DE GRUYTER OPEN

DEMONSTRATIO MATHEMATICA
Vol. XLVII No 4 2014

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THE EXISTENCE OF A UNIQUE SOLUTION OF THE HYPERBOLIC FUNCTIONAL DIFFERENTIAL EQUATION

Abstract. We consider the Z. Szmydt problem for the hyperbolic functional differential equation. We prove a theorem on existence of a unique classical solution and the Carathéodory solution of the hyperbolic equation.

1. Introduction

Existence of classical solutions for the Z. Szmydt problem for hyperbolic differential equations has been studied by Szmydt in [8, 9] and Lasota in [7]. In this paper, we want to investigate the Z. Szmydt problem for the functional differential equation. We deal with the classical and the Caratheodory solutions for the considered problem. Theorems of the existence of Carathéodory solutions for hyperbolic equations can be found in [2]–[4]. The papers [5, 6] discusses the problem of Z. Szmydt and its relation with classical problems. The Z. Szmydt problem contains the problem of Darboux, Cauchy, Picard and Goursat as special cases.

Put $I = [0, a] \times [0, b]$, where a, $b > 0$. Let $(x, y) \in I$ and

$$A[x, y] = \{(s, t) \in \mathbb{R}^2\colon\ (x + s, y + t) \in I\}.$$

For $u : I \to \mathbb{R}$, we define $u_{[x,y]} : A[x, y] \to \mathbb{R}$ by the formula

$$u_{[x,y]}(s, t) = u(x + s, y + t) \ \text{ for } (s, t) \in A[x, y].$$

Let $E = [-a, a] \times [-b, b]$. Then $A[x, y] \subset E$ for $(x, y) \in I$. We denote by $C^1(E, \mathbb{R})$ the space of all continuous functions $u : E \to \mathbb{R}$ such that $\partial u/\partial x$ and $\partial u/\partial y$ exist and are continuous. The norm $||\cdot||_*$ is defined by

$$||u||_* = \sup_{(x,y)\in E} |u(x, y)| + \sup_{(x,y)\in E} \left|\frac{\partial u}{\partial x}(x, y)\right| + \sup_{(x,y)\in E} \left|\frac{\partial u}{\partial y}(x, y)\right|.$$

2010 *Mathematics Subject Classification*: 35D05, 35L70, 35R10.
Key words and phrases: existence theorem, functional differential equation, hyperbolic equations.

DOI: 10.2478/dema-2014-0070

Put $\Xi = I \times C^1(E, \mathbb{R})$. Let $f : \Xi \to \mathbb{R}$, $g : [0,a] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $h : [0,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\psi : [0,a] \to [0,b]$, $\phi : [0,b] \to [0,a]$, $u_0 \in \mathbb{R}$ and $(x_0, y_0) \in I$ be given. We will consider the Z. Szmydt problem

$$(1)\qquad \begin{cases} \frac{\partial^2 u}{\partial x \partial y}(x,y) = f\left(x, y, u_{[x,y]}\right) & \text{in } I, \\ \frac{\partial u}{\partial x}(x, \psi(x)) = g\left(x, u(x, \psi(x)), \frac{\partial u}{\partial y}(x, \psi(x))\right) & \text{in } [0,a], \\ \frac{\partial u}{\partial y}(\phi(y), y) = h\left(y, u(\phi(y), y), \frac{\partial u}{\partial x}(\phi(y), y)\right) & \text{in } [0,b], \\ u(x_0, y_0) = u_0. \end{cases}$$

Remark 1. The third variable of f belongs to the functional space $C^1(E, \mathbb{R})$, so the right-hand side of the differential equation depends on first order derivatives of u even thought it is not explicitly seen.

We will need the following assumption.

Assumption 1. *If $(x,y) \in I$ and w, $\overline{w} \in C^1(E, \mathbb{R})$ are such that $w(\tilde{x}, \tilde{y}) = \overline{w}(\tilde{x}, \tilde{y})$ for $(\tilde{x}, \tilde{y}) \in A[x,y]$ then $f(x,y,w) = f(x,y,\overline{w})$.*

The Assumption 1 means that the values of f at the point $(x,y,w) \in \Xi$ depend on (x,y) and on the restriction of w to the set $A[x,y]$, only. We give examples of equations which can be obtained from (1).

