DE GRUYTER OPEN

DEMONSTRATIO MATHEMATICA
Vol. XLVII No 4 2014

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ON SOME QUALITATIVE PROPERTIES OF MILD SOLUTIONS OF NONLOCAL SEMILINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. In the present paper, we investigate the qualitative properties such as existence, uniqueness and continuous dependence on initial data of mild solutions of first and second order nonlocal semilinear functional differential equations with delay in Banach spaces. Our analysis is based on semigroup theory and modified version of Banach contraction theorem.

1. Introduction

The problems of existence, uniqueness and other qualitative properties of solutions for semilinear differential equations in Banach spaces has been studied extensively in the literature for last many years, see [1]–[4], [7]–[12], [18]. On the other hand, as nonlocal condition is more precise to describe natural phenomena than classical initial condition, the Cauchy problem with nonlocal condition also received much attention in recent years, see [1]–[3], [8], [10]. These type of problems were first studied by L. Byszewski. Also, the problems of qualitative properties of solutions of second order functional differential equations have been studied by many authors, see [5]–[7], [9], [12], [14]–[16]. It is advantageous to treat second order abstract differential equations directly rather than to convert into first order differential system. For direct applications of second order differential system, one may refer Fitzgibbon [6].

In the present paper, we consider semilinear functional differential problem of first order of the type:

$$(1)\qquad x'(t) = A_1 x(t) + f(t, x_t), \quad t \in [0, T],$$

2010 *Mathematics Subject Classification*: Primary 34A12; Secondary 45J05, 45N05, 47H10, 47B38.

Key words and phrases: existence, uniqueness, functional differential equation, fixed point, semigroup theory.

DOI: 10.2478/dema-2014-0069

(2) $$x(t) + (g(x_{t_1}, \dots, x_{t_p}))(t) = \phi(t), \quad -r \le t \le 0,$$

and second order differential system of the type:

(3) $$x''(t) = A_2 x(t) + f(t, x_t), \quad t \in [0, T],$$

(4) $$x(t) + (g(x_{t_1}, \dots, x_{t_p}))(t) = \phi(t), \quad -r \le t \le 0,$$

(5) $$x'(0) = \eta \in X,$$

where $0 < t_1 < t_2 < \cdots < t_p \le T$, $p \in N$; A_1 is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t), t \ge 0$ on X; A_2 is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $\{C(t)\}_{t \in \mathbb{R}}$ on X; f, g, ϕ are given functions satisfying some assumptions and $x_t(\theta) = x(t+\theta)$, for $\theta \in [-r, 0]$ and $t \in [0, T]$. Recently, L. Byszewski and H. Akca [1] studied existence, uniqueness and continuous dependence of a mild solution on initial data of problem (1)–(2) by Banach contraction theorem. The objective of this paper is to improve their results. We are achieving the same results with less restrictions by using modified version of Banach contraction principle. Also we obtain existence and uniqueness of a mild solution on initial data of second order differential system (3)–(5), using the theory of strongly continuous cosine family of operators.

The paper is organized as follows: Section 2 presents preliminaries and hypothesises. In Sections 3 and 4, we prove existence, uniqueness and continuous dependence on initial data of mild solutions of first order and second order differential system, respectively. Finally in Section 5, we give applications based on our results.

2. Preliminaries and hypotheses

Let X be a Banach space with the norm $\|\cdot\|$. Let $C = \mathcal{C}([-r, 0], X), 0 < r < \infty$ be the Banach space of all continuous functions $\psi : [-r, 0] \to X$ endowed with supremum norm

$$\|\psi\|_C = \sup\{\|\psi(t)\| : -r \le t \le 0\}.$$

Let $B = \mathcal{C}([-r, T], X), T > 0$ be the Banach space of all continuous functions $x : [-r, T] \to X$ with the supremum norm $\|x\|_B = \sup\{\|x(t)\| : -r \le t \le T\}$. For any $x \in B$ and $t \in [0, T]$, we denote x_t the element of C given by $x_t(\theta) = x(t+\theta)$, for $\theta \in [-r, 0]$ and ϕ is a given element of C.

