(\tau)d\tau\right|^p\cdot\left|\alpha(t_i)-\alpha(t_{i-1})\right|^{p-1},$$

where ρ_i^+ and ρ_i^- are points in the intervals (t_i,t_{i+1}) and (t_{i-1},t_i), respectively.

Thus

$$V^2_{(p,\alpha)}(f;[a,b]):=\sup_\pi\sigma^2_{(p,\alpha)}(f;[a,b])\le\|f''_\alpha\|^p_{L_{p,\alpha}}.\ \blacksquare$$

By Lemma 3.7 and Corollary 3.9, we obtain the main results:

Theorem 3.10. *A real function f defined on the interval $[a,b]$ belongs to the class $\alpha\text{-}AC^2_p[a,b]$, $1<p<+\infty$, if and only if $f\in BV^2_{(p,\alpha)}[a,b]$. Moreover*

$$V^2_{(p,\alpha)}(f;[a,b])=\|f''_\alpha\|^p_{L_{p,\alpha}[a,b]}=\int_a^b\left|f''_\alpha(\xi)\right|^p d\alpha(\xi).$$

Analogously, we shall obtain the following embedding

$$BV^2_{(p,\alpha)}[a,b]\hookrightarrow RV_{p,\alpha}[a,b],\quad (1<p<+\infty\ \text{ and }\alpha\in\mathcal{C}[a,b]\text{ increasing})$$

i.e., there exists a constant $M > 0$ such that

(5) $$\|f\|_p \leq M\|f\|_{(p,2,\alpha)}, \quad (f \in BV^2_{(p,\alpha)}[a,b]).$$

4. Main results

In this section, we shall present a characterization of functions $h : [a,b] \times \mathbb{R} \to \mathbb{R}$, for which the Nemytskii operator $H = H_h$, generated by h, maps the space $BV^2_{(p,\alpha)}[a,b]$ into $BV^2_{(q,\alpha)}[a,b]$, where $1 < q < p$, and is globally Lipschitzian. On the other hand, if $1 < p < q$ then the Nemytskii operator H is constant. We present the first theorem:

Theorem 4.1. *Let $1 < q < p$ and $\alpha \in AC^2_p[a,b]$. Then the Nemytskii operator H generated by a function $h : [a,b] \times \mathbb{R} \to \mathbb{R}$ maps the space $BV^2_{(p,\alpha)}[a,b]$ into space $BV^2_{(q,\alpha)}[a,b]$ and if it is globally Lipschitzian, i.e., there exists a constant $M > 0$ such that*

$$\left\|H_{f_1} - H_{f_2}\right\|_{(q,2,\alpha)} \leq M\|f_1 - f_2\|_{(p,2,\alpha)}, \quad f_1, f_2 \in BV^2_{(p,\alpha)}[a,b]$$

if and only if

$$h(t,x) = A(t)x + B(t), \quad t \in [a,b], \ x \in \mathbb{R},$$

where $A, B \in BV^2_{(q,\alpha)}[a,b]$.

Proof. Since $H : BV^2_{(p,\alpha)}[a,b] \to BV^2_{(q,\alpha)}[a,b]$ is globally Lipschitzian and the embedding (5) holds, there exist constants $M > 0$ and $N > 0$ such that

$$\left\|Hf_1 - Hf_2\right\|_q \leq N\left\|Hf_1 - Hf_2\right\|_{(q,2,\alpha)} \leq M\|f_1 - f_2\|_{(p,2,\alpha)},$$

where $f_1, f_2 \in BV^2_{(p,\alpha)}[a,b]$.

Fix $t, t' \in [a,b]$, $t < t'$; let $y_1, y_2, y'_1, y'_2 \in \mathbb{R}$ and define two polynomial functions $u_i : [a,b] \to \mathbb{R}$ $i = 1,2$ by

$$u_i(s) := \frac{y'_i - y_i}{\alpha(t') - \alpha(t)}\Bigg[(\alpha(s) - \alpha(a))^2 + \left(1 - \frac{(\alpha(t') - \alpha(a))^2 - (\alpha(t) - \alpha(a))^2}{\alpha(t') - \alpha(t)}\right)$$
$$(\alpha(s) - \alpha(a)) - (\alpha(t) - \alpha(a))^2 - \left(1 - \frac{(\alpha(t') - \alpha(a))^2 - (\alpha(t) - \alpha(a))^2}{\alpha(t') - \alpha(t)}\right)$$
$$(\alpha(t) - \alpha(a))\Bigg] + y_i, \qquad s \in [a,b].$$

Also, u_i satisfies the following conditions $u_i(t) = y_i$, $u_i(t') = y'_i$, for $i = 1,2$.

