DE GRUYTER
OPEN

DEMONSTRATIO MATHEMATICA
Vol. XLVII No 4 2014

Jarosław Widomski

ON THE COEFFICIENT PROBLEM OF MEROMORPHIC HARMONIC MAPPINGS

Abstract. In this paper, we shall study estimates for the coefficients a_n, $n = 1, 2$ of a class of univalent harmonic mappings defined on the exterior of the unit disk $\tilde{\mathbb{U}} = \{z : |z| > 1\}$, which keep infinity fixed. For this purpose, we apply Faber polynomials and an inequality of the Grunsky type.

1. Introduction

In [1], Hengartner and Schober introduced the class Σ_H of all complex-valued, harmonic, orientation preserving, univalent mappings f defined on $\tilde{\mathbb{U}} = \{z : |z| > 1\}$, which are normalized at infinity by $f(\infty) = \infty$. Such functions admit the representation

$$f(z) = h(z) + \overline{g(z)} + A \log |z|, \tag{1}$$

where $A \in \mathbb{C}$ and

$$h(z) = \alpha z + \sum_{n=0}^{\infty} a_n z^{-n}, \quad g(z) = \beta z + \sum_{n=1}^{\infty} b_n z^{-n} \tag{2}$$

are analytic in $\tilde{\mathbb{U}}$, $0 \le |\beta| < |\alpha|$, and $a = \overline{f_{\overline{z}}}/f_z$ is analytic and satisfies $|a(z)| < 1$ for $z \in \tilde{\mathbb{U}}$.

Since the affine transformation $(\overline{\alpha} f - \overline{\beta f} - \overline{\alpha} a_0 + \overline{\beta a_0})/(|\alpha|^2 - |\beta|^2)$ is again in the class, we may let $\alpha = 1$, $\beta = 0$, and $a_0 = 0$ in (2). Therefore, let $\Sigma_H^{'}$ be the set of all harmonic, orientation preserving, univalent mappings f given by (1), where

$$h(z) = z + \sum_{n=1}^{\infty} a_n z^{-n}, \quad g(z) = \sum_{n=1}^{\infty} b_n z^{-n} \tag{3}$$

are analytic in $\tilde{\mathbb{U}}$. We further consider a special case $\Sigma_H^{''} = \{f \in \Sigma_H^{'} : A = 0\}$, that is, the subclass without logarithmic singularity.

2010 *Mathematics Subject Classification*: 30C45.
Key words and phrases: harmonic meromorphic mapping, coefficient bounds.

DOI: 10.2478/dema-2014-0068

Finally, let Σ_H^0 denote the non-vanishing class defined by

$$\Sigma_H^0 = \left\{ f - c : f \in \Sigma_H^{'} \text{ and } c \notin f(\tilde{\mathbb{U}}) \right\}.$$

The families Σ_H^0, $\Sigma_H^{'}$, and $\Sigma_H^{''}$ are compact with respect to the topology of locally uniform convergence (see Theorem 3.6, [1]).

Using Schwarz's lemma, Hengartner and Schober ([1]) proved the following theorem.

Theorem A.

(i) *If* $f \in \Sigma_H^{'}$ *then* $|A| \le 2$ *and* $|b_1| \le 1$.
(ii) *If* $f \in \Sigma_H^{''}$ *then* $|b_1| \le 1$ *and* $|b_2| \le \frac{1}{2}(1 - |b_1|^2) \le \frac{1}{2}$.
(iii) *If* $f \in \Sigma_H^{'}$ *has the representation* (1) *and it satisfies* (3) *then* $\sum_{n=1}^{\infty} n(|a_n|^2 - |b_n|^2) \le 1 + 2\,\mathrm{Re}\, b_1$.

All the results are sharp.

Furthermore, they also proved the distortion theorem, which we shall use in this paper.

Theorem B. *If* $f \in \Sigma_H^0$ *then* $|f(z)| \le 4(1 + |z|)^2/|z|$ *for all* $z \in \tilde{\mathbb{U}}$. *Moreover,* $f(\tilde{\mathbb{U}})$ *contains the set* $\{w : |w| > 16\}$, *and* $|c| \le 16$.

Their work gave rise to several fascinating problems and conjectures. Though several researchers solved some of these problems and conjectures, yet many questions are still unanswered and need to be investigated. One of these is the problem of finding best possible bounds for modulus of the coefficients of function $f \in \Sigma_H$.