Example 1. Suppose that $F : I \times \mathbb{R} \to \mathbb{R}$ is a given function. Set

$$f(x,y,w) = F(x, y, w(\alpha_1(x,y) - x, \alpha_2(x,y) - y)),$$

where $\alpha_1 : I \to \mathbb{R}$, $\alpha_2 : I \to \mathbb{R}$ are given and $(\alpha_1(x,y) - x, \alpha_2(x,y) - y) \in A[x,y]$ for $(x,y) \in I$. Consequently, as a special case of (1) we get the following equation with a deviated argument

$$(2)\qquad \frac{\partial^2 u}{\partial x \partial y}(x,y) = F(x, y, u(\alpha_1(x,y), \alpha_2(x,y))) \quad \text{in } I.$$

If we want to get an integro-differential equation then we define

$$f(x,y,w) = F\left(x, y, \int\int_{A[x,y]} w(s,t)\, ds dt\right)$$

and consequently we have

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) = F\left(x, y, \int\int_{A[x,y]} u_{[x,y]}(s,t)\, ds dt\right) \quad \text{in } I.$$

Remark 2. Equation (1) contains as special cases equations, in which the right-hand side depends on the first order derivatives of an unknown function even though in f there is no explicit dependence on $\partial u/\partial x$ and $\partial u/\partial y$. This is due to the fact that the last variable of f belong to the functional space $C^1(E, \mathbb{R})$.

2. The classical solutions

A function $u : I \to \mathbb{R}$ is called a classical solution of (1) if $u \in C^1(I,\mathbb{R})$ has the derivative $\partial^2 u/\partial x \partial y$ and it satisfies (1) for all $(x,y) \in I$.

Theorem 1. *We assume that the functions $f : \Xi \to \mathbb{R}$, $g : [0,a] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $h : [0,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\psi : [0,a] \to [0,b]$, $\phi : [0,b] \to [0,a]$ are continuous and $u_0 \in \mathbb{R}$ is a constant. Moreover, Assumption 1 is satisfied and there exist constants L, K, $M \geq 0$ such that*

$$|f(x,y,\tilde{\omega}) - f(x,y,\omega)| \leq L\|\tilde{\omega} - \omega\|_*, \tag{3}$$

$$|g(x,\tilde{\rho},\tilde{\nu}) - g(x,\rho,\nu)| \leq K|\tilde{\rho} - \rho| + M|\tilde{\nu} - \nu|, \tag{4}$$

$$|h(y,\tilde{\rho},\tilde{\mu}) - g(y,\rho,\mu)| \leq K|\tilde{\rho} - \rho| + M|\tilde{\mu} - \mu|, \tag{5}$$

for $(x,y) \in I$, ω, $\tilde{\omega} \in C(E,\mathbb{R})$ and ρ, $\tilde{\rho}$, ν, $\tilde{\nu}$, μ, $\tilde{\mu} \in \mathbb{R}$. Moreover,

$$2K + Ka + Kb + La + Lb + 2Lab < 1, \tag{6}$$

$$M(1+b) + L(a+b+2ab) < 1, \tag{7}$$

$$M(1+a) + L(a+b+2ab) < 1. \tag{8}$$

Then there exists exactly one solution of problem (1).

Proof. Now, we define the operators $S : C^1(I,\mathbb{R}) \to C^1(I,\mathbb{R})$ by

$$\begin{aligned}(Su)(x,y) = u_0 &+ \int_{x_0}^{x} g\left(s, u(s,\psi(s)), \frac{\partial u}{\partial y}(s,\psi(s))\right) ds \\ &+ \int_{y_0}^{y} h\left(t, u(\phi(t),t), \frac{\partial u}{\partial x}(\phi(t),t)\right) dt \\ &+ \int_{y_0}^{y} \int_{\phi(t)}^{x_0} f\left(s,t,u_{[s,t]}\right) ds dt \\ &+ \int_{x_0}^{x} \int_{\psi(s)}^{y} f\left(s,t,u_{[s,t]}\right) dt ds \quad \text{in } I.\end{aligned}$$

It is easy to see that under our assumptions, the operator S is well defined and $Su \in C^1(I,\mathbb{R})$ for $u \in C^1(I,\mathbb{R})$. Moreover, we have

$$\frac{\partial}{\partial x}(Su)(x,y) = g\left(x, u(x,\psi(x)), \frac{\partial u}{\partial y}(x,\psi(x))\right) + \int_{\psi(x)}^{y} f\left(x,t,u_{[x,t]}\right) dt \quad \text{in } I,$$

$$\begin{aligned}\frac{\partial}{\partial y}(Su)(x,y) &= h\left(y, u(\phi(y),y), \frac{\partial u}{\partial x}(\phi(y),y)\right) \\ &\quad + \int_{\phi(y)}^{x_0} f\left(s,y,u_{1[s,y]}\right) ds + \int_{x_0}^{x} f\left(s,y,u_{1[s,y]}\right) ds \\ &= h\left(y, u(\phi(y),y), \frac{\partial u}{\partial x}(\phi(y),y)\right) + \int_{\phi(y)}^{x} f\left(s,y,u_{1[s,y]}\right) ds \quad \text{in } I.\end{aligned}$$

By the above, it is easily seen that equation $u(x,y) = (Su)(x,y)$ is equivalent to problem (1).