Definition 2.1. A function $x \in B$ satisfying the equations:

$$x(t) = T(t)\phi(0) - T(t)(g(x_{t_1}, \dots, x_{t_p}))(0) + \int_0^t T(t-s) f(s, x_s) ds, \quad t \in [0, T],$$

$$x(t) + (g(x_{t_1}, \dots, x_{t_p}))(t) = \phi(t), \quad -r \le t \le 0,$$

is said to be the mild solution of the initial value problem (1)–(2).

Definition 2.2. A one parameter family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators in the Banach space X is called strongly continuous cosine family if and only if

(a) $C(0) = I$ is the identity operator,
(b) $C(t+s) + C(t-s) = 2C(t)C(s) \quad \forall t, \quad s \in \mathbb{R}$,
(c) the map $t \mapsto C(t)(x)$ is strongly continuous for each $x \in X$.

The associated sine function is the family $\{S(t)\}_{t\in\mathbb{R}}$ of operators defined by $S(t)x = \int_0^t C(s)xds$, for $x \in X, t \in \mathbb{R}$. The infinitesimal generator $A_2 : X \to X$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by $A_2 x = \frac{d^2}{dt^2}C(t)x|_{t=0}$, $x \in D(A_2)$, where $D(A_2) = \{x \in X : C(.)x \in C^2(\mathbb{R}, X)\}$. For more information on strongly continuous cosine and sine families, we refer the reader to [[5], [15]–[17]].

In this paper, we assume that there exist positive constants $K \geq 1, M \geq 1$ and N such that $\|T(t)\| \leq K, \|C(t)\| \leq M$ and $\|S(t)\| \leq N$, for every $t \in [0, T]$.

Definition 2.3. A function $x : [-r, T] \to X$ is called a mild solution of the system (3)–(5), if it satisfies the following equations

$$\begin{aligned} x(t) &= C(t)[\phi(0) - (g(x_{t_1}, \dots, x_{t_p}))(0)] + S(t)\eta \\ &\quad + \int_0^t S(t-s)f(s, x_s)ds, \quad t \in [0, T], \\ x(t) &+ (g(x_{t_1}, \dots, x_{t_p}))(t) = \phi(t), \quad -r \leq t \leq 0, \\ x'(0) &= \eta \in X. \end{aligned}$$

Our results are based on the modified version of Banach contraction principle.

Lemma 2.4. [[13], p. 196] *Let X be a Banach space. Let D be an operator which maps the elements of X into itself for which D^r is a contraction, where r is a positive integer. Then D has a unique fixed point.*

We list the following hypotheses for our convenience.

(H_1) Let $f : [0, T] \times C \to X$ be such that for every $w \in B$ and $t \in [0, T]$, $f(., w_t) \in B$ and there exists a constant $L > 0$ such that

$$\|f(t, \psi) - f(t, \phi)\| \leq L(\|\psi - \phi\|_C), \qquad \phi, \psi \in C.$$

(H_2) Let $g : C^p \to C$ be such that exists a constant $G \geq 0$ satisfying

$$\|(g(x_{t_1}, x_{t_2}, \dots, x_{t_p}))(t) - (g(y_{t_1}, y_{t_2}, \dots, y_{t_p}))(t)\| \leq G\|x - y\|_B, \quad t \in [-r, 0].$$

3. First order nonlocal differential system

3.1. Existence and uniqueness

Theorem 3.1.1. *Suppose that the hypotheses* (H_1) *and* (H_2) *are satisfied. Then the initial-value problem* (1)–(2) *has a unique mild solution* x *on* $[-r, T]$.