Moreover

$$(u_i(s))' = \frac{(y_i' - y_i)}{\alpha(t') - \alpha(t)} \times \left[2(\alpha(s) - \alpha(a)) + \left(1 - \frac{(\alpha(t') - \alpha(t))^2 - (\alpha(t) - \alpha(a))^2}{\alpha(t') - \alpha(t)}\right)\right]\alpha'(s)$$

and

$$(u_i(s))'' = \frac{y_i' - y_i}{\alpha(t') - \alpha(t)}\left[2(\alpha'(s))^2 + (\alpha(s) - \alpha(a))\alpha''(s) + \left(1 - \frac{(\alpha(t') - \alpha(t))^2 - (\alpha(t) - \alpha(a))^2}{\alpha(t') - \alpha(t)}\right)\alpha''(s)\right]$$

for every $s \in [a, b]$, $i = 1, 2$. So $(u_i)' \in \alpha\text{-}AC^2_{p,\alpha}[a, b]$ and verifies

$$\int_a^b \left|\frac{y_i' - y_i}{\alpha(t') - \alpha(t)}\left[2(\alpha'(s))^2 + (\alpha(s) - \alpha(a))\alpha''(s) + \left(1 - \frac{(\alpha(t') - \alpha(t))^2 - (\alpha(t) - \alpha(a))^2}{\alpha(t') - \alpha(t)}\right)\alpha''(s)\right]\right|^p d\alpha(s) < +\infty, \; i = 1, 2$$

by applying Hölder and Minkowski's inequalities.

Then the functions $u_i \in BV^2_{(p,\alpha)}[a, b]$, $i = 1, 2$ (see Thm. 3.10) and

$$(u_1 - u_2)(a) = y_1 - y_2 - \frac{y_1' - y_1 - y_2' + y_2}{\alpha(t') - \alpha(t)}(\alpha(t) - \alpha(a))\left[(\alpha(t) - \alpha(a)) + \left(1 - \frac{(\alpha(t') - \alpha(t))^2 - (\alpha(t) - \alpha(a))^2}{\alpha(t') - \alpha(t)}\right)\right]$$

and

$$(u_1 - u_2)'(a) = \frac{y_1' - |!y_1 - y_2' + y_2}{\alpha(t') - \alpha(t)}\left(1 - \frac{(\alpha(t') - \alpha(t))^2 - (\alpha(t) - \alpha(a))^2}{\alpha(t') - \alpha(t)}\right)\alpha'(a).$$

Also

$$(u_1 - u_2)'(s) = \frac{y_1' - y_1 - y_2' + y_2}{\alpha(t') - \alpha(t)}\left[2(\alpha(s) - \alpha(a)) + \left(1 - \frac{(\alpha(t') - \alpha(a))^2 - (\alpha(t) - \alpha(a))^2}{\alpha(t') - \alpha(t)}\right)\right]\alpha'(s), \qquad s \in [a, b],$$

$$\|(u_1 - u_2)'\|_\infty \le L_1\left|y_1' - y_1 - y_2' + y_2\right|; \qquad L_1 = 2|\alpha(b) - \alpha(a)|\|\alpha'\|_\infty,$$

and

$$\|(u_1 - u_2)''\|_\infty^q \le L_2^q|y_1' - y_1 - y_2' + y_2|^q;$$

where $L_2 = (\alpha(b) - \alpha(a))\left[2\|\alpha'\|_\infty + |1 + \alpha(b) - \alpha(a)|\|\alpha''\|_\infty\right]$. So, Hu_1 and Hu_2 are in $BV_{(p,\alpha)}[a, b] \subset \alpha\text{-}Lip[a, b]$. Moreover

$$(Hu_i)(t) = h(t, u_i(t)) = h(t, y_i), \quad i = 1, 2$$

and

$$(Hu_i)(t') = h(t', u_i(t')) = h(t', y_i'), \quad i = 1, 2,$$

which we may rewrite in the form

$$\begin{aligned}\frac{(Hu_1 - Hu_2)(t') - (Hu_1 - Hu_2)(t)}{\alpha(t') - \alpha(t)} &\leq K\|Hu_1 - Hu_2\|_{\alpha - L_p[a,b]} \\ &\leq KL\|u_1 - u_2\|_{BC_{p,\alpha}[a,b]}.\end{aligned}$$