Now, we prove that S is a contraction. Let u, $\tilde{u} \in C^1(I,\mathbb{R})$. Then for any $(x,y) \in I$, we have

$$
\begin{aligned}
& |(S\tilde{u})(x,y) - (Su)(x,y)| \\
& \leq \int_0^a \left| g\left(s, \tilde{u}(s,\psi(s)), \frac{\partial \tilde{u}}{\partial y}(s,\psi(s))\right) - g\left(s, u(s,\psi(s)), \frac{\partial u}{\partial y}(s,\psi(s))\right)\right| ds \\
& + \int_0^b \left| h\left(, t, \tilde{u}(\phi(t),t), \frac{\partial \tilde{u}}{\partial x}(\phi(t),t)\right) - h\left(, t, u(\phi(t),t), \frac{\partial u}{\partial x}(\phi(t),t)\right)\right| dt \\
& + \int_0^b \int_0^a \left| f\left(s,t,\tilde{u}_{[s,t]}\right) - f\left(s,t,u_{[s,t]}\right)\right| dsdt \\
& + \int_0^a \int_0^b \left| f\left(s,t,\tilde{u}_{[s,t]}\right) - f\left(s,t,u_{[s,t]}\right)\right| dtds \\
& \leq \int_0^a \left(K|\tilde{u}(s,\psi(s)) - u(s,\psi(s))| + M\left|\frac{\partial \tilde{u}}{\partial y}(s,\psi(s)) - \frac{\partial u}{\partial y}(s,\psi(s))\right|\right) ds \\
& + \int_0^b \left(K|\tilde{u}(\phi(t),t) - u(\phi(t),t)| + M\left|\frac{\partial \tilde{u}}{\partial x}(\psi(t),t) - \frac{\partial u}{\partial x}(\phi(t),t)\right|\right) dt \\
& + 2\int_0^a \int_0^b L\left|\left|\tilde{u}_{[s,t]} - u_{[s,t]}\right|\right|_* dtds \\
& \leq Ka||\tilde{u} - u|| + Ma\left|\left|\frac{\partial \tilde{u}}{\partial y} - \frac{\partial u}{\partial y}\right|\right| + Kb||\tilde{u} - u|| + Mb\left|\left|\frac{\partial \tilde{u}}{\partial x} - \frac{\partial u}{\partial x}\right|\right| \\
& + 2abL\left[||\tilde{u} - u|| + \left|\left|\frac{\partial \tilde{u}}{\partial x} - \frac{\partial u}{\partial x}\right|\right| + \left|\left|\frac{\partial \tilde{u}}{\partial y} - \frac{\partial u}{\partial y}\right|\right|\right] \\
& = (Ka + Kb + 2Lab)||\tilde{u} - u|| + (Nb + 2Lab)\left|\left|\frac{\partial \tilde{u}}{\partial x} - \frac{\partial u}{\partial x}\right|\right| \\
& + (Ma + 2Lab)\left|\left|\frac{\partial \tilde{u}}{\partial y} - \frac{\partial u}{\partial y}\right|\right|.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
& \left|\frac{\partial}{\partial x}(S\tilde{u})(x,y) - \frac{\partial}{\partial x}(Su)(x,y)\right| \\
& \leq \left| g\left(x, \tilde{u}(x,\psi(x)), \frac{\partial \tilde{u}}{\partial y}(x,\psi(x))\right) - g\left(x, u(s,\psi(x)), \frac{\partial u}{\partial y}(x,\psi(x))\right)\right| \\
& \quad + \int_0^b \left| f\left(x,t,\tilde{u}_{[x,t]}\right) - f\left(x,t,u_{[x,t]}\right)\right| dt \\
& \leq (K + Lb)||\tilde{u} - u|| + Lb\left|\left|\frac{\partial \tilde{u}}{\partial x} - \frac{\partial u}{\partial x}\right|\right| + (M + Lb)\left|\left|\frac{\partial \tilde{u}}{\partial y} - \frac{\partial u}{\partial y}\right|\right|
\end{aligned}
$$

and

$$\left|\frac{\partial}{\partial y}(S\tilde{u})(x,y)-\frac{\partial}{\partial y}(Su)(x,y)\right|$$
$$\leq (K+La)\,||\tilde{u}-u||+(M+La)\left|\left|\frac{\partial\tilde{u}}{\partial x}-\frac{\partial u}{\partial x}\right|\right|+La\left|\left|\frac{\partial\tilde{u}}{\partial y}-\frac{\partial u}{\partial y}\right|\right|.$$

