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ON m-ω_1-$p^{\omega+n}$-PROJECTIVE ABELIAN p-GROUPS

Abstract. For any non-negative integers m and n, we define the classes of *m-ω_1-$p^{\omega+n}$-projective groups* and *strongly m-ω_1-$p^{\omega+n}$-projective groups*, which properly encompass the classes of ω_1-$p^{\omega+n}$-projectives introduced by Keef in J. Algebra Numb. Th. Acad. (2010) and strongly ω_1-$p^{\omega+n}$-projectives introduced by the present author in Hacettepe J. Math. Stat. (2014), respectively. The new group structures share many interesting properties, which are closely related to these of the aforementioned two own subclasses. Moreover, certain basic results in this direction are also established.

1. Introduction and terminology

Let all groups considered in this paper be p-torsion abelian groups for some arbitrary fixed prime p. Our notions and notations are in the most part standard and follow those from the classical books [11], [12] and [14], as the not well-known of them will be explicitly explained below.

A class of groups that plays a prominent role in primary abelian group theory is the one consisting of all $p^{\omega+n}$-projectives, where $n \geq 0$ is an integer, defined by Nunke in [16] like this: The group G is called *$p^{\omega+n}$-projective* if there exists a p^n-bounded subgroup $P \leq G$ such that G/P is a direct sum of cyclic groups (note that P is necessarily nice in G because the quotient G/P is separable $=$ p^ω-bounded). This is tantamount to the fact that G is isomorphic to S/B, where S is a direct sum of cyclic groups and B is p^n-bounded (cf. [13]).

Using the specific nature of countable subgroups, Keef successfully generalized in [15] the last concept to the class of so-termed ω_1-$p^{\omega+n}$-projective groups: A group G is said to be *ω_1-$p^{\omega+n}$-projective* if there is a countable subgroup $C \leq G$ such that G/C is $p^{\omega+n}$-projective. Notice that such a sub-

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group C can be chosen to satisfy the inequalities $p^{\omega+n}G \subseteq C \subseteq p^{\omega}G$, and resultantly C is of necessity nice in G.

On another vein, Keef showed in [15] that ω_1-$p^{\omega+n}$-projectives can be characterized in a different way as follows: The group G is ω_1-$p^{\omega+n}$-projective if there exists a p^n-bounded subgroup $H \leq G$ such that G/H is the direct sum of a countable group and a direct sum of cyclics. As observed in [3] and [4], such a subgroup H need not always be nice in G, and so there was given the following definition: A group G is called *strongly* ω_1-$p^{\omega+n}$*-projective* if there is a p^n-bounded nice subgroup $N \leq G$ with G/N a direct sum of a countable group and a direct sum of cyclic groups. Note that $p^{\omega+n}$-projectives are strongly ω_1-$p^{\omega+n}$-projective. Some principal results concerning strongly ω_1-$p^{\omega+n}$-projectives were established in [4].

On the other hand, Keef showed in ([15], Theorem 3.6) that any group G for which $p^{\omega}G$ is countable and $G/p^{\omega}G$ is $p^{\omega+n}$-projective has to be ω_1-$p^{\omega+n}$-projective. Our paper is mainly motivated by the observation that, however, a group G such that $G/p^{\omega}G$ is $p^{\omega+n}$-projective with countable $p^{\omega+n}G$ need not necessarily be ω_1-$p^{\omega+n}$-projective, provided $p^{\omega}G$ is uncountable. Actually, it belongs to a more general class of groups called in [8] as *n-simply presented* groups; a group G is said to be n-simply presented if G/T is simply presented for some $T \leq G$ with $p^nT = \{0\}$. If, in addition, T is nice in G, then G is called *strongly n-simply presented*.

Incidentally, in [10], a group G was defined to be *m,n-simply presented* if there exists a p^n-bounded subgroup P of G such that G/P is strongly m-simply presented. By analogy, a group G is said to be *strongly m,n-simply presented* if there exists a p^n-bounded nice subgroup N of G such that G/N is strongly m-simply presented as well as a group G is called *weakly m,n-simply presented* if there is a p^m-bounded nice subgroup M of G such that G/M is n-simply presented (see also [5] and [6]).

