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EXISTENCE AND CONTROLLABILITY RESULTS FOR MIXED FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH INFINITE DELAY

Abstract. Sufficient conditions are established for the existence of solution for mixed neutral functional integrodifferential equations with infinite delay. The results are obtained using the theory of fractional powers of operators and the Sadovskii's fixed point theorem. As an application we prove a controllability result for the system.

1. Introduction

In this article, we establish existence results of the following mixed functional integrodifferential equations with infinite delay:

$$\frac{d}{dt}[x(t)-g(t,x_t)]+Ax(t) = f\left(t,x_t,\int_0^t w(t,s,x_s)ds,\int_0^b h(t,s,x_s)ds\right),\quad t\in J, \tag{1.1}$$

$$x_0=\phi\in\mathcal{B}, \tag{1.2}$$

where $J=[0,b]$, $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $T(t), t\geq 0$ in a Banach space $(X,\|\cdot\|)$, the nonlinear functions $g: J\times\mathcal{B}\to X$, $f: J\times\mathcal{B}\times X\times X\to X$, $w,h:\Delta\times\mathcal{B}\to X$, $\Delta=\{(t,s): 0\leq s\leq t\leq b\}$ are continuous functions and $\mathcal{B}$ is a phase space defined later. The histories $x_t:(-\infty,0]\to X$, $x_t(s)=x(t+s)$, $s\leq 0$, belong to an abstract phase space $\mathcal{B}$.

Due to the application in many areas of applied mathematics, there has been an increasing interest in the investigation of neutral functional differential and integrodifferential equations with the effect of infinite delay. The work on first order abstract neutral functional differential equations with

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infinite delay was initiated by Hernández and Henríquez in [15, 16]. Afterwards, several works reporting existence results, controllability problem and other properties of mild solution for first-order abstract neutral functional differential and functional integrodifferential equations with infinite delay have been published, see for example, [1]–[6], [10]–[12], [14]–[16], [20] and the reference cited therein. The problems related to existence, uniqueness and other properties of mild solutions of mixed functional integrodifferential equations with bounded delay have been studied in [7, 8, 18, 19] by using different techniques.

The aim of this paper is to study the existence and controllability of the problem (1.1)–(1.2) by using the theory of fractional powers of operators and the Sadovskii's fixed point theorem. The results of this paper are based on the analytic semi group theory and the ideas and techniques in Fu [11] and Li et al. [20]. The results obtained here generalizes the results of [11, 20].

The paper is organized as follows. In Section 2, we recall some preliminaries and list the hypotheses that will be used through out. In Section 3, we establish the existence result for the system (1.1)–(1.2), in Section 4, we deal with controllability problem and finally in Section 5, an example is considered to illustrate the application of our result.

2. Preliminaries and hypotheses

We introduce some preliminaries from [9, 13, 17, 22] and hypotheses that will be used in our further analysis.

We suppose that $(X, \|\cdot\|)$ is a Banach space, and $-A : D(-A) \to X$ is the infinitesimal generator of a compact analytic semigroup $\{T(t)\}_{t\geq 0}$ of uniformly bounded linear operators. Let $0 \in \rho(A)$. Then it is possible to define the fractional power A^α, for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(A^\alpha)$. Furthermore, the subspace $D(A^\alpha)$ is dense in X and $\|x\|_\alpha = \|A^\alpha x\|, \quad x \in D(A^\alpha)$ defines a norm on $D(A^\alpha)$. We denote by X_α the Banach space $D(A^\alpha)$ normed with $\|x\|_\alpha$.

Lemma 2.1. ([22]) *The following properties hold:*

(i) *If $0 < \beta < \alpha \leq 1$, then $X_\alpha \subset X_\beta$ and the imbedding is compact whenever the resolvent operator of A is compact.*
(ii) *There exists constant $M \geq 1$ such that $\|T(t)\| \leq M$, for all $t \in J$.*
(iii) *For every $0 < \alpha \leq 1$ there exists $C_\alpha > 0$ such that*

$$\|(-A)^\alpha T(t)\| \leq \frac{C_\alpha}{t^\alpha}, \quad 0 < t \leq b.$$

To study the system (1.1)–(1.2), we assume that the histories $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$ belong to a seminormed abstract linear space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ of functions mapping $(-\infty, 0]$ into X, which has been introduced by Hale and Kato [13] and widely discussed in [17]. We assume that $\mathcal{B}$ satisfies the following axioms:

(A1) If $x : (-\infty, \sigma + b) \to X$, $b > 0$ is continuous on $[\sigma, \sigma + b)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + b)$ the following statements hold:
  (i) x_t is in $\mathcal{B}$;
  (ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$;
  (iii) $\|x_t\|_{\mathcal{B}} \leq K(t-\sigma)\sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}_\alpha}$.
  Here $H \geq 0$ is a constant, $K, M : [0, +\infty) \to [0, +\infty)$, $K(\cdot)$ is continuous and $M(\cdot)$ is locally bounded, and H, $K(\cdot)$, $M(\cdot)$ are independent of $x(t)$.

(A2) For the function $x(\cdot)$ in (A1), x_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma + b]$.

(A3) The space $\mathcal{B}$ is complete.

Definition 2.1. A function $x : (-\infty, b] \to X$ is called a mild solution of the problem (1.1)–(1.2) if $x_0 = \phi \in \mathcal{B}$ on $(-\infty, 0]$, the restriction of $x(\cdot)$ to the interval J is continuous, and for each $s \in [0, t)$, the function $AT(t-s)g(s, x_s)$, $0 \leq t < b$ is integrable and the integral equation

$$x(t) = T(t)[\phi(0) - g(0, \phi)] + g(t, x_t) - \int_0^t AT(t-s)g(s, x_s)ds + \int_0^t T(t-s)f\left(s, x_s, \int_0^s w(s, \tau, x_\tau)d\tau, \int_0^b h(s, \tau, x_\tau)d\tau\right)ds, \quad t \in J \tag{2.1}$$

is satisfied.