By the above,

$$\begin{aligned}||S\tilde{u}-Su||_{*}\leq{}&(K+K+Ka+Kb+La+Lb+2Lab)\,||\tilde{u}-u||\\&+[M(1+b)+L(a+b+2ab)]\left|\left|\frac{\partial\tilde{u}}{\partial x}-\frac{\partial u}{\partial x}\right|\right|\\&+[M(1+a)+L(a+b+2ab)]\left|\left|\frac{\partial\tilde{u}}{\partial y}-\frac{\partial u}{\partial y}\right|\right|.\end{aligned}$$

Let

$$\begin{aligned}\alpha=\max\{&2K+Ka+Kb+La+Lb+2Lab,\\&M(1+b)+L(a+b+2ab)\,,\,M(1+a)+L(a+b+2ab)\}.\end{aligned}$$

Then

$$||S\tilde{u}^{*}-Su^{*}||_{*}<\alpha||\tilde{u}-u||_{*}.$$

We see from (6)–(8) that $\alpha<1$. Using the Banach fixed point theorem, we get the existence of exactly one solution of problem (1). ■

Now, we give an example with demonstrates that in Theorem 1 assumptions (6)–(8) are necessary.

Example 2. Let a_0, b_0, x_0, $y_0 \in [0,a]\times[0,b]$ and

(9) $$u_{xy}(x,y)=0 \text{ in } [0,a]\times[0,b],$$

(10) $$u_x(x,b_0)=ku(x,b_0) \text{ in } [0,a],$$

(11) $$u_y(a_0,y)=lu(a_0,y) \text{ in } [0,b],$$

(12) $$u(x_0,y_0)=u_0, \quad \text{where } u_0 \text{ is a constant.}$$

It is an easy matter to prove that the solution of the problem (9)–(11) is

$$u(x,y)=C(e^{k(x-a_0)}+e^{l(y-b_0)}-1), \text{ where } C=\text{const}.$$

From (12), we see

$$C(e^{k(x_0-a_0)}+e^{l(y_0-b_0)}-1)=u_0.$$

Let $x_0=y_0=1-\ln 2>0$, $a_0=b_0=1$ and $k=l=1$. Then

$$C(e^{k(x_0-a_0)}+e^{l(y_0-b_0)}-1)=C(e^{-\ln 2}+e^{-\ln 2}-1)=C\left(\frac{1}{2}+\frac{1}{2}-1\right)=0.$$

Therefore, for $u_0\neq 0$ the solution of our problem doesn't exist. For $u_0=0$, there exist infinitely many solutions of this problem.

3. The Carathéodory solutions

Now, we discuss Carathéodory solutions of hyperbolic functional differential equations. In order to define this solutions we need an appropriate definition of an absolutely continuous function of two variables. For this purpose, we first introduce suitable notation. Given a rectangle $J = [a_1, a_2] \times [b_1, b_2]$ contained in I and $u : I \to \mathbb{R}$, let

$$\Delta_J (u) = u (a_1, b_1) - u (a_2, b_1) - u (a_1, b_2) + u (a_2, b_2) .$$

A rectangle is called subrectangle of I if its sides are parallel to the coordinate axes. Let m denote Lebesgue measure on $\mathbb{R}^2$. We say that $u : I \to \mathbb{R}$ is absolutely continuous if the following two conditions are satisfied:

a) Given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{J \in \mathbb{J}} |\Delta_J (u)| < \epsilon,$$

whenever $\mathbb{J}$ is a finite collection of pairwise non-overlapping subrectangles of I with

$$\sum_{J \in \mathbb{J}} m (J) < \delta.$$

b) The marginal functions $u (\cdot, b)$ and $u (a, \cdot)$ are absolutely continuous functions of a single variable on $[0, a]$ and $[0, b]$, respectively.

Let $AC (I, \mathbb{R})$ denote the set of absolutely continuous functions on I. In [1], we can find that the following statements are equivalent:

a) $u \in AC (I, \mathbb{R})$,
b) there exist $g \in AC ([0, a], \mathbb{R})$, $h \in AC ([0, b], \mathbb{R})$ and $L \in L^1 (I, \mathbb{R})$ such that

$$u (x, y) = g (x) + h (y) + \int_0^x \int_0^y L (s, t) \, dsdt.$$

We denote by $C_x (I, \mathbb{R})$, the space of functions ν of the variables (x, y) defined on I, continuous in $x \in [0, a]$ for almost all $y \in [0, b]$ and measurable in $y \in [0, b]$ for all $x \in [0, a]$ and such that