Proof. Let $x(t)$ be a mild solution of the problem (1)–(2). Then it satisfies the equivalent integral equation

$$x(t) = T(t)\phi(0) - T(t)(g(x_{t_1}, \ldots, x_{t_p}))(0) + \int_0^t T(t-s)f(s, x_s)ds, \quad t \in [0, T], \tag{6}$$

$$x(t) + (g(x_{t_1}, \ldots, x_{t_p}))(t) = \phi(t), \quad -r \leq t \leq 0. \tag{7}$$

Now, we rewrite solution of initial value problem (1)–(2) as follows: For $\phi \in C$, define $\widehat{\phi} \in B$ by

$$\widehat{\phi}(t) = \begin{cases} \phi(t) - (g(x_{t_1}, \ldots, x_{t_p}))(t), & \text{if } -r \leq t \leq 0, \\ T(t)[\phi(0) - (g(x_{t_1}, \ldots, x_{t_p}))(0)], & \text{if } 0 \leq t \leq T. \end{cases}$$

If $y \in B$ and $x(t) = y(t) + \widehat{\phi}(t)$, $t \in [-r, T]$, then it is easy to see that y satisfies

$$y(t) = 0; \quad -r \leq t \leq 0 \quad \text{and} \tag{8}$$

$$y(t) = \int_0^t T(t-s)f\left(s, y_s + \widehat{\phi}_s\right) ds, \quad t \in [0, T], \tag{9}$$

if and only if $x(t)$ satisfies the equations (6)–(7).

We define the operator $F : B \to B$, by

$$(Fy)(t) = \begin{cases} 0, & \text{if} \quad -r \leq t \leq 0, \\ \int_0^t T(t-s)f\left(s, y_s + \widehat{\phi}_s\right) ds, & \text{if } t \in [0, T]. \end{cases} \tag{10}$$

From the definition of an operator F defined by the equation (10), it is to be noted that the equations (8)–(9) can be written as

$$y = Fy.$$

Now, we show that F^n is a contraction on B for some positive integer n. Let $y, w \in B$ and using hypotheses (H_1) and (H_2), we get

$$\begin{aligned} \|(Fy)(t) - (Fw)(t)\| &\leq \int_0^t \|T(t-s)\| \|f(s, y_s + \widehat{\phi}_s) - f(s, w_s + \widehat{\phi}_s)\| ds \\ &\leq \int_0^t KL\|(y_s + \widehat{\phi}_s) - (w_s + \widehat{\phi}_s)\|_C ds \end{aligned}$$

$$\leq KL\int_0^t \|y_s - w_s\|_C ds \leq KL\int_0^t \|y-w\|_B ds$$
$$\leq KL\|y-w\|_B t,$$

$$\begin{aligned}
\|(F^2y)(t)-(F^2w)(t)\| &= \|(F(Fy))(t)-(F(Fw))(t)\| \\
&= \|(F(y_1))(t)-(F(w_1))(t)\| \\
&\leq \int_0^t \|T(t-s)\|\|f(s,y_{1_s}+\widehat{\phi}_s)-f(s,w_{1s}+\widehat{\phi}_s)\|_C ds \\
&\leq \int_0^t KL[\|(y_{1_s}+\widehat{\phi}_s)-(w_{1_s}+\widehat{\phi}_s)\|_C]ds \\
&\leq KL\int_0^t \|y_{1_s}-w_{1s}\|_C ds \\
&\leq KL\int_0^t \|y_1-w_1\|_{\mathcal{C}([-r,s],X)} ds \\
&\leq KL\int_0^t \sup_{\tau\in[-r,s]}[\|y_1(\tau)-w_1(\tau)\|]ds \\
&\leq KL\int_0^t \sup_{\tau\in[-r,s]}[\|Fy(\tau)-Fw(\tau)\|]ds \\
&\leq KL\int_0^t \sup_{\tau\in[-r,s]}(KL\|y-w\|_B\tau)ds \\
&\leq (KL)^2\int_0^t \|y-w\|_B s ds \leq \frac{(KLt)^2}{2!}\|y-w\|_B.
\end{aligned}$$

Continuing in this way, we get,

$$\|(F^ny)(t)-(F^nw)(t)\| \leq \frac{(KLt)^n}{n!}\|y-w\|_B \leq \frac{(KLT)^n}{n!}\|y-w\|_B.$$

For n large enough, $\frac{(KLT)^n}{n!} < 1$. Thus there exists a positive integer n such that F^n is a contraction in B. By virtue of Lemma 2.4, the operator F has a unique fixed point $\tilde{y}$ in B. Then $\tilde{x} = \tilde{y} + \widehat{\phi}$ is a solution of the Cauchy problem (1)–(2). This completes the proof. ∎