Substituting, we have

$$\begin{aligned}&\frac{|h(t', y_1') - h(t', y_2') - h(t, y_1) + h(t, y_2)|^q}{|\alpha(t') - \alpha(t)|^{q-1}} \\ &\quad \leq KL\Bigg[\bigg|y_1 - y_2 - \frac{y_1' - y_1 - y_2' + y_2}{\alpha(t') - \alpha(t)}(\alpha(t) - \alpha(a))\Big((\alpha(t) - \alpha(a)) \\ &\qquad + \Big(1 - \frac{(\alpha(t') - \alpha(a))^2 - (\alpha(t) - \alpha(a))^2}{\alpha(t') - \alpha(t)}\Big)\Big)\bigg| \\ &\qquad + \bigg|\frac{y_1' - y_2' - y_1 + y_2}{\alpha(t') - \alpha(t)}\Big(1 - \frac{(\alpha(t') - \alpha(a))^2 - (\alpha(t) - \alpha(a))^2}{\alpha(t') - \alpha(t)}\Big)\alpha'(a)\bigg| \\ &\qquad + L_2\big|y_1' - y_2' - y_1 + y_2\big|\Bigg],\end{aligned}$$

multiplying the inequality by $|\alpha(t') - \alpha(t)|$ and applying the triangular inequality, we get

$$\begin{aligned}&\big|h(t', y_1') - h(t', y_2') - h(t, y_1) + h(t, y_2)\big|^q \\ &\quad \leq KL\Big[|y_1 - y_2|^q|\alpha(t') - \alpha(t)|^{q-1} + \ell_1|y_1' - y_2' - y_1 + y_2|^q|\alpha(t) - \alpha(a)|^q \\ &\qquad + \ell_2|y_1' - y_2' - y_1 + y_2|^q|\alpha(b) - \alpha(a)|^q + L_2^p|y_1' - y_2' - y_1 + y_2|^q\Big],\end{aligned}$$

for every fixed $y \in \mathbb{R}$, the constant function $u_0(t) = y$, $t \in [a, b]$ belongs to $BV^2_{(p,\alpha)}[a, b]$, and from hypothesis $Hu_0(t) = h(t, u_0(t)) = h(t, y)$ belongs to $BV^2_{(p,\alpha)}[a, b]$. Consequently, $h(\cdot, y)$ is continuous on $[a, b]$. Therefore, letting $t' \to t$ in the above inequality and considering the continuity of the α, we obtain

$$\text{(6)} \qquad \big|h(t', y_1') - h(t', y_2') - h(t, y_1) + h(t, y_2)\big| \leq 4N|y_1' - y_2' - y_1 + y_2|,$$

for all $t \in [a, b]$ and $y_1, y_2, y_1', y_2' \in \mathbb{R}$.

Let us fix $t \in [a, b]$ and define the function $P_t : \mathbb{R} \to \mathbb{R}$ by $P_t(y) := h(t, y) - h(t, 0)$.

Setting $y_1 = v + w$, $y_2 = v$, $y_1' = w$ and $y_2' = 0$ in the inequality (6), we get

$$h(t, v + w) = h(t, v) + h(t, w) + h(t, 0)$$

rewritten in terms of P_t, results $P_t(v + w) = P_t(v) + P_t(w)$ $v, w \in \mathbb{R}$. Setting $y_1' = y_2' = 0$ in the inequality (6), we obtain

$$|P_t(y_1) - P_t(y_2)| \leq 4N|y_1 - y_2|, \ \ y_1, y_2 \in \mathbb{R}.$$

Thus, the function P_t is additive and continuous on $\mathbb{R}$, consequently is linear on $\mathbb{R}$ and there exists a function $A : [a, b] \to \mathbb{R}$ such that $P_t(y) = A(t)y$, $y \in \mathbb{R}$. Defined the function $B : [a, b] \to \mathbb{R}$ by $B(t) = h(t, 0)$, $t \in [a, b]$, it follows from the definition of P_t that