$$||\nu||_x = \int_0^b \max_{x \in [0,a]} |\nu (x, y)| dy < \infty.$$

$C_y (I, \mathbb{R})$ is the space of functions μ of the variables (x, y) defined on I, continuous in $y \in [0, b]$ for almost all $x \in [0, a]$ and measurable in $x \in [0, a]$ for all $y \in [0, b]$ and such that

$$||\mu||_y = \int_0^a \max_{y \in [0,b]} |\mu (x, y)| dx < \infty.$$

Note that if $u \in AC(I,\mathbb{R})$ then $\partial u/\partial x$, $\partial u/\partial y$ and $\partial^2 u/\partial x \partial y$ exist almost everywhere on I. Furthermore, $\partial u/\partial x \in C_y(I,\mathbb{R})$ and $\partial u/\partial y \in C_x(I,\mathbb{R})$. As in [3], we can verify that $||\cdot||_x$, $||\cdot||_y$ are norms and $(C_x(I,\mathbb{R}), ||\cdot||_x)$, $(C_y(I,\mathbb{R}), ||\cdot||_y)$ are Banach spaces. We consider the following problem:

$$(13) \qquad \begin{cases} \frac{\partial^2 u}{\partial x \partial y}(x,y) = f\left(x, y, u_{[x,y]}, \frac{\partial u}{\partial x}(x,y), \frac{\partial u}{\partial y}(x,y)\right) & \text{a.e. in } I, \\ \frac{\partial u}{\partial x}(x, \psi(x)) = g(x, u(x, \psi(x))) & \text{a.e. in } [0,a], \\ \frac{\partial u}{\partial y}(\phi(y), y) = h(y, u(\phi(y), y)) & \text{a.e. in } [0,b], \\ u(x_0, y_0) = u_0. \end{cases}$$

Remark 3. If $\frac{\partial u}{\partial x}(\cdot, y) : [0,a] \to \mathbb{R}$ is measurable for fixed $y \in [0,b]$ and $\phi : [0,b] \to [0,a]$ is continuous then $\frac{\partial u}{\partial x}(\phi(\cdot), y) : [0,b] \to \mathbb{R}$ may not be measurable. Therefore, in the case of Carathéodory solutions, we will consider the problem (13) instead of (1).

Assumption 2. *Assume that functions $f : I \times C(E,\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : [0,a] \times \mathbb{R} \to \mathbb{R}$ and $h : [0,b] \times \mathbb{R} \to \mathbb{R}$ of the variables (x,y,ω,μ,ν), (x,ρ) and (y,ρ) are such that*

1° *$f(\cdot,\cdot,\omega,\mu,\nu) : I \to \mathbb{R}$, $g(\cdot,\rho) : [0,a] \times \mathbb{R} \to \mathbb{R}$ and $h(\cdot,\rho) : [0,b] \times \mathbb{R} \to \mathbb{R}$ are measurable for all $\omega \in C(E,\mathbb{R})$, $\mu \in \mathbb{R}$, $\nu \in \mathbb{R}$, $\rho \in \mathbb{R}$.*
2° *There are $l_i \in L^1(I,\mathbb{R}_+)$ for $i = 1, 2, 3$, $l_4 \in L^1([0,a],\mathbb{R}_+)$, $l_5 \in L^1([0,b],\mathbb{R}_+)$ and constants c_2, $c_3 \geq 0$ such that*

$$(14) \qquad \begin{aligned} |f(x,y,\tilde{\omega},\tilde{\mu},\tilde{\nu}) - f(x,y,\omega,\mu,\nu)| &\leq l_1(x,y)\,||\tilde{\omega} - \omega|| \\ &+ g\left(c_2 + \int_0^x l_2(s,y)\,dsg\right)|\tilde{\mu} - \mu| + g\left(c_3 + \int_0^y l_3(x,t)\,dt\right)|\tilde{\nu} - \nu|, \end{aligned}$$

$$(15) \qquad |g(x,\tilde{\rho}) - g(x,\rho)| \leq l_4(x)\,||\tilde{\rho} - \rho||,$$

$$(16) \qquad |h(y,\tilde{\rho}) - h(x,\rho)| \leq l_5(y)\,||\tilde{\rho} - \rho||,$$

for a.e. $(x,y) \in I$ and all $(\omega,\mu,\nu), (\tilde{\omega},\tilde{\mu},\tilde{\nu}) \in C\left(I,\mathbb{R}^k\right) \times \mathbb{R}^k \times \mathbb{R}^k$.