Lemma 2.2. ([24], Sadovskii) *Let D be a convex, bounded and closed subset of a Banach space X. If $F : D \to D$ is a condensing operator, then F has a fixed point in D.*

We list the following hypotheses in order to establish our existence result.

(H_1) There exist constants G, $L_g > 0$, such that
  (i) $\|A^\beta g(t_2, \psi_2) - A^\beta g(t_1, \psi_1)\| \leq G(|t_2 - t_1| + \|\psi_2 - \psi_1\|_{\mathcal{B}})$, $t_1, t_2 \in J$, $\psi_1, \psi_2 \in \mathcal{B}$.
  (ii) $\|A^\beta g(t, \psi)\| \leq L_g(\|\psi\|_{\mathcal{B}} + 1)$, $t \in J$, $\psi \in \mathcal{B}$.

(H_2) There exist $p, q \in C(J, [0, \infty))$, such that

$$\left\|\int_0^t \omega(t, s, \psi)ds\right\| \leq p(t)\|\psi\|_{\mathcal{B}} \text{ and } \left\|\int_0^b h(t, s, \psi)ds\right\| \leq q(t)\|\psi\|_{\mathcal{B}},$$

for every $(t, s) \in \Delta$ and $\psi \in \mathcal{B}$.

(H_3) For every positive integer k, there exists $\alpha_k \in L^1(J, [0, \infty))$, such that

$$\sup_{\|\psi\|_{\mathcal{B}}, \|x\|, \|y\| \leq k} \|f(t, \psi, x, y)\| \leq \alpha_k(t) \text{ for a.e. } t \in J$$

and

$$\liminf_{k\to+\infty}\frac{1}{k}\int_0^b \alpha_k(s)ds = \mu < \infty.$$

(H_4) (i) For each $(t,s)\in\Delta$, the functions $\omega(t,s,.)$, $h(t,s,.) : \mathcal{B}\to X$ are continuous and for each $\psi\in\mathcal{B}$ the functions $\omega(.,.,\psi)$, $h(.,.,\psi) : \Delta\to X$ are strongly measurable.

(ii) For each $t\in J$, the function $f(t,.,.,.) : \mathcal{B}\times X\times X\to X$ is continuous and for each $(\psi,\ x,\ y)\in\mathcal{B}\times X\times X$, the function $f(.,\psi,\ x,\ y) : J\to X$ is strongly measurable.

3. Existence result

Theorem 3.1. *Let the hypotheses (H_1)–(H_4) be satisfied. Then the system (1.1)–(1.2) has a mild solution on $(-\infty,b]$ if*

$$L_0 := GK_b\left(\|A^{-\beta}\| + \frac{C_{1-\beta}b^{\beta}}{t^{\beta}}\right) < 1, \tag{3.1}$$

$$\left[\left(\|A^{-\beta}\| + \frac{C_{1-\beta}b^{\beta}}{\beta}\right)L_g + M\vartheta\mu\right]K_b < 1, \tag{3.2}$$

where $\vartheta = \max_{t\in J}\{1,p(t),q(t)\}$ and $K_b = \sup\{K(t) : t\in[0,b]\}$.

Proof. Consider the operator Γ defined by

$$\Gamma(x)(t) = \begin{cases}\phi(t), \quad t\in(-\infty,0],\\ T(t)[\phi(0)-g(0,\phi)] + g(t,x_t) - \int_0^t AT(t-s)g(s,x_s)ds\\ +\int_0^t T(t-s)f\left(s,x_s,\int_0^s w(s,\tau,x_\tau)d\tau,\int_0^b h(s,\tau,x_\tau)d\tau\right)ds,\ t\in J.\end{cases}$$

Then the equation (2.1) with the initial condition $x_0=\phi\in\mathcal{B}$ on $(-\infty,0]$ can be written as $x=\Gamma x$. We show that Γ has a fixed point $x(\cdot)$, then this fixed point $x(\cdot)$ is a mild solution of the system (1.1)–(1.2).

Let $y(\cdot) : (-\infty,b)\to X$ be the function defined by

$$y(t) = \begin{cases}T(t)\phi(0), & t\geq 0,\\ \phi(t), & -\infty<t<0.\end{cases}$$

Then $y_0=\phi$ and the map $t\to y_t$ is continuous. We can assume that

$$N = \sup\{\|y_t\|_{\mathcal{B}} : 0\leq t\leq b\}.$$

For each $z\in C(J;X), z(0)=0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases}z(t), & t\in J,\\ 0, & -\infty<t<0.\end{cases}$$

If $x(\cdot)$ satisfies (2.1) with $x_0=\phi\in\mathcal{C}$, we can decompose it as $x(t) = z(t)+y(t)$, $0\leq t\leq b$, which implies that $x_t=\overline{z}_t+y_t$ for every $t\in J$ and the function $z(\cdot)$ satisfies

$$z(t) = -T(t)g(0,\phi) + g(t,\overline{z}_t + y_t) - \int_0^t AT(t-s)g(s,\overline{z}_s + y_s)ds$$
$$+ \int_0^t T(t-s)$$
$$\times f\left(s, \overline{z}_s + y_s, \int_0^s w(s,\tau,\overline{z}_\tau + y_\tau)d\tau, \int_0^b h(s,\tau,\overline{z}_\tau + y_\tau)d\tau\right) ds, \ t \in J.$$

Let F be the operator on $C([0,b];X)$ defined by

$$F(z)(t) = -T(t)g(0,\phi) + g(t,\overline{z}_t + y_t) - \int_0^t AT(t-s)g(s,\overline{z}_s + y_s)ds$$
$$+ \int_0^t T(t-s)$$
$$\times f\left(s, \overline{z}_s + y_s, \int_0^s w(s,\tau,\overline{z}_\tau + y_\tau)d\tau, \int_0^b h(s,\tau,\overline{z}_\tau + y_\tau)d\tau\right) ds.$$

Obviously, the operator Γ having a fixed point is equivalent to F having one, so it turns out to prove that F has a fixed point.