In this note, we give some estimates of the coefficients a_n, $n = 1, 2$ when $f \in \Sigma_H^{'}$ or $f \in \Sigma_H^{''}$. The bounds are not best possible, but so far nothing is known about the behavior of the coefficients of the analytic part h.

2. Main results

We shall first recall some facts concerning the so called Faber polynomials. Given a function

$$g(z) = cz + c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots, \quad c > 1, \quad z \in \tilde{\mathbb{U}} \tag{4}$$

of class Σ of meromorphic univalent functions, consider the expansion

$$\xi g'(\xi) = (g(\xi) - w) \sum_{n=0}^{\infty} F_n(w)\xi^{-n}, \tag{5}$$

valid for all ξ in some neighborhood od ∞. $F_n(w)$ is a monic polynomial of degree n, called the nth Faber polynomial of the function g.

Comparing the coefficients of like powers of ξ on both sides of (5), we obtain $F_0(w) = 1$, $F_1(w) = \frac{1}{c}(w - c_0)$ and $F_2(w) = \frac{1}{c}(-2c_1 + (w - c_0)^2\frac{1}{c})$.

After additional assumption $|g(z)| > 1$, we can transform the function $g(z) \in \Sigma$ to

$$h(z) = (g(z^2))^{1/2} = b_0 z + \frac{b_1}{z} + \frac{b_3}{z^3} + \cdots . \tag{6}$$

Then $h \in \Sigma$, and a calculation gives

$$b_0 = \sqrt{c}, \quad b_1 = \frac{c_0}{2\sqrt{c}}, \quad b_3 = \left(c_1 - \frac{c_0^2}{4c}\right)\frac{1}{2\sqrt{c}}.$$

Putting $g(\xi) = h(\xi)$ in (5), we get $F_0(w) = 1$, $F_1(w) = \frac{w}{b_0}$, $F_2(w) = \frac{1}{b_0}(-2b_1 + \frac{w^2}{b_0})$ and $F_3(w) = \frac{w}{b_0^2}(-3b_1 + \frac{w^2}{b_0})$.

Now observe that since g is univalent, the function

$$\frac{\xi g'(\xi)}{g(\xi) - g(z)} - \frac{\xi}{\xi - z} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \beta_{nk} z^{-k}\xi^{-n} \tag{7}$$

is analytic for $|z| > 1$ and $|\xi| > 1$. In view of (5), the relation (7) gives

$$\sum_{n=0}^{\infty} F_n(g(z))\xi^{-n} = 1 + \sum_{n=1}^{\infty}\left\{z^n + \sum_{k=1}^{\infty}\beta_{nk}z^{-k}\right\}\xi^{-n}.$$

Thus the Faber polynomials satisfy

$$F_n(g(z)) = z^n + \sum_{k=1}^{\infty}\beta_{nk}z^{-k}, \quad n = 1, 2, \ldots . \tag{8}$$

The coefficients β_{nk} are known as the Grunsky coefficients of g. The property (8) characterizes the nth Faber polynomial of g among all polynomials of degree n.

Let us put

$$F_n\left(\frac{1}{\overline{g(\xi)}}\right) = \sum_{k=0}^{\infty}\delta_{nk}\overline{\xi}^{-k}. \tag{9}$$

Singh in [4] showed that if $w = g(z)$ maps $\tilde{\mathbb{U}}$ conformally onto a domain Ω contained in $|w| > 1$ and $\lambda_1, \ldots, \lambda_N$ are arbitrary complex numbers then

$$\left|\sum_{n=1}^{N}\sum_{k=1}^{N} k\lambda_n\lambda_k\beta_{nk}\right| \le \sum_{n=1}^{N} n|\lambda_n|^2 - \sum_{n=1}^{N}\sum_{k=1}^{N} k\lambda_n\overline{\lambda_k}\delta_{nk}, \quad N = 1, 2, \ldots . \tag{10}$$

Now, we use (10) to prove the following theorem.

Theorem 1. *Let g belong to the class Σ of univalent functions of the form (4) and satisfying $|g(z)| > 1$, $z \in \tilde{\mathbb{U}}$. Then the following inequalities hold:*

(a) $|c_0| \le 2(c-1)$,
(b) $|c_1| \le c(1 - \frac{1}{c^2})$,

(c) i) $|c_2| \leq \frac{2}{3}\left(c - \frac{1}{c^2}\right)$, *for* $1 < c \leq 4$,

ii) $|c_2| \leq \frac{2}{3}\left(c + \frac{c^3-6c^2+9c-5}{c^2}\right)$, *for* $c > 4$.