$$h(t, y) = A(t)y + B(t), \ \ t \in [a, b], \ y \in \mathbb{R}.$$

Since the composition operator H maps the space $BV^2_{(p,\alpha)}[a, b]$ into $BV^2_{(q,\alpha)}[a, b]$, the function $B(\cdot) = h(\cdot, 0)$ belongs to $BV^2_{(q,\alpha)}[a, b]$, and the function $A(\cdot) = h(\cdot, 1) - h(\cdot, 0)$ also belongs to $BV^2_{(q,\alpha)}[a, b]$. Thus the function h has the following form

$$h(t, y) = A(t)y + B(t), \ \ t \in [a, b], \ y \in \mathbb{R},$$

where $A, B \in BV^2_{(q,\alpha)}[a, b]$.

Reciprocally, if we suppose that the function h is additive; i.e., $h(t, y) = A(t)y + B(t)$, where $A, B \in BV^2_{(q,\alpha)}[a, b]$ is an algebra, we get

$$\|Hf_1 - Hf_2\|_{(p,2,\alpha)} \leq \|A\|_{(q,2,\alpha)}\|f_1 - f_2\|_{(q,2,\alpha)} \ \ f_1, f_2 \in BV^2_{(q,\alpha)}[a, b].$$

Therefore, the composition operator H generated by the function h maps the space $BV^2_{(p,\alpha)}[a, b]$ into $BV^2_{(q,\alpha)}[a, b]$ and satisfies the global Lipschitz condition. ∎

5. Uniformly continuous composition operator

Now we shall weaken the hypothesis of Theorem 4.1 and we get a proposition that holds only the necessary condition for the Composition Operator. For this we need to recall some definitions and results that we will use for this purpose.

We shall say

$$\mathbf{p}(f) := \mathbf{p}(f; [a, b]) = \inf\left\{\epsilon > 0 : \ V^2_{(p,\alpha)}(f/\epsilon) \leq 1\right\}; \quad f \in BV^2_{(p,\alpha)}[a, b],$$

so

$$\|f\|_{(p,2,\alpha)} := |f(a)| + |f'(a)| + [\mathbf{p}(f)]^{1/p}.$$

Lemma 5.1. *Let $f \in BV^2_{(p,\alpha)}[a,b]$ and $1 < p < +\infty$. We have:*

(1) *if $\mathbf{p}(f) > 0$, then $V^2_{(p,\alpha)}(f/\mathbf{p}(f)) \leq 1$;*
(2) *if $\rho > 0$, then $V^2_{(p,\alpha)}(f/\rho) \leq 1$ iff $\mathbf{p}(f) \leq \rho$;*
(3) *if $\rho > 0$ and $V^2_{(p,\alpha)}(f/\rho) \leq 1$, then $\mathbf{p}(f) = \rho$.*

Proof. (1) The definition of $\mathbf{p}(f)$ implies $V^2_{(p,\alpha)} \leq 1$ for all $\rho > \mathbf{p}(p)$. Choose a sequence $\rho_n > \mathbf{p}(f)$, $n \in \mathbb{N}$, which converges to $\mathbf{p}(f)$ as $n \to +\infty$. Then $f/\rho_n \to f/\mathbf{p}(f)$ uniformly on $[a,b]$. So that

$$V^2_{(p,\alpha)}\big(f/\mathbf{p}(f)\big) \leq \liminf_{n\to+\infty} V^2_{(p,\alpha)}\big(f/\rho_n\big) \leq 1.$$

It follows that $\mathbf{p}(f) \in \{\rho > 0 : \ V^2_{(p,\alpha)}(f/\mathbf{p}(f)) \leq 1\}$ and

$$\mathbf{p}(f) = \{\rho > 0 : \ V^2_{(p,\alpha)}(f/\mathbf{p}(f)) \leq 1\}.$$

(2) If $V^2_{(p,\alpha)}(f/\mathbf{p}(f)) \leq 1$, the definition of $\mathbf{p}(f)$ implies $\mathbf{p}(f) \leq \rho$. If $\mathbf{p}(f) = \rho$ then $V^2_{(p,\alpha)}(f/\mathbf{p}(f)) \leq 1$ by (1). Let us show that

$$\text{if } \mathbf{p}(f) < 1 \text{ then } V^2_{(p,\alpha)}(f/\rho) < 1. \tag{7}$$