For each positive integer k, define

$$B_k = \{z \in C(J;X) : z(0) = 0, \ \|z(t)\| \leq k, \ t \in J\}.$$

Then for each k, B_k is clearly convex, bounded and closed subset of $C(J;X)$. Note that, for any $z \in B_k$ and $t \in J$, we have $\overline{z}_t + y_t \in \mathcal{B}$. Thus from assumption (A2)(iii), we have

$$\begin{aligned} \|\overline{z}_t + y_t\|_{\mathcal{B}} &\leq \|\overline{z}_t\|_{\mathcal{B}} + \|y_t\|_{\mathcal{B}} \\ &\leq K(t)\sup\{\|\overline{z}(s)\| : 0 \leq s \leq t\} + M(t)\|\overline{z}_0\|_{\mathcal{B}} + \|y_t\|_{\mathcal{B}} \\ &\leq kK_b + N, \end{aligned} \tag{3.3}$$

since $\|\overline{z}_0\|_{\mathcal{B}} = 0$. Using Lemma 2.1, hypothesis $(H_1(\text{ii}))$ and the condition (3.3), we get

$$\begin{aligned} \|AT(t-s)g(s,\overline{z}_s + y_s)\| &\leq \|A^{1-\beta}T(t-s)A^\beta g(s,(\overline{z})_s + y_s)\| \\ &\leq \frac{C_{1-\beta}}{(t-s)^{1-\beta}} L_g(kK_b + N + 1). \end{aligned}$$

Thus from Bochner Theorem [21], it follows that $AT(t-s)g(s,\overline{z}_s + y_s)$ is integrable on $[0,t)$, and hence, F is well defined on B_k.

We claim that there exists a positive integer k such that $FB_k \subseteq B_k$. If this is not true, then for each positive integer k, there is a function $z_k \in B_k$, but $F(z_k) \notin B_k$, that is, $\|F(z_k)(t)\| > k$ for some $t(k) \in J$, where $t(k)$ denotes

t depending on k. However, on the other hand, we have

$$
\begin{aligned}
(3.4)\qquad & k < \|F(z_k)(t)\| \\
&\leq \|T(t)\|\|A^{-\beta}\|\|A^{\beta}g(0,\phi)\| + \|A^{-\beta}\|\|A^{\beta}g(t,(\overline{z}_k)_t + y_t)\| \\
&\quad + \int_0^t \|A^{1-\beta}T(t-s)\| \left\|A^{\beta}g(s,(\overline{z}_k)_s + y_s)\right\| ds + \int_0^t \|T(t-s)\| \\
&\quad \times \left\| f\left(s,(\overline{z}_k)_s + y_s, \int_0^s w(s,\tau,(\overline{z}_k)_\tau + y_\tau)d\tau, \int_0^b h(s,\tau,(\overline{z}_k)_\tau + y_\tau)d\tau\right)\right\| ds.
\end{aligned}
$$

Note that, from the condition (3.3) and hypothesis (H_2), for each $t \in J$, we have

$$
\begin{aligned}
(3.5)\qquad & \max\left\{\|(\overline{z}_k)_t + y_t\|, \left\|\int_0^t w(t,s,(\overline{z}_k)_s + y_s)ds\right\|, \left\|\int_0^b h(t,s,(\overline{z}_k)_s + y_s)ds\right\|\right\} \\
&\leq (kK_b + N)\max_{t\in J}\{1, p(t), q(t)\} = (kK_b + N)\vartheta.
\end{aligned}
$$

Using hypothesis (H_1)–(H_3) and the inequalities (3.3) and (3.5), from (3.4), we obtain

$$
\begin{aligned}
k <& M\|A^{-\beta}\|L_g(\|\phi\|_{\mathcal{B}} + 1) + \|A^{-\beta}\|L_g(kK_b + N + 1) \\
&+ \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} L_g(kK_b + N + 1)ds + M\int_0^t \alpha_{(kK_b+N)\vartheta}(s)ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
1 <& \frac{M\|A^{-\beta}\|L_g(\|\phi\|_{\mathcal{B}} + 1)}{k} + \|A^{-\beta}\|L_g\left(K_b + \frac{N+1}{k}\right) \\
&+ \frac{C_{1-\beta}b^{\beta}L_g\left(K_b + \frac{N+1}{k}\right)}{\beta} \\
&+ M\left(K_b + \frac{N}{k}\right)\vartheta\frac{1}{(kK_b + N)\vartheta}\int_0^b \alpha_{(kK_b+N)\vartheta}(s)ds.
\end{aligned}
$$

Taking lower limit and using (H_3), we get

$$
\left[\left(\|A^{-\beta}\| + \frac{C_{1-\beta}b^{\beta}}{\beta}\right)L_g + M\vartheta\mu\right]K_b \geq 1.
$$

This contradicts (3.1). Hence, for some positive integer k, $FB_k \subseteq B_k$.

Next, we show that the operator F has a fixed point in B_k by applying Sadovskii's fixed point theorem, which implies that (1.1)–(1.2) has a mild solution. We decompose F as $F = F_1 + F_2$, where the operators F_1, F_2 are defined on B_k, respectively by

$$(F_1 z)(t) = -T(t)g(0,\phi) + g(t, \overline{z}_t + y_t) - \int_0^t AT(t-s)g(s, \overline{z}_s + y_s)ds,$$

$$(F_2 z)(t) = \int_0^t T(t-s) \times f\left(s, \overline{z}_s + y_s, \int_0^s w(s,\tau,\overline{z}_\tau + y_\tau)d\tau, \int_0^b h(s,\tau,\overline{z}_\tau + y_\tau)d\tau\right) ds,$$

for $z \in B_k$ and $t \in J$. To prove F is a condensing operator [24], we will show that F_1 verifies a contraction condition, while F_2 is a compact operator.