Proof. Singh in [4] considered the class of regular analytic functions in $|z| < 1$ having the power series expansion $f(z) = b_1 z + b_2 z^2 + \cdots$, $b_1 > 0$, which are bounded so that $|f(z)| < 1$ for $|z| < 1$. He showed that

$$|b_2| \leq 2b_1(1 - b_1).$$

Similar inequality for the coefficient b_2 was earlier proved by Pick [3]. This inequality is sharp, equality is attained for the function f given by $\frac{f(z)}{(1-f(z))^2} = \frac{b_1 z}{(1-z)^2}$ and its rotations. By the simple inversion $g(z) = \frac{1}{f(1/z)}$, $z \in \tilde{\mathbb{U}}$, we have $b_1 = \frac{1}{c}$ and $b_2 = \frac{-c_0}{c^2}$. Hence (a) holds.

With the choices $N = 2$ and $\lambda_2 = 0$, the inequality (10) takes the form $|\beta_{11}| \leq 1 - \delta_{11}$. If we use (8), (9) and the fact that $F_1(w) = (w - c_0)\frac{1}{c}$, then we obtain $\beta_{11} = \frac{c_1}{c}$, $\delta_{11} = \frac{1}{c^2}$, which gives the required condition (b). Equality holds in (b) for domain bounded by the unit circumference with a rectilinear slit in the direction away from the center.

Let us apply the inequality (10) to the function $h(z)$ given by (6). On taking $N = 3$ in (10) and setting $\lambda_1 = \lambda_2 = 0$, we have $|\beta_{33}| \leq 1 - \delta_{33}$. Using once again equalities (8), (9) and the Faber polynomial $F_3(w)$ for $w = h(z)$, we get

$$\left|\frac{3c_2}{2c} + \frac{c_0^3}{8c^3}\right| \leq 1 - \left(\frac{3|c_0|^2}{4c^3} + \frac{1}{c^3}\right).$$

From this, we conclude that

$$|c_2| \leq \frac{|c_0|^3}{12c^2} - \frac{|c_0|^2}{2c^2} - \frac{2}{3c^2} + \frac{2c}{3}.$$

It remains to find the maximal value of the right hand of above inequality. Maximinizing the function $w(t) = \frac{t^3}{12c^2} - \frac{t^2}{2c^2} - \frac{2}{3c^2} + \frac{2c}{3}$, $0 \leq t \leq 2(c-1)$, $c > 1$, we have (c). In the case $1 < c \leq 4$, the inequality is sharp, the extremal domain is the domain bounded by the unit circumference with three symmetrically situated rectilinear slits of equal length. ■

We can now formulate our main result.

Theorem 2. *If* $f \in \Sigma'_H$ *has expansion* (1) *along with* (3) *then*

(i) $|a_1| < d + 3$,
(ii) $|a_2| < 2d + \frac{27}{4}$ *for* $1 < c \leq 4$, *and* $|a_2| < \frac{5}{2}d + \frac{31}{3}$ *for* $c > 4$,

where d *is the smallest constant for which* $f(\tilde{\mathbb{U}})$ *contains the set* $\{w : |w| > d\}$.

Proof. The point of departure is the covering theorem for Σ'_H (Theorem B). It says that the range of each harmonic mapping $f \in \Sigma'_H$ contains the domain

$|w| > d$, where $d \le 16$. Let $\triangle$ be the preimage of this domain under f. Let ϕ be the meromorphic conformal mapping of $\tilde{\mathbb{U}}$ onto $\triangle$, so that $|\phi(z)| > 1$, $z \in \tilde{\mathbb{U}}$ with $\phi(\infty) = \infty$ and $\phi'(\infty) = 1$. Then the composition $F = \frac{1}{d}(f \circ \phi)$ is a harmonic automorphism of $\tilde{\mathbb{U}}$.

Now, let $\phi(z) = z + d_0 + \frac{d_1}{z} + \frac{d_2}{z^2} + \cdots$. Thus,

$$\begin{aligned} F(z) = \frac{1}{d}\Bigg[&\phi(z) + \frac{a_1}{\phi(z)} + \frac{a_2}{(\phi(z))^2} + \cdots + \frac{\overline{b_1}}{\overline{\phi(z)}} + \frac{\overline{b_2}}{(\overline{\phi(z)})^2} + \cdots \\ &+ \frac{A}{2}(\log \phi(z) + \log \overline{\phi(z)})\Bigg] \\ = \frac{1}{d}\Bigg[&z + d_0 + Ak\pi i + \frac{1}{z}\left(d_1 + a_1 + \frac{1}{2}Ad_0\right) \\ &+ \frac{1}{z^2}\left(d_2 + a_2 - a_1 d_0 + \frac{1}{2}Ad_1 - \frac{1}{4}Ad_0^2\right) + \cdots + \frac{1}{\overline{z}}\left(\overline{b_1} + \frac{1}{2}A\overline{d_0}\right) \\ + \frac{1}{\overline{z}^2}&\left(\overline{b_2} - \overline{b_1 d_0} + \frac{1}{2}A\overline{d_1} - \frac{1}{4}A\overline{d_0}^2\right) + \cdots + A\log|z|\Bigg]. \end{aligned}$$