If $\mathbf{p}(f) = 0$, then f is a constant mapping and $V^2_{(p,\alpha)}(f/\rho) = 0$, so assume that $\mathbf{p}(f) > 0$. From the convexity of $V^2_{(p,\alpha)}(f)$ and item above, we have:

$$V^2_{(p,\alpha)}(f/\rho) \leq \mathbf{p}(f)/\rho\, V^2_{(p,\alpha)}(f/\mathbf{p}(f)) \leq \mathbf{p}(f)/\rho < 1.$$

(3) Let $V^2_{(p,\alpha)}\Big(f/\rho\Big) = 1$. By (2), if $\mathbf{p}(f) > \rho$ then $V^2_{(p,\alpha)}\big(f\big) > 1$, which is impossible. Taking into account (7), we conclude that $\mathbf{p}(f) = \rho$. ∎

Our second main result reads as follows:

Theorem 5.2. *Let $1 < q < p < +\infty$ and $h : [a,b] \times \mathbb{R} \to \mathbb{R}$. If a composition operator H, maps the space $BV^2_{(p,\alpha)}[a,b]$ into $BV^2_{(q,\alpha)}[a,b]$ generated by h, is uniformly continuous, i.e.,*

$$\big\|Hf_1 - Hf_2\big\|_{q,\alpha} \leq \omega(\big\|f_1 - f_2\big\|_{p,\alpha}, \quad f_1, f_2 \in BV^2_{(p,\alpha)}[a,b],$$

where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is the modulus continuity of H. Then

$$h(t,x) = A(t)x + B(t), \quad t \in [a,b], \ x \in \mathbb{R},$$

where $A, B \in BV^2_{(q,\alpha)}[a,b]$.

Proof. For every $x \in \mathbb{R}$, the constant function $u(t) = x$, $t \in [a,b]$ belongs to $BV^2_{(p,\alpha)}[a,b]$. Since the Nemytskii operator H maps the space $BV^2_{(p,\alpha)}[a,b]$ into $BV^2_{(q,\alpha)}[a,b]$, it follows that the function $t \mapsto h(t,u(t)) = h(t,x)$ belongs to $BV^2_{(p,\alpha)}[a,b]$.

The uniform continuity of H on $BV_{(p,\alpha)}[a,b]$ and the embedding (5) imply

$$\|Hf_1 - Hf_2\|_q \le N\|Hf_1 - Hf_2\|_{q,\alpha} \le \omega(\|f_1 - f_2\|_{p,\alpha}), \tag{8}$$

where $f_1, f_2 \in BV^2_{(p,\alpha)}[a,b]$ and $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is the modulus continuity of H, i.e.

$$\omega(\beta) := \sup\left\{\|H(f_1) - H(f_2)\|_{q,\alpha} : \|f_1 - f_2\|_{p,\alpha} \le \beta;\ f_1, f_2 \in BV^2_{(p,\alpha)}[a,b]\right\},$$

for $\beta > 0$. From the definition of the norm $\|\cdot\|_{p,\alpha}$, we obtain

$$\mathbf{p}\big(H(f_1) - H(f_2)\big) \le \|H(f_1) - H(f_2)\|_{q,\alpha}, \tag{9}$$

for $f_1, f_2 \in BV^2_{(p,\alpha)}[a,b]$. Hence, in view of Lemma 5.1 and (9), if $\mathbf{p}\big(H(f_1) - H(f_2)\big) \le \omega\big(\|f_1 - f_2\|_{p,\alpha}\big)$ then

$$V^2_{(p,\alpha)}\left(\frac{H(f_1) - H(f_2)}{\omega\big(\|f_1 - f_2\|_{p,\alpha}\big)}\right) \le 1.$$