Firstly, we prove that F_1 is a contraction. Take any $z_1, z_2 \in B_k$. Then for each $t \in J$, by Lemma 2.1, hypothesis $(H_1)(i)$ and (3.3), we have

$$\begin{aligned}
&\|(F_1 z_1)(t) - (F_1 z_2)(t)\| \\
&\quad \le \|A^{-\beta}\| \|A^\beta g(t, (\overline{z}_1)_t + y_t) - A^\beta g(t, (\overline{z}_2)_t + y_t)\| \\
&\qquad + \int_0^t \|A^{1-\beta}T(t-s)\| \left\| A^\beta g(s, (\overline{z}_1)_s + y_s) - A^\beta g(s, (\overline{z}_2)_s + y_s) \right\| ds \\
&\quad \le \|A^{-\beta}\| G \|(\overline{z}_1)_t - (\overline{z}_2)_t\|_{\mathcal{B}} + \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} G \|(\overline{z}_1)_s - (\overline{z}_2)_s\|_{\mathcal{B}} ds \\
&\quad \le \|A^{-\beta}\| G \{K(t) \sup\{\|\overline{z}_1(s) - \overline{z}_2(s)\| : 0 \le s \le t\} \\
&\qquad + M(t) \left[\|(\overline{z}_1)_0\|_{\mathcal{B}} + \|(\overline{z}_2)_0\|_{\mathcal{B}}\right]\} \\
&\qquad + \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} G\{K(s) \sup\{\|\overline{z}_1(\tau) - \overline{z}_2(\tau)\| : 0 \le \tau \le s\} \\
&\qquad + M(t) \left[\|(\overline{z}_1)_0\|_{\mathcal{B}} + \|(\overline{z}_2)_0\|_{\mathcal{B}}\right]\} ds \\
&\quad \le GK_b \left(\|A^{-\beta}\| + \frac{C_{1-\beta} b^\beta}{t^\beta} \right) \sup\{\|z_1(s) - z_2(s)\| : 0 \le t \le b\},
\end{aligned}$$

since $\|(\overline{z}_1)_0\|_{\mathcal{B}} = 0$ and $\|(\overline{z}_2)_0\|_{\mathcal{B}} = 0$. This implies,

$$\|F_1 z_1 - F_1 z_2\| \le L_0 \|z_1 - z_2\|.$$

Therefore, by assumption (3.1), we see that F_1 is a contraction.

To prove F_2 is a compact operator, first we prove that F_2 is continuous on B_k. Let $\{z_n\} \subseteq B_k$ with $z_n \to z$ in B_k; then for each $t \in J$, $(\overline{z}_n)_t \to \overline{z}_t$ and by using hypothesis (H_4), we have

$$\begin{aligned}
&f\left(t, \overline{z}_{n_t} + y_t, \int_0^t w(t,s,(\overline{z}_n)_s + y_s)ds, \int_0^b h(t,s,(\overline{z}_n)_s + y_s)ds\right) \\
&\quad \to f\left(t, \overline{z}_t + y_t, \int_0^t w(t,s,\overline{z}_s + y_s)ds, \int_0^b h(t,s,\overline{z}_s + y_s)ds\right) \quad \text{as } n \to \infty,
\end{aligned}$$

for each $t \in J$ and since

$$\left\| f\left(t, \overline{z}_{n_t} + y_t, \int_0^t w(t,s,(\overline{z}_n)_s + y_s)ds, \int_0^b h(t,s,(\overline{z}_n)_s + y_s)ds\right) - f\left(t, \overline{z}_t + y_t, \int_0^t w(t,s,\overline{z}_s + y_s)ds, \int_0^b h(t,s,\overline{z}_s + y_s)ds\right)\right\| \leq 2\alpha_{(kK_b+N)\vartheta}(t),$$

we have, by dominated convergence theorem

$$\begin{aligned}
&\|Fz_n - Fz\| \\
&= \sup_{t\in J}\left\| \int_0^t T(t-s)\Big[f\Big(s, (\overline{z}_n)_s + y_s, \int_0^s w(s,\tau,(\overline{z}_n)_\tau + y_\tau)d\tau, \int_0^b h(s,\tau,(\overline{z}_n)_\tau + y_\tau)d\tau\Big) \right.\\
&\quad \left. - f\Big(s, \overline{z}_s + y_s, \int_0^s w(s,\tau,\overline{z}_\tau + y_\tau)d\tau, \int_0^b h(s,\tau,\overline{z}_\tau + y_\tau)d\tau\Big)\Big]ds \right\| \\
&\leq M\int_0^t \left\| f\left(s, (\overline{z}_n)_s + y_s, \int_0^s w(s,\tau,(\overline{z}_n)_\tau + y_\tau)d\tau, \int_0^b h(s,\tau,(\overline{z}_n)_\tau + y_\tau)d\tau\right)\right.\\
&\quad \left. - f\left(s, \overline{z}_s + y_s, \int_0^s w(s,\tau,\overline{z}_\tau + y_\tau)d\tau, \int_0^b h(s,\tau,\overline{z}_\tau + y_\tau)d\tau\right)\right\| ds
\end{aligned}$$

$\to 0$ as $n \to \infty$. Therefore, F is continuous.