Using the notation

$$F(z) = \alpha z + \beta\overline{z} + \sum_{n=0}^{\infty} A_n z^{-n} + \overline{\sum_{n=1}^{\infty} B_n z^{-n}} + A\log|z|,$$

we get formulas

$$\alpha = \frac{1}{d}, \quad \beta = 0, \quad A_1 = \frac{1}{d}\left(d_1 + a_1 + \frac{1}{2}Ad_0\right),$$

$$A_2 = \frac{1}{d}\left(d_2 + a_2 - a_1 d_0 + \frac{1}{2}Ad_1 - \frac{1}{4}Ad_0^2\right), \quad B_1 = \frac{1}{d}\left(b_1 + \frac{1}{2}\overline{A}d_0\right).$$

Since $F \in \Sigma_H$ and it maps $\tilde{\mathbb{U}}$ harmonically onto itself, it follows that

$$|\alpha + \overline{B_1}| \le 1, \quad |\beta + \overline{A_1}| \le 1, \quad |A_n| \le \frac{1}{n}, \quad |B_n| \le \frac{1}{n} \quad \text{for} \quad n \ge 2,$$

as was shown in [2]. Above estimates along with an application of the triangle inequality yield

$$|a_1| \le d + \left|d_1 + \frac{1}{2}Ad_0\right| \quad \text{and} \quad |a_2| \le \frac{1}{2}d + \left|d_2 - a_1 d_0 + \frac{1}{2}A\left(d_1 - \frac{1}{2}d_0^2\right)\right|.$$

Theorem 1, when applied to the function ϕ, gives

$$|d_0| \le 2\left(1 - \frac{1}{c}\right), \quad |d_1| \le 1 - \frac{1}{c^2}.$$

Thus, in view of $|A| \le 2$, the estimate for $|a_1|$ reduces to $|a_1| < d + 3 \le 19$.

Similarly, by Theorem 1, the coefficient d_2 of ϕ satisfies the inequalities

$$|d_2| \leq \frac{2}{3}\left(1 - \frac{1}{c^3}\right) \quad \text{and} \quad |d_2| \leq \frac{2}{3}\left(1 + \frac{c^3 - 6c^2 + 9c - 5}{c^3}\right),$$

for $1 < c \leq 4$ and $c > 4$, respectively. It finally leads to the estimate $|a_2| < 2d + \frac{27}{4} \leq 38.75$ for $1 < c \leq 4$, and $|a_2| < \frac{5}{2}d + \frac{31}{3} \leq 50.(3)$ for $c > 4$. This completes the proof. ■

Remark 1. If we suppose that $f \in \Sigma_H''$, then similar calculations lead to slightly better bounds $|a_1| < 17$, $|a_2| < 34.0625$ for $c \in (1, 4\rangle$ and $|a_2| < 43.(3)$ for $c > 4$.

References

[1] W. Hengartner, G. Schober, *Univalent harmonic functions*, Trans. Amer. Math. Soc. 299 (1987), 1–31.

[2] S. H. Jun, *Univalent harmonic mappings on* $\triangle = \{z : |z| > 1\}$, Proc. Amer. Math. Soc. 119(1) (1993), 109–114.

[3] G. Pick, *Über die konforme Abbildung eines Kreises auf ein schlichtes und zugleich beschränktes Gebiet*, Wien. Ber. 126 (1917), 247–263.

[4] V. Singh, *Grunsky inequalities and coefficients of bounded schlicht functions*, Ann. Acad. Sci. Fenn. Math. 310 (1962), 22.

J. Widomski
INSTITUTE OF MATHEMATICS
MARIA CURIE-SKŁODOWSKA UNIVERSITY
20-031 LUBLIN, POLAND
E-mail: jwidomski@hektor.umcs.lublin.pl

Received June 26, 2013; revised version February 5, 2014.

Communicated by J. Kotus.