3.2. Continuous dependence on initial data

Theorem 3.2.1. *Suppose that the functions f and g satisfy the hypotheses (H_1) and (H_2). Then for each $\phi_1, \phi_2 \in C$ and for the corresponding mild solutions x_1, x_2 of the problems*

(11) $$x'(t) = A_1x(t) + f(t,x_t), \quad t \in [0,T],$$

(12) $$x(t) + (g(x_{t_1},\ldots,x_{t_p}))(t) = \phi_i(t), \quad -r \leq t \leq 0, \quad (i=1,2),$$

the inequality

$$\|x_1 - x_2\|_B \le Ke^{TKL}[\|\phi_1 - \phi_2\|_C + G\|x_1 - x_2\|_B] \tag{13}$$

is true.

Additionally, if $G < \frac{1}{Ke^{TKL}}$ *then*

$$\|x_1 - x_2\|_B \le \frac{Ke^{TKL}}{1 - GKe^{TKL}}\|\phi_1 - \phi_2\|_C. \tag{14}$$

Proof was given in paper [1], so we omit details here.

Remark. If $G = 0$, inequality (13) is reduced to classical inequality

$$\|x_1 - x_2\|_B \le Ke^{TKL}\|\phi_1 - \phi_2\|_C,$$

which is the characteristic for the continuous dependence of the mild solution of the semilinear functional differential equation with classical initial condition.

4. Second order nonlocal differential system

4.1. Existence and uniqueness

Theorem 4.1.1. *Suppose that the hypotheses* (H_1) and (H_2) *are satisfied. Then the second order differential problem* (3)–(5) *has a unique mild solution* x *on* $[-r, T]$.

Proof. Let $x(t)$ be a mild solution of the problem (3)–(5). Then it satisfies the equivalent integral equation

$$\begin{aligned} x(t) &= C(t)\phi(0) - C(t)(g(x_{t_1}, \dots, x_{t_p}))(0) + S(t)\eta \\ &\quad + \int_0^t S(t-s)f(s, x_s)ds, \quad t \in [0, T], \end{aligned} \tag{15}$$

$$x(t) + (g(x_{t_1}, \dots, x_{t_p}))(t) = \phi(t), \quad -r \le t \le 0, \tag{16}$$

$$x'(0) = \eta \in X. \tag{17}$$

Now, we rewrite solution of initial value problem (3)–(5) as follows: For $\phi \in C$, define $\widehat{\phi} \in B$ by

$$\widehat{\phi}(t) = \begin{cases} \phi(t) - (g(x_{t_1}, \dots, x_{t_p}))(t), & \text{if } -r \le t \le 0, \\ C(t)[\phi(0) - (g(x_{t_1}, \dots, x_{t_p}))(0)], & \text{if } 0 \le t \le T. \end{cases} \tag{18}$$

If $y \in B$ and $x(t) = y(t) + \widehat{\phi}(t)$, $t \in [-r, T]$, then it is easy to see that y satisfies

$$y(t) = 0; \quad -r \le t \le 0 \quad \text{and} \tag{19}$$

$$y(t) = S(t)\eta + \int_0^t S(t-s) f\left(s, y_s + \widehat{\phi}_s\right) ds, \quad t \in [0,T], \tag{20}$$

if and only if $x(t)$ satisfies the equations (15)–(17).

We define the operator $F : B \to B$, by

$$(Fy)(t) = \begin{cases} 0, & \text{if} \quad -r \le t \le 0, \\ S(t)\eta + \int_0^t S(t-s) f\left(s, y_s + \widehat{\phi}_s\right) ds, & \text{if } t \in [0,T]. \end{cases} \tag{21}$$