Next, we prove that the family $\{Fz : z \in B_k\}$ is an equicontinuous family of functions. Let $\epsilon > 0$ be small and $0 < t_1 < t_2$. Then we have

$$\begin{aligned}
&\|F_2(z)(t_2) - F_2(z)(t_1)\| \\
&\leq \int_0^{t_1-\epsilon} \|T(t_2-s) - T(t_1-s)\| \\
&\qquad \times \left\| f\left(s, \overline{z}_s + y_s, \int_0^s w(s,\tau,\overline{z}_\tau + y_\tau)d\tau, \int_0^b h(s,\tau,\overline{z}_\tau + y_\tau)d\tau\right)\right\| ds \\
&\quad + \int_{t_1-\epsilon}^{t_1} \|T(t_2-s) - T(t_1-s)\| \\
&\qquad \times \left\| f\left(s, \overline{z}_s + y_s, \int_0^s w(s,\tau,\overline{z}_\tau + y_\tau)d\tau, \int_0^b h(s,\tau,\overline{z}_\tau + y_\tau)d\tau\right)\right\| ds \\
&\quad + \int_{t_1}^{t_2} \|T(t_2-s)\| \\
&\qquad \times \left\| f\left(s, \overline{z}_s + y_s, \int_0^s w(s,\tau,\overline{z}_\tau + y_\tau)d\tau, \int_0^b h(s,\tau,\overline{z}_\tau + y_\tau)d\tau\right)\right\| ds
\end{aligned}$$

$$\leq \int_0^{t_1-\epsilon} \|T(t_2-s)-T(t_1-s)\| \alpha_{(kK_b+N)\vartheta}(s)ds$$
$$+ \int_{t_1-\epsilon}^{t_1} \|T(t_2-s)-T(t_1-s)\| \alpha_{(kK_b+N)\vartheta}(s)ds$$
$$+ \int_{t_1}^{t_2} \|T(t_2-s)\| \alpha_{(kK_b+N)\vartheta}(s)ds.$$

Since $T(t),\ t>0$ is compact and hence continuous in the uniform operator topology, and $\alpha_{(kK_b+N)\vartheta} \in L^1$, we see that $\|F_2(z)(t_2)-F_2(z)(t_1)\| \to 0$ independent of $z \in B_k$ as $(t_2-t_1) \to 0$ with ϵ sufficiently small. This shows that F maps B_k into an equicontinuous family of functions.

Now, it remains to prove that $V(t)=\{F_2(z)(t) : z \in B_k\}$ is relatively compact in X. Let $0<t\leq b$ be fixed and ϵ be a real number satisfying $0<\epsilon<t$; for $z \in B_k$, we define

$$F_{2\epsilon}(z)(t) = \int_0^{t-\epsilon} T(t-s) f\left(s, \overline{z}_s + y_s, \int_0^s w(s,\tau,\overline{z}_\tau + y_\tau)d\tau, \int_0^b h(s,\tau,\overline{z}_\tau + y_\tau)d\tau\right) ds$$
$$= T(\epsilon)\int_0^{t-\epsilon} T(t-\epsilon-s) f\left(s, \overline{z}_s + y_s, \int_0^s w(s,\tau,\overline{z}_\tau + y_\tau)d\tau, \int_0^b h(s,\tau,\overline{z}_\tau + y_\tau)d\tau\right) ds.$$

Since

$$\left\| \int_0^{t-\epsilon} T(t-\epsilon-s) \times f\left(s, \overline{z}_s + y_s, \int_0^s w(s,\tau,\overline{z}_\tau + y_\tau)d\tau, \int_0^b h(s,\tau,\overline{z}_\tau + y_\tau)d\tau\right) ds \right\| \leq M \int_0^{t-\epsilon} \alpha_{(kK_b+N)\vartheta}(s)ds,$$

the compactness of $T(t)\ (t>0)$, implies that the set $V\epsilon(t)=\{F_{2\epsilon}(z)(t) : z \in B_k\}$ is relative compact in X for every $\epsilon,\quad 0<\epsilon<t$. Moreover, by making use of Lemma 2.1, hypotheses $(H_2),\ (H_3)$ and the condition (3.3), for every $z \in B_k$, we have

$$\|F_2(z)(t)-F_{2\epsilon}(z)(t)\|$$
$$= \int_{t-\epsilon}^{t} \left\| T(t-s) f\left(s, \overline{z}_s + y_s, \int_0^s w(s,\tau,\overline{z}_\tau + y_\tau)d\tau, \int_0^b h(s,\tau,\overline{z}_\tau + y_\tau)d\tau\right) \right\| ds$$
$$\leq \int_{t-\epsilon}^{t} M \alpha_{(kK_b+N)\vartheta}(s)ds.$$

Therefore, there are relative compact sets arbitrarily close to the set $V(t) = \{F_2(z)(t) : z \in B_k\}$; hence the set $V(t)$ is also relative compact in X. Thus, by the Arzela–Ascoli theorem, F_2 is a compact operator. These arguments enable us to conclude that $F = F_1 + F_2$ is condensing operator on B_k, and by the fixed point theorem of Sadovskii's, there exists a fixed point $z(\cdot)$ for F on B_k. If we define $x(t) = \overline{z}(t) + y(t), -\infty < t \leq b$, it is easy to see that $x(\cdot)$ is a mild solution of (1.1)–(1.2) satisfying $x_0 = \phi$. ∎