Fix $t, \bar{t} \in [a,b]$, $t < \bar{t}$; let $y_1, y_2, \overline{y}_1, \overline{y}_2 \in \mathbb{R}$ and define two polynomial functions $u_i : [a,b] \to \mathbb{R}$, $i = 1, 2$ by

$$\begin{aligned} f_i(s) := \frac{\overline{y}_i - y_i}{\alpha(\bar{t}) - \alpha(t)}\Bigg[&(\alpha(s) - \alpha(a))^2 + \left(1 - \frac{(\alpha(\bar{t}) - \alpha(a))^2 - (\alpha(t) - \alpha(a))^2}{\alpha(\bar{t}) - \alpha(t)}\right) \\ &(\alpha(s) - \alpha(a)) - (\alpha(t) - \alpha(a))^2 - \left(1 - \frac{(\alpha(\bar{t}) - \alpha(a))^2 - (\alpha(t) - \alpha(a))^2}{\alpha(\bar{t}) - \alpha(t)}\right) \\ &(\alpha(t) - \alpha(a))\Bigg] + y_i, \qquad s \in [a,b]. \end{aligned}$$

The inequality (8) for the above functions gives

$$\|Hf_1 - Hf_2\|_q \le M\big(|\overline{y}_1 - y_2|^q + \big(2^q|\overline{y}_1 - \overline{y_2} - y_1 + y_2|^q\big)\big)$$

where $M = KL$.

Since f_i satisfies the following conditions $f_i(t) = y_i$, $f_i(\bar{t}) = \overline{y}_i$, for $i = 1, 2$; then by definition of the norm $\|\cdot\|_{p,\alpha}$, we get

$$\left(\frac{\big|h(\bar{t}, f_1(\bar{t})) - h(\bar{t}, f_2(\bar{t})) - h(t, f_1(t)) + h(t, f_2(t))\big|^q}{\big|\big(\omega(\|f_1 - f_2\|)\big)^q\alpha(\bar{t}) - \alpha(t)\big|^{q-1}}\right) \le 2^{q-1}M^q(|\overline{y}_1 - y_2|^q + 2^q|\overline{y}_1 - \overline{y_2} - y_1 + y_2|^q)$$

or equivalently

$$\begin{aligned} &\big|h(\bar{t}, f_1(\bar{t})) - h(\bar{t}, f_2(\bar{t})) - h(t, f_1(t)) + h(t, f_2(t))\big|^q \\ &\le (\omega(\|f_1 - f_2\|))^q 2^{q-1}M^q\big(|\overline{y}_1 - y_2|^q + 2^q|\overline{y}_1 - \overline{y_2} - y_1 + y_2|^q\big)|\alpha(\bar{t}) - \alpha(t)|^q. \end{aligned}$$

Passing to the limits on both sides of this inequality as $\alpha(\bar{t}) \nearrow \alpha(t)$, we get

(10) $$\left|h(\bar{t}, f_1(\bar{t})) - h(\bar{t}, f_2(\bar{t})) - h(t, f_1(t)) + h(t, f_2(t))\right| \leq 0.$$

Let us fix $t \in [a, b]$ and define the function $P_t : \mathbb{R} \to \mathbb{R}$ by the following formula

(11) $$P_t(s) := h(t, s) - h(t, 0), \quad s \in \mathbb{R}.$$

Setting

$$y_1 = w + z, \quad y_2 = w, \quad \overline{y}_1 = z, \quad \overline{y}_2 = 0$$

in the inequality (10), we have

(12) $$h(\bar{t}, z) - h(\bar{t}, 0) - h(t, w + z) + h(t, w) = 0.$$

By using (11), we can write (12) in terms of the function P_t:

$$P_t(w + z) = P_t(w) + P_t(z), \qquad w, z \in \mathbb{R}.$$

Thus, for each $t \in [a, b]$, the function $P_t(\cdot) = h(t, \cdot)$ satisfies the *Cauchy functional equation* in $\mathbb{R}$. Modifying a little the standard argument (cf. Kuczma [8]), we conclude that, for each $t \in [a, b]$, there exist $A(t), B(t) \in BC^2_{(q,\alpha)}[a, b]$ such that

$$h(t, s) = A(t)s + B(t).$$

The uniform continuity of operator $H : BV^2_{(p,\alpha)}[a, b] \to BV^2_{(q,\alpha)}[a, b]$ implies the continuity of the additive function $A(t)$. This completes the proof. ■

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