Theorem 2. *Suppose that Assumption 2 is satisfied. Moreover,*

$$(17) \qquad \int_0^a l_4(s)\,ds + \int_0^b l_5(t)\,dt + 2\int_0^a \int_0^b l_1(s,t)\,dtds < 1,$$

$$(18) \qquad 2g\left(c_2 b + \int_0^b \int_0^a l_2(s,y)\,dsdt\right) < 1,$$

$$(19) \qquad 2g\left(c_3 a + \int_0^a \int_0^b l_3(x,t)\,dtdxg\right) < 1.$$

Then problem (13) has a unique solution.

Proof. We define the operator $S : AC(I,\mathbb{R}) \to AC(I,\mathbb{R})$ by

$$\begin{aligned}(Su)(x,y) = u_0 &+ \int_{x_0}^{x} g(s,u(s,\psi(s)))\,ds + \int_{y_0}^{y} h(,t,u(\phi(t),t))\,dt \\ &+ \int_{y_0}^{y}\int_{\phi(t)}^{x_0} f\left(s,t,u_{[s,t]},\frac{\partial u}{\partial x}(s,t),\frac{\partial u}{\partial y}(s,t)\right) dsdt \\ &+ \int_{x_0}^{x}\int_{\psi(s)}^{y} f\left(s,t,u_{[s,t]},\frac{\partial u}{\partial x}(s,t),\frac{\partial u}{\partial y}(s,t)\right) dtds \quad \text{on } I.\end{aligned}$$

It is easy to see that under our assumptions, the operator S is well defined and $Su \in AC(I,\mathbb{R})$ for $u \in AC(I,\mathbb{R})$. Moreover, we have

$$\begin{aligned}\frac{\partial}{\partial x}(Su)(x,y) &= g(x,u(x,\psi(x))) \\ &\quad + \int_{\psi(x)}^{y} f\left(x,t,u_{[x,t]},\frac{\partial u}{\partial x}(x,t),\frac{\partial u}{\partial y}(x,t)\right) dt \quad \text{a.e. on } I, \\ \frac{\partial}{\partial y}(Su)(x,y) &= h(y,u(\phi(y),y)) \\ &\quad + \int_{\phi(y)}^{x} f\left(s,y,u_{1[s,y]},\frac{\partial u}{\partial x}(s,y),\frac{\partial u}{\partial y}(s,y)\right) ds \quad \text{a.e. on } I.\end{aligned}$$

By the above, it is easily seen that equation $u(x,y) = (Su)(x,y)$ is equivalent to problem (13). Now, we define the norm $||\cdot||_*$ by

$$||u||_* = ||u|| + \left|\left|\frac{\partial u}{\partial x}\right|\right|_y + \left|\left|\frac{\partial u}{\partial y}\right|\right|_x .$$

We prove that S is contraction. Let $u, \tilde{u} \in AC(I,\mathbb{R})$. Then

$$\begin{aligned}|(S\tilde{u})(x,y) - (Su)(x,y)| \le &\int_0^a |g(s,\tilde{u}(s,\psi(s))) - g(s,u(s,\psi(s)))|\,ds \\ &+ \int_0^b |h(t,\tilde{u}(\phi(t),t)) - h(t,u(\phi(t),t))|dt \\ &+ \int_0^b\int_0^a \left| f\left(s,t,\tilde{u}_{[s,t]},\frac{\partial \tilde{u}}{\partial x}(s,t),\frac{\partial \tilde{u}}{\partial y}(s,t)\right)\right. \\ &\left. - f\left(s,t,u_{[s,t]},\frac{\partial u}{\partial x}(s,t),\frac{\partial u}{\partial y}(s,t)\right)\right| dsdt \\ &+ \int_0^a\int_0^b \left| f\left(s,t,\tilde{u}_{[s,t]},\frac{\partial \tilde{u}}{\partial x}(s,t),\frac{\partial \tilde{u}}{\partial y}(s,t)\right)\right. \\ &\left. - f\left(s,t,u_{[s,t]},\frac{\partial u}{\partial x}(s,t),\frac{\partial u}{\partial y}(s,t)\right)\right| dtds\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^a \left(l_4\left(s\right)\left|\tilde{u}\left(s,\psi\left(s\right)\right)-u\left(s,\psi\left(s\right)\right)\right|\right)ds \\
&+\int_0^b \left(l_5\left(t\right)\left|\tilde{u}\left(\phi\left(t\right),t\right)-u\left(\phi\left(t\right),t\right)\right|\right)dt \\
&+2\int_0^a\int_0^b \Bigg[l_1\left(s,t\right)\left|\left|\tilde{u}_{[s,t]}-u_{[s,t]}\right|\right| \\
&+g\left(c_2+\int_0^x l_2\left(s,y\right)dsg\right)\left|\frac{\partial\tilde{u}}{\partial x}\left(s,t\right)-\frac{\partial u}{\partial x}\left(s,t\right)\right| \\
&+g\left(c_3+\int_0^y l_3\left(x,t\right)dt\right)\left|\frac{\partial\tilde{u}}{\partial y}\left(s,t\right)-\frac{\partial u}{\partial y}\left(s,t\right)\right|\Bigg]dtds \\
&\leq \int_0^a l_4\left(s\right)ds||\tilde{u}-u||+\int_0^b l_5\left(t\right)dt||\tilde{u}-u|| \\
&+2\int_0^a\int_0^b l_1\left(s,t\right)dtds||\tilde{u}-u|| \\
&+2\int_0^b\left[g\left(c_2+\int_0^a l_2\left(s,y\right)ds\right)\int_0^a\left|\frac{\partial\tilde{u}}{\partial x}\left(s,t\right)-\frac{\partial u}{\partial x}\left(s,t\right)\right|ds\right]dt \\
&+2\int_0^a\left[g\left(c_3+\int_0^b l_3\left(x,t\right)dt\right)\int_0^b\left|\frac{\partial\tilde{u}}{\partial y}\left(s,t\right)-\frac{\partial u}{\partial y}\left(s,t\right)\right|dt\right]ds \\
&\leq\left(\int_0^a l_4\left(s\right)ds+\int_0^b l_5\left(t\right)dt+2\int_0^a\int_0^b l_1\left(s,t\right)dtds\right)||\tilde{u}-u|| \\
&+2\left(c_2b+\int_0^b\int_0^a l_2\left(s,y\right)dsdt\right)\left|\left|\frac{\partial\tilde{u}}{\partial x}-\frac{\partial u}{\partial x}\right|\right|_y \\
&+2\left(c_3a+\int_0^a\int_0^b l_3\left(x,t\right)dtdxg\right)\left|\left|\frac{\partial\tilde{u}}{\partial y}-\frac{\partial u}{\partial y}\right|\right|_x .
\end{aligned}
$$