4. Application

As an application of Theorem 3.1, we shall consider the system (1.1)–(1.2) with control parameter such as:

$$\frac{d}{dt}[x(t) - g(t, x_t)] + Ax(t) = Cu(t) + f\left(t, x_t, \int_0^t w(t, s, x_s)ds, \int_0^b h(t, s, x_s)ds\right), \quad t \in J, \tag{4.1}$$

$$x_0 = \phi \in \mathcal{B}, \tag{4.2}$$

where the control function $u(\cdot)$ is given in $L^2(J, U)$ - the Banach space of admissible control functions with U as a Banach space and C is a bounded linear operator from U into X. The mild solution of the system (4.1)–(4.2) is given by

$$\begin{aligned} x(t) &= T(t)[\phi(0) - g(0, \phi)] + g(t, x_t) - \int_0^t AT(t-s)g(s, x_s)ds \\ &\quad + \int_0^t T(t-s) \\ &\quad \times \left[Cu(s) + f\left(s, x_s, \int_0^s w(s, \tau, x_\tau)d\tau, \int_0^b h(s, \tau, x_\tau)d\tau\right)\right]ds, \ t \in J, \end{aligned} \tag{4.3}$$

$$x_0 = \phi \in \mathcal{B}. \tag{4.4}$$

Definition 4.1. The system (4.1)–(4.2) is said to be controllable on the interval J if for every initial function $\phi \in \mathcal{B}$ and $x_1 \in X$, there exists a control $u \in L^2(J, U)$ such that the mild solution $x(\cdot)$ of (4.1)–(4.2) satisfies $x(b) = x_1$.

To establish the controllability result, we need the following additional hypothesis:

(H_6) The linear operator $W : L^2(J; U) \to X$ defined by

$$Wu = \int_0^b T(b-s)Cu(s)ds$$

has an induced inverse operator $\tilde{W}^{-1}$, which takes values in $L^2(J; U)/kerW$.

For the construction of the operator W and its inverse, see [23].

THEOREM 4.1. *Let the hypotheses* (H_1)–(H_6) *be satisfied. Then the system* (4.1)–(4.2) *is controllable on interval* J *if*

$$L_0 := GK_b\left(\|A^{-\beta}\| + \frac{C_{1-\beta}b^\beta}{t^\beta}\right) < 1, \tag{4.5}$$

$$\left(1 + bM\|C\|\|\tilde{W}^{-1}\|\right)\left[\left(\|A^{-\beta}\| + \frac{C_{1-\beta}b^\beta}{\beta}\right)L_g + M\vartheta\mu\right]K_b < 1, \tag{4.6}$$

where $\vartheta = \max_{t\in J}\{1, p(t), q(t)\}$ *and* $K_b = \sup\{K(t) : t \in [0, b]\}$.

Proof. By use of the hypothesis (H_6), for an arbitrary function $x(\cdot)$, we define the control

$$u(t) = \tilde{W}^{-1}\bigg[x_1 - T(b)[\phi(0) - g(0,\phi)] - g(b, x_b) + \int_0^b AT(b-s)g(s, x_s)ds$$
$$- \int_0^b T(b-s)\, f\left(s, x_s, \int_0^s w(s,\tau,x_\tau)d\tau, \int_0^b h(s,\tau,x_\tau)d\tau\right)ds\bigg](t).$$

Using this control, we show that the operator Ψ defined by

$$\Psi(x)(t) = \begin{cases} \phi(t), \quad t \in (-\infty, 0], \\ T(t)[\phi(0) - g(0,\phi)] + g(t, x_t) - \int_0^t AT(t-s)g(s, x_s)ds \\ + \int_0^t T(t-s) \\ \times \left[Cu(s) + f\left(s, x_s, \int_0^s w(s,\tau,x_\tau)d\tau, \int_0^b h(s,\tau,x_\tau)d\tau\right)\right]ds, \ t \in J, \end{cases}$$

has a fixed point $x(\cdot)$. Then this fixed point $x(\cdot)$ is a mild solution of the problem (4.1)–(4.2), and we can easily verify that $x(b) = \Psi(x)(b) = x_1$. This means that the control u steers the system from the initial function ϕ to x_1 in time b, which implies that the system is controllable. As in the proof of Theorem 3.1, the equivalent operator for Ψ is given by

$$\Omega(z)(t) = -T(t)g(0,\phi) + g(t, \overline{z}_t + y_t) - \int_0^t AT(t-s)g(s, \overline{z}_s + y_s)ds$$
$$+ \int_0^t T(t-s)\bigg[Cu(s)$$
$$+ f\left(s, \overline{z}_s + y_s, \int_0^s w(s,\tau,\overline{z}_\tau + y_\tau)d\tau, \int_0^b h(s,\tau,\overline{z}_\tau + y_\tau)d\tau\right)\bigg]ds.$$

Our aim is to prove that there exists a positive integer k such that $\Omega B_k \subseteq B_k$. If possible, for each positive integer k, there is a function $z_k \in B_k$ such that $\Omega(z_k) \notin B_k$, then $\|\Omega(z_k)(t)\| > k$ for some t depending on k. But by using the hypotheses (H_1)–(H_3), Lemma 2.1 and the inequalities (3.3) and (3.5), we obtain

$$
\begin{aligned}
(4.7)\qquad k &< \|\Omega(z_k)(t)\| \\
&\le M\|A^{-\beta}\|L_g(\|\phi\|_{\mathcal{B}}+1)+\|A^{-\beta}\|L_g(kK_b+N+1) \\
&\quad+\int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}}L_g(kK_b+N+1)ds+\int_0^t M\|C\|\|u_k(s)\|ds \\
&\quad+M\int_0^t \alpha_{(kK_b+N)\vartheta}(s)ds,
\end{aligned}
$$