From the definition of an operator F defined by the equation (21), it is to be noted that the equations (19)–(20) can be written as

$$y = Fy.$$

Now, we show that F^n is a contraction on B for some positive integer n. Let $y, w \in B$. Using hypotheses (H_1) and (H_2), we get

$$\begin{aligned} \|(Fy)(t) - (Fw)(t)\| &\le \|S(t)\eta - S(t)\eta\| \\ &\quad + \int_0^t \|S(t-s)\| \|f(s, y_s + \widehat{\phi}_s) - f(s, w_s + \widehat{\phi}_s)\| ds \\ &\le \int_0^t ML[\|(y_s + \widehat{\phi}_s) - (w_s + \widehat{\phi}_s)\|] ds \\ &\le ML \int_0^t \|y_s - w_s\|_C ds \\ &\le ML \int_0^t \|y - w\|_B ds \\ &\le ML \|y - w\|_B t. \end{aligned}$$

By similar calculations as in the Section 3, we get

$$\|(F^2 y)(t) - (F^2 w)(t)\| \le \frac{(MLt)^2}{2!} \|y - w\|_B.$$

Continuing in this way, we get

$$\|(F^n y)(t) - (F^n w)(t)\| \le \frac{(MLt)^n}{n!} \|y - w\|_B \le \frac{(MLT)^n}{n!} \|y - w\|_B.$$

For n large enough, $\frac{(MLT)^n}{n!} < 1$. Thus there exists a positive integer n such that F^n is a contraction in B. By virtue of Lemma 2.4, the operator F has a unique fixed point $\tilde{y}$ in B. Then $\tilde{x} = \tilde{y} + \widehat{\phi}$ is a solution of second order differential problem (3)–(5). This completes the proof. ∎

4.2. Continuous dependence on initial data

Theorem 4.2.1. *Suppose that the functions f and g satisfies the hypotheses (H_1) and (H_2). Then for each $\phi_1, \phi_2 \in C$ and for the corresponding mild* *solutions x_1, x_2 of the problems*

$$x''(t) = A_2 x(t) + f(t, x_t), \quad t \in [0, T], \tag{22}$$

$$x(t) + (g(x_{t_1}, \dots, x_{t_p}))(t) = \phi_i(t), \quad -r \leq t \leq 0, \quad (i = 1, 2), \tag{23}$$

$$x'(0) = \eta_i \in X, \quad (i = 1, 2), \tag{24}$$

the inequality

$$\|x_1 - x_2\|_B \leq [M\|\phi_1 - \phi_2\|_C + MG\|x_1 - x_2\|_B + N\|\eta_1 - \eta_2\|]e^{TNL} \tag{25}$$

is true.

Additionally if $G < \frac{1}{Me^{TNL}}$ *then*

$$\|x_1 - x_2\|_B \leq \frac{Me^{TNL}}{1 - GMe^{TNL}}\|\phi_1 - \phi_2\|_C + \frac{Ne^{TNL}}{1 - GMe^{TNL}}\|\eta_1 - \eta_2\|. \tag{26}$$

Proof. Let $\phi_i (i = 1, 2)$ be arbitrary functions in C and let $x_i (i = 1, 2)$ be the mild solutions of the problem (22)–(24).

Then for $t \in [-r, 0]$

$$\begin{aligned} & x_1(t) - x_2(t) \\ & \quad = \phi_1(t) - (g(x_{1t_1}, \dots, x_{1t_p}))(t) - \phi_2(t) + (g(x_{2t_1}, \dots, x_{2t_p}))(t) \end{aligned} \tag{27}$$

and for $t \in [0, T]$

$$\begin{aligned} & x_1(t) - x_2(t) \\ & = C(t)[\phi_1(0) - \phi_2(0) - (g(x_{1t_1}, \dots, x_{1t_p}))(0) + (g(x_{2t_1}, \dots, x_{2t_p}))(0)] \\ & \quad + S(t)(\eta_1 - \eta_2) + \int_0^t S(t - s)(f(s, x_{1s}) - f(s, x_{2s}))ds. \end{aligned} \tag{28}$$