Moreover

$$
\begin{aligned}
&\left|\frac{\partial}{\partial x}\left(S\tilde{u}\right)\left(x,y\right)-\frac{\partial}{\partial x}\left(Su\right)\left(x,y\right)\right| \\
&\qquad\leq\left|g\left(x,\tilde{u}\left(x,\psi\left(x\right)\right)\right)-g\left(x,u\left(s,\psi\left(x\right)\right)\right)\right| \\
&\qquad\quad+\left|\int_{\psi(x)}^y\left|f\left(x,t,\tilde{u}_{[x,t]},\frac{\partial\tilde{u}}{\partial x}\left(x,t\right),\frac{\partial\tilde{u}}{\partial y}\left(x,t\right)\right)\right.\right. \\
&\qquad\quad\left.\left.-f\left(x,t,u_{[x,t]},\frac{\partial u}{\partial x}\left(x,t\right),\frac{\partial u}{\partial y}\left(x,t\right)\right)\right|dt\right|.
\end{aligned}
$$

By the above

$$\begin{aligned}
\left\|\frac{\partial}{\partial x}S\tilde{u}-\frac{\partial}{\partial x}Su\right\|_y \leq & \int_0^a \max_{y\in[0,b]}\left|g\left(s,\tilde{u}\left(s,\psi\left(s\right)\right)\right)-g\left(s,u\left(s,\psi\left(s\right)\right)\right)\right|ds \\
& +\int_0^a \max_{y\in[0,b]}\left|\int_0^b \left|f\left(s,t,\tilde{u}_{[s,t]},\frac{\partial\tilde{u}}{\partial x}\left(s,t\right),\frac{\partial\tilde{u}}{\partial y}\left(s,t\right)\right)\right.\right. \\
& \left.\left.-f\left(s,t,u_{[s,t]},\frac{\partial u}{\partial x}\left(s,t\right),\frac{\partial u}{\partial y}\left(s,t\right)\right)\right|dt\right|ds \\
\leq & \int_0^a \left|g\left(s,\tilde{u}\left(s,\psi\left(s\right)\right)\right)-g\left(s,u\left(s,\psi\left(s\right)\right)\right)\right|ds \\
& +\int_0^a\int_0^b \left|f\left(s,t,\tilde{u}_{[s,t]},\frac{\partial\tilde{u}}{\partial x}\left(s,t\right),\frac{\partial\tilde{u}}{\partial y}\left(s,t\right)\right)\right. \\
& \left.-f\left(s,t,u_{[s,t]},\frac{\partial u}{\partial x}\left(s,t\right),\frac{\partial u}{\partial y}\left(s,t\right)\right)\right|dsdt \\
\leq & \left(\int_0^a l_4\left(s\right)ds+\int_0^a\int_0^b l_1\left(s,t\right)dtds\right)\left|\left|\tilde{u}-u\right|\right| \\
& +g\left(c_2 b+\int_0^b\int_0^a l_2\left(s,y\right)dsdt\right)\left\|\frac{\partial\tilde{u}}{\partial x}-\frac{\partial u}{\partial x}\right\|_y \\
& +g\left(c_3 a+\int_0^a\int_0^b l_3\left(x,t\right)dtdxg\right)\left\|\frac{\partial\tilde{u}}{\partial y}-\frac{\partial u}{\partial y}\right\|_x .
\end{aligned}$$