where u_k is corresponding control of x_k, $x_k=\overline{z}_k+y$. Note that

$$
\begin{aligned}
(4.8)\qquad & \|u_k(s)\| \\
&\le \|\tilde{W}^{-1}\|\Big\{\|x_1\|+\|T(b)\|[\|\phi\|_{\mathcal{B}}+\|A^{-\beta}\|\|A^{\beta}g(0,\phi)\|] \\
&\quad+\|A^{-\beta}\|\|A^{\beta}g(b,(\overline{z}_k)_b+y_b)\| \\
&\quad+\int_0^b \|A^{1-\beta}T(b-\tau)\|\left\|A^{\beta}g(\tau,(\overline{z}_k)_\tau+y_\tau)\right\|d\tau \\
&\quad+\int_0^b \|T(b-\tau)\|\Big\|f\Big(\tau,(\overline{z}_k)_\tau+y_\tau, \\
&\qquad\qquad \int_0^\tau w(\tau,\sigma,(\overline{z}_k)_\sigma+y_\sigma)d\sigma,\int_0^b h(\tau,\sigma,(\overline{z}_k)_\sigma+y_\sigma)d\sigma\Big)\Big\|d\tau\Big\} \\
&\le \|\tilde{W}^{-1}\|\Big\{\|x_1\|+M[\|\phi\|_{\mathcal{B}}+\|A^{-\beta}\|L_g(\|\phi\|_{\mathcal{B}}+1)]+\|A^{-\beta}\|L_g(kK_b+N+1) \\
&\quad+\int_0^b \frac{C_{1-\beta}}{(b-\tau)^{1-\beta}}L_g(kK_b+N+1)d\tau+M\int_0^b \alpha_{(kK_b+N)\vartheta}(\tau)d\tau\Big\}.
\end{aligned}
$$

Using the estimate (4.8) in (4.7), we get

$$
\begin{aligned}
(4.9)\qquad k &< M\|A^{-\beta}\|L_g(\|\phi\|_{\mathcal{B}}+1)+\|A^{-\beta}\|L_g(kK_b+N+1) \\
&\quad+\frac{C_{1-\beta}b^{\beta}L_g\left(kK_b+N+1\right)}{\beta} \\
&\quad+bM\|C\|\|\tilde{W}^{-1}\|\Big\{\|x_1\|+M[\|\phi\|_{\mathcal{B}}+\|A^{-\beta}\|L_g(\|\phi\|_{\mathcal{B}}+1)] \\
&\quad+\|A^{-\beta}\|L_g(kK_b+N+1)\frac{C_{1-\beta}b^{\beta}L_g\left(K_b+N+1\right)}{\beta} \\
&\quad+M\int_0^b \alpha_{(kK_b+N)\vartheta}(s)ds\Big\}+M\int_0^b \alpha_{(kK_b+N)\vartheta}(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&= M\|A^{-\beta}\|L_g(\|\phi\|_{\mathcal{B}}+1) \\
&\quad + bM\|C\|\|\tilde{W}^{-1}\|\Big\{\|x_1\| + M[\|\phi\|_{\mathcal{B}} + \|A^{-\beta}\|L_g(\|\phi\|_{\mathcal{B}}+1)]\Big\} \\
&\quad + (1+bM\|C\|\|\tilde{W}^{-1}\|)\|A^{-\beta}\|L_g(kK_b+N+1) \\
&\quad + (1+bM\|C\|\|\tilde{W}^{-1}\|)\frac{C_{1-\beta}b^{\beta}L_g\,(kK_b+N+1)}{\beta} \\
&\quad + (1+bM\|C\|\|\tilde{W}^{-1}\|)M\int_0^b \alpha_{(kK_b+N)\vartheta}(s)ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
1 < \frac{M^*}{k} &+ (1+bM\|C\|\|\tilde{W}^{-1}\|)\left(\|A^{-\beta}\| + \frac{C_{1-\beta}b^{\beta}}{\beta}\right)L_g\left(K_b + \frac{N+1}{k}\right) \\
&+ (1+bM\|C\|\|\tilde{W}^{-1}\|)M\left(K_b+\frac{N}{k}\right)\vartheta\frac{1}{(kK_b+N)\vartheta}\int_0^b \alpha_{(kK_b+N)\vartheta}(s)ds,
\end{aligned}
$$

where

$$
\begin{aligned}
M^* &= M\|A^{-\beta}\|L_g(\|\phi\|_{\mathcal{B}}+1) \\
&\quad + bM\|C\|\|\tilde{W}^{-1}\|\Big\{\|x_1\| + M[\|\phi\|_{\mathcal{B}} + \|A^{-\beta}\|L_g(\|\phi\|_{\mathcal{B}}+1)]\Big\}
\end{aligned}
$$

is independent of k. Taking lower limit and using (H_3), we get

$$
\Big(1+bM\|C\|\|\tilde{W}^{-1}\|\Big)\left[\left(\|A^{-\beta}\| + \frac{C_{1-\beta}b^{\beta}}{\beta}\right)L_g + M\vartheta\mu\right]K_b \geq 1.
$$

This contradicts (4.6). Hence, for some positive integer k, we must have $\Omega B_k \subseteq B_k$.

In order to apply Sadoviskii's fixed point theorem, we decompose Ω as $\Omega = F_1 + \Omega_1$, where the operators F_1, Ω_1 are defined on B_k, respectively by

$$
\begin{aligned}
F_1(z)(t) &= -T(t)g(0,\phi) + g(t,\overline{z}_t + y_t) - \int_0^t AT(t-s)g(s,\overline{z}_s+y_s)ds, \\
\Omega_1(z)(t) &= \int_0^t T(t-s)\Big[Cu(s) \\
&\quad + f\left(s,\overline{z}_s+y_s,\int_0^s w(s,\tau,\overline{z}_\tau+y_\tau)d\tau,\int_0^b h(s,\tau,\overline{z}_\tau+y_\tau)d\tau\right)\Big]ds
\end{aligned}
$$

$$= \int_0^t T(t-s)C\tilde{W}^{-1}\Big[x_1 - T(b)[\phi(0) - g(0,\phi)] - g(b, \overline{z}_b + y_b)$$
$$+ \int_0^b AT(b-\tau)g(\tau, \overline{z}_\tau + y_\tau)d\tau$$
$$- \int_0^b T(b-\tau)f\Big(\tau, \overline{z}_\tau + y_\tau, \int_0^\tau w(\tau,\sigma,\overline{z}_\sigma + y_\sigma)d\sigma,$$
$$\int_0^b h(\tau,\sigma,\overline{z}_\sigma + y_\sigma)d\sigma\Big)d\tau\Big](s)ds$$
$$+ \int_0^t T(t-s)f\Big(s, \overline{z}_s + y_s, \int_0^s w(s,\tau,\overline{z}_\tau + y_\tau)d\tau, \int_0^b h(s,\tau,\overline{z}_\tau + y_\tau)d\tau\Big)ds,$$

for $z \in B_k$ and $t \in J$. We have already proved that F_1 verifies a contraction condition. The proof that Ω_1 is a compact operator can be completed by closely looking at the proof of compactness of the operator F_2. ∎