From (28) and hypothesises (H_1) and (H_2), we get, for $\tau \in [0, t]$,

$$\begin{aligned} \|x_1(\tau) - x_2(\tau)\| &= \|C(t)\|\|\phi_1 - \phi_2\|_C + G\|C(t)\|\|x_1 - x_2\|_B \\ & \quad + \|S(t)\|\|\eta_1 - \eta_2\| + \int_0^\tau \|S(t - s)\|\|f(s, x_{1s}) - f(s, x_{2s})\|ds \\ & \leq M\|\phi_1 - \phi_2\|_C + MG\|x_1 - x_2\|_B + N\|\eta_1 - \eta_2\| \\ & \quad + NL\int_0^\tau \|x_{1s} - x_{2s}\|_C ds, \end{aligned}$$

$$\begin{aligned} & \sup_{\tau \in [0,t]} \|x_1(\tau) - x_2(\tau)\| \\ & \qquad \leq M\|\phi_1 - \phi_2\|_C + MG\|x_1 - x_2\|_B + N\|\eta_1 - \eta_2\| \\ & \qquad + NL\int_0^t \|x_1 - x_2\|_{\mathcal{C}([-r,s],X)} ds, \quad 0 \leq \tau \leq t \leq T. \end{aligned} \tag{29}$$

Simultaneously, by equation (27) and hypothesis (H_2), we get

$$\|x_1(t) - x_2(t)\| \leq \|\phi_1 - \phi_2\|_C + G\|x_1 - x_2\|_B, \quad \text{for} \quad t \in [-r, 0]. \tag{30}$$

Since $M \geq 1$, inequalities (29) and (30) imply

$$\|x_1 - x_2\|_{C([-r,t],X)} \leq M\|\phi_1 - \phi_2\|_C + MG\|x_1 - x_2\|_B + N\|\eta_1 - \eta_2\| + NL\int_0^t \|x_1 - x_2\|_{\mathcal{C}([-r,s],X)} ds, \quad t \in [-r, T]. \tag{31}$$

Now, applying Grownwall's inequality to the above inequality (31), we get

$$\|x_1 - x_2\|_B \leq [M\|\phi_1 - \phi_2\|_C + MG\|x_1 - x_2\|_B + N\|\eta_1 - \eta_2\|]e^{TNL}. \tag{32}$$

Hence the inequality (25) holds. Finally inequality (26) is a consequence of the inequality (32). Hence the proof is complete. ∎

Remark. If $G = 0$, inequality (25) is reduced to classical inequality

$$\|x_1 - x_2\|_B \leq [M\|\phi_1 - \phi_2\|_C + N\|\eta_1 - \eta_2\|]e^{TNL},$$

which is the characteristic for the continuous dependence of the mild solution of the second order semilinear functional differential evolution equation with classical initial condition.

5. Applications

5.1. First order nonlocal differential system. To illustrate the application of our result proved in Section 3, consider the following semilinear partial functional differential problem of the form

$$\frac{\partial}{\partial t}w(u,t) = \frac{\partial^2}{\partial u^2}w(u,t) + H\left(t, w(u, t-r))\right), \quad 0 \leq u \leq \pi,\ t \in [0,T] \tag{33}$$

$$w(0,t) = w(\pi, t) = 0, \quad 0 \leq t \leq T, \tag{34}$$

$$w(u,t) + \sum_{i=1}^{p} w(u, t_i + t) = \phi(u,t), \quad 0 \leq u \leq \pi, \quad -r \leq t \leq 0, \tag{35}$$

where $0 < t_1 \leq t_2 \leq t_p \leq T$, the function $H : [0,T], \times\mathbb{R} \to \mathbb{R}$ is continuous. We assume that the function H satisfy the following condition:
For every $t \in [0,T]$ and $v, y \in \mathbb{R}$, there exists a constant $l > 1$ such that

$$|H(t,v) - H(t,y)| \leq l(|v - y|).$$

Let us take $X = L^2[0,\pi]$. Define the operator $A_1 : X \to X$ by $A_1 z = z''$ with domain $D(A_1) = \{z \in X : z,\ \ z'$ are absolutely continuous, $z'' \in X$ and $z(0) = z(\pi) = 0\}$. Then the operator A_1 can be written as

$$A_1 z = \sum_{n=1}^{\infty} -n^2(z, z_n)z_n, \quad z \in D(A_1),$$

where $z_n(u) = (\sqrt{2/\pi}) \sin nu,\ n = 1, 2, ..$ is the orthogonal set of eigenvectors of A_1 and A_1 is the infinitesimal generator of an analytic semigroup $T(t)$, $t \geq 0$ and is given by