We see that

$$\begin{aligned}
& \left|\frac{\partial}{\partial y}\left(S\tilde{u}\right)\left(x,y\right)-\frac{\partial}{\partial y}\left(Su\right)\left(x,y\right)\right| \\
& \qquad \leq \left|h\left(y,\tilde{u}\left(\phi\left(y\right),y\right)\right)-h\left(y,u\left(\psi\left(y\right),y\right)\right)\right| \\
& \qquad\quad +\left|\int_{\phi(y)}^x \left|f\left(s,y,\tilde{u}_{[s,y]},\frac{\partial\tilde{u}}{\partial x}\left(s,y\right),\frac{\partial\tilde{u}}{\partial y}\left(s,y\right)\right)\right.\right. \\
& \qquad\quad \left.\left.-f\left(s,y,u_{1[s,y]},\frac{\partial u}{\partial x}\left(s,y\right),\frac{\partial u}{\partial y}\left(s,y\right)\right)\right|ds\right| .
\end{aligned}$$

Therefore

$$\begin{aligned}
\left\|\frac{\partial}{\partial y}S\tilde{u}-\frac{\partial}{\partial y}Su\right\|_x \leq & \int_0^b \max_{x\in[0,a]}\left|h\left(t,\tilde{u}\left(\phi\left(t\right),t\right)\right)-h\left(t,u\left(\psi\left(t\right),t\right)\right)\right|dt \\
& +\int_0^b \max_{x\in[0,a]}\left|\int_0^a \left|f\left(s,t,\tilde{u}_{[s,t]},\frac{\partial\tilde{u}}{\partial x}\left(s,t\right),\frac{\partial\tilde{u}}{\partial y}\left(s,t\right)\right)\right.\right. \\
& \left.\left.-f\left(s,t,u_{[s,t]},\frac{\partial u}{\partial x}\left(s,t\right),\frac{\partial u}{\partial y}\left(s,t\right)\right)\right|ds\right|dt
\end{aligned}$$

$$\leq \left(\int_0^b l_5(t)\,dt + \int_0^a \int_0^b l_1(s,t)\,dtds \right) ||\tilde{u} - u||$$
$$+ g\left(c_2 b + \int_0^b \int_0^a l_2(s,y)\,dsdt \right) \left\| \frac{\partial \tilde{u}}{\partial x} - \frac{\partial u}{\partial x} \right\|_y$$
$$+ g\left(c_3 a + \int_0^a \int_0^b l_3(x,t)\,dtdx \right) \left\| \frac{\partial \tilde{u}}{\partial y} - \frac{\partial u}{\partial y} \right\|_x .$$

Now, we see

$$||S\tilde{u} - Su||_* \leq \left(\int_0^a l_4(s)\,ds + \int_0^b l_5(t)\,dt + 2\int_0^a \int_0^b l_1(s,t)\,dtds \right) ||\tilde{u} - u||$$
$$+ 2\left(c_2 b + \int_0^b \int_0^a l_2(s,y)\,dsdt \right) \left\| \frac{\tilde{\partial u}}{\partial x} - \frac{\partial u}{\partial x} \right\|_y$$
$$+ 2\left(c_3 a + \int_0^a \int_0^b l_3(x,t)\,dtdx \right) \left\| \frac{\partial \tilde{u}}{\partial y} - \frac{\partial u}{\partial y} \right\|_x .$$

By the above,

$$||S\tilde{u} - Su||_* < \alpha ||\tilde{u} - u||_*,$$

where

$$\alpha = \max\left\{ \int_0^a l_4(s)\,ds + \int_0^b l_5(t)\,dt + 2\int_0^a \int_0^b l_1(s,t)\,dtds\ , \right.$$
$$\left. 2c_2 b + 2\int_0^b \int_0^a l_2(s,y)\,dsdt,\ 2c_3 a + 2\int_0^a \int_0^b l_3(x,t)\,dtdx \right\}.$$

We see from (17)–(19) that $\alpha < 1$. Using the Banach fixed point theorem, we get the existence of exactly one solution of problem (2). ∎

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Received April 29, 2013; revised version April 29, 2014.
Communicated by K. Chełmiński.