5. Example

Consider the mixed partial differential equation system

$$\frac{\partial}{\partial t}\Big[z(t,w) + \int_{-\infty}^t \int_0^\pi b(s-t,y,w)z(s,y)\,dy\,ds\Big] \tag{5.1}$$
$$= \frac{\partial^2}{\partial w^2}z(t,w) + a(w)z(t,w) + \int_0^t \int_{-\infty}^s c(\sigma - s,y,w)z(\sigma,w)\,d\sigma\,ds$$
$$+ \int_0^b \int_{-\infty}^s d(\sigma - s,y,w)z(\sigma,w)\,d\sigma\,ds,\ 0 \le t \le b,\ 0 \le w \le \pi,$$

$$z(t,0) = z(t,\pi) = 0, \tag{5.2}$$

$$z(\theta,w) = \phi(\theta,w), \quad \theta \le 0, \quad 0 \le w \le \pi. \tag{5.3}$$

To write the system (5.1)–(5.3) in the form (1.1)–(1.2), we shall take $X = L^2[0,\pi]$, and define $x(t) = z(t,\cdot)$ The operator A is defined by $Av = -v''$ with the domain

$$D(A) = \{v \in X : v,\ v' \text{ absolutely continuous},\ v'' \in X,\ v(0) = v(\pi) = 0\}.$$

It is well known that $-A$ generates a strongly continuous semigroup $\{T(t)\}_{t\ge 0}$ which is analytic, compact and self-adjoint. Furthermore, $-A$ has a discrete spectrum, the eigenvalues are $-n^2$, $n \in N$, with the corresponding normalized eigenvectors $z_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$. Then the following properties hold:

(a) If $v \in D(A)$ then $Av = \sum_{n=1}^\infty n^2\langle v, z_n\rangle z_n$.

(b) For every $v \in X$,

$$T(t)v = \sum_{n=1}^{\infty} e^{-n^2 t}\langle v, z_n\rangle z_n,$$

$$A^{-1/2}v = \sum_{n=1}^{\infty} \frac{1}{n}\langle v, z_n\rangle z_n.$$

In particular, $\|T(t)\| \le e^{-t}$ and $\|A^{-1/2}\| = 1$.

(c) The operator $A^{1/2}$ is given by $A^{1/2}v = \sum_{n=1}^{\infty} n\langle v, z_n\rangle z_n$, on the space $D(A^{1/2}) = \{v \in X, \ \sum_{n=1}^{\infty} n\langle v, z_n\rangle z_n \in X\}$.

Consider the phase space $\mathcal{B} = C_r \times L^p(g; X)$, $r \ge 0$, $1 \le p \le \infty$, with $r = 0$, $p = 2$ and $X = L^2[0,\pi]$, which has been discussed in [17]. Assume that the following conditions hold:

(i) The function $b(\cdot)$ is measurable, and

$$\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi}\Big(b^2(\theta,y,w)/g(\theta)\Big)dy\,d\theta\,dw < \infty.$$

(ii) The function $\left(\frac{\partial}{\partial w}\right)b(\theta,y,w)$ is measurable, $b(\theta,y,0) = b(\theta,y,\pi) = 0$, and

$$\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi}\frac{1}{g(\theta)}\Big(\frac{\partial}{\partial w}b(\theta,y,w)\Big)^2 dy\,d\theta\,dw < \infty.$$

(iii) The function $c(\cdot)$ is measurable with $\int_{-\infty}^{0}(c^2(\theta)/g(\theta))d\theta < \infty$.

(iv) The function $d(\cdot)$ is measurable with $\int_{-\infty}^{0}(d^2(\theta)/g(\theta))d\theta < \infty$.

(v) The function ϕ defined by $\phi(\theta)(w) = \phi(\theta, w)$ belongs to $\mathcal{B}$.
Define the functions $g : J \times \mathcal{B} \to X$, $f : J \times \mathcal{B} \times X \times X \to X$, $w, h : \Delta \times \mathcal{B} \to X$ by

$$g(t,\psi)(\xi) = \int_{-\infty}^{0}\int_0^{\pi} b(\theta,y,w)\psi(\theta,w)\,dw\,d\theta,$$

$$w(t,s,\psi)(\xi) = \int_{-\infty}^{0} c(\theta,y,w)\psi(\theta,w)d\theta,$$

$$h(t,s,\psi)(\xi) = \int_{-\infty}^{0} d(\theta,y,w)\psi(\theta,w)d\theta,$$

$$f\left(t,\psi,\int_0^t k(t,s,\psi)ds,\int_0^b h(t,s,\psi)ds\right)(\xi)$$
$$= a(\xi)\psi(0,\xi) + \int_0^t k(t,s,\psi)(\xi)ds + \int_0^b h(t,s,\psi)(\xi)ds.$$

Under the conditions (i) to (v), the above defined functions verifie the hypothesis of the Theorem 3.1 (see [11, 20] for details). Additionally, if (3.1) and (3.2) hold, the system (5.1)–(5.3) has mild solution on $[-r, \infty)$.

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