$$T(t)z = \sum_{n=1}^{\infty} exp(-n^2 t)(z, z_n)z_n, \quad z \in X.$$

Now, for the analytic semigroup $T(t)$ being compact, there exists constant K such that

$$|T(t)| \leq K, \quad \text{for each} \quad t \in [0, T].$$

Define the function $f : [0, T] \times C \to X$, as follows

$$f(t, \psi)(u) = H(t, \psi(-r)u),$$

for $t \in [0, T], \psi \in C$ and $0 \leq u \leq \pi$. With these choices of the functions, the equations (33)–(35) can be formulated as an abstract integro-differential equation in Banach space X:

$$x'(t) = A_1 x(t) + f\Big(t, x_t\Big), \quad t \in [0, T],$$

$$x(t) + (g(x_{t_1}, \dots, x_{t_p}))(t) = \phi(t), \quad t \in [-r, 0].$$

Since all the hypotheses of the Theorem 3.1.1. are satisfied, Theorem 3.1.1. can be applied to guarantee the existence of mild solution $w(u, t) = x(t)u$, $t \in [0, T], u \in [0, \pi]$, of the semilinear partial differential problem (33)–(35).

5.2. Second order nonlocal differential system. To illustrate the application of our result proved in Section 4, consider the following semilinear partial functional integro-differential problem of the form

(36) $$\frac{\partial^2}{\partial t^2} w(u, t) = \frac{\partial^2}{\partial u^2} w(u, t) + H\left(t, w(u, t - r)\right), \quad 0 \leq u \leq \pi,\ t \in [0, T],$$

(37) $$w(0, t) = w(\pi, t) = 0, \quad 0 \leq t \leq T,$$

(38) $$w(u, t) + \sum_{i=1}^{p} w(u, t_i + t) = \phi(u, t), \quad 0 \leq u \leq \pi, \quad -r \leq t \leq 0,$$

(39) $$\frac{\partial}{\partial t} w(u, 0) = \eta(u), \quad 0 \leq u \leq \pi,$$

where $0 < t_1 \leq t_2 \leq t_p \leq T$, the function $H : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous. We assume that the function H satisfies the following condition:
For every $t \in [0, T]$ and $v, y \in \mathbb{R}$, there exists an $l > 1$ such that

$$|H(t, v) - H(t, y)| \leq l(|v - y|).$$

Let us take $X = L^2[0, \pi]$. Define the operator $A_2 : X \to X$ by $A_2 z = z_{uu}$ with domain $D(A_2) = \{z \in X : z, \quad z_u$ are absolutely continuous, $z_{uu} \in X$ and $z(0) = z(\pi) = 0\}$. Then the operator A_2 is the infinitesimal generator of a strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$ on X. Moreover, A_2 has a discrete spectrum, the eigenvalues are $-n^2$, $n \in \mathbb{N}$, with corresponding eigenvectors $z_n(u) = (\sqrt{2/\pi})\sin(nu)$. The set $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X and the following properties hold:

(a) If $z \in D(A_2)$ then $A_2 z = -\sum_{n=1}^{\infty} n^2(z, z_n)z_n$.

(b) For every $z \in X$, $C(t)z = \sum_{n=1}^{\infty} \cos nt(z, z_n)z_n$.

(c) For every $z \in X$, $S(t)z = \sum_{n=1}^{\infty} \frac{\sin nt}{n}(z, z_n)z_n$.

Consequently, $\|C(t)\| = \|S(t)\| \leq 1$ and $S(t)$ is compact for $t \in \mathbb{R}$.

Now, as in Section 5.1, the equations (36)–(39) can be formulated as an abstract integro differential equations (3)–(5) in Banach space X. Since all the hypotheses of Theorem 4.1.1. are satisfied, Theorem 4.1.1. can be applied to guarantee the existence of mild solution $w(u,t) = x(t)u$, $t \in [0,T]$, $u \in [0,\pi]$ of the semilinear partial differential problem (36)–(39).

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Received March 11, 2013.

Communicated by A. Fryszkowski.