So, the objective of the present article is to develop in this direction some new concepts and to find suitable relationships between them and the mentioned above group classes.

Definition 1.1. Suppose $m \geq 0$ and $n \geq 0$ are integers. The group G is said to be *m-ω_1-$p^{\omega+n}$-projective* if there is a p^m-bounded subgroup A of G such that G/A is strongly ω_1-$p^{\omega+n}$-projective.

In particular, if A is nice in G, then G is called *strongly m-ω_1-$p^{\omega+n}$-projective.*

If $m = 0$, we obtain strongly ω_1-$p^{\omega+n}$-projectives, while we obtain ω_1-$p^{\omega+m}$-projectives when $n = 0$. Likewise, it is evident that m-ω_1-$p^{\omega+n}$-projectives are n,m-simply presented.

In the case when we require "niceness" of the supported subgroup, the substitution $m = 0$ implies strongly ω_1-$p^{\omega+n}$-projectives, whereas $n = 0$ implies strongly ω_1-$p^{\omega+m}$-projectives. Also, it is apparent that strongly m-ω_1-$p^{\omega+n}$-projectives are strongly n, m-simply presented.

Ignoring the word "strongly" in the last definition, we state:

Definition 1.2. Suppose $m \geq 0$ and $n \geq 0$ are integers. The group G is said to be *weakly m-ω_1-$p^{\omega+n}$-projective* if there is a p^m-bounded nice subgroup X of G such that G/X is ω_1-$p^{\omega+n}$-projective.

Substituting $m = 0$, we yield ω_1-$p^{\omega+n}$-projective groups, while if $n = 0$, we yield strongly ω_1-$p^{\omega+m}$-projective groups. Moreover, when $m = n$, we obtain the notion of weakly n-ω_1-$p^{\omega+n}$-projective groups posed in [3]. Even more, it follows at once that the group G with properties $p^{\omega+n}G$ is countable (while $p^{\omega}G$ is uncountable) and $G/p^{\omega}G$ is $p^{\omega+n}$-projective is n-ω_1-$p^{\omega+n}$-projective. Besides, weakly m-ω_1-$p^{\omega+n}$-projective groups are obviously weakly m, n-simply presented.

Notice that Theorem 1.5 (a) from [15] guarantees that the quotient G/X must be a subgroup of the direct sum of a countable group and a $p^{\omega+n}$-projective group.

By the same token, we can extend the original concept of weak ω_1-$p^{\omega+n}$-projectivity from [2] to an analogue of the above weak m-ω_1-$p^{\omega+n}$-projectivity for any integer $m \geq 0$ just provided that G/X is weak ω_1-$p^{\omega+n}$-projective, but this will be the theme of some other research project.

Definition 1.3. Suppose $m \geq 0$ and $n \geq 0$ are integers. The group G is called *almost m-ω_1-$p^{\omega+n}$-projective* if there is a nice subgroup V of G such that p^mV is countable, $p^mV \subseteq p^{\omega+m}G$ and G/V is $p^{\omega+n}$-projective.

The setting $m = 0$ implies ω_1-$p^{\omega+n}$-projectives, whereas $n = 0$ implies ω_1-$p^{\omega+m}$-projectives.

Definition 1.4. Suppose $m \geq 0$ and $n \geq 0$ are integers. The group G is said to be *decomposably m-ω_1-$p^{\omega+n}$-projective* if there is a p^m-bounded subgroup S of G with the property that G/S is a direct sum of a countable group and a $p^{\omega+n}$-projective group.

In particular, if S is nice in G, then G is called *nice decomposably m-ω_1-$p^{\omega+n}$-projective.*

If $m = 0$, we identify the direct sums between countable groups and $p^{\omega+n}$-projective groups. If $n = 0$, we unify all ω_1-$p^{\omega+m}$-projectives; in fact, owing to ([15], Theorem 1.2 (a1)) each ω_1-$p^{\omega+m}$-projective group is even decomposably m-ω_1-$p^{\omega+n}$-projective for any $n \geq 0$.

Besides, it is clear that Definition 1.4 yields Definition 1.1, because the direct sum of a countable group and a $p^{\omega+n}$-projective group is strongly ω_1-$p^{\omega+n}$-projective (see [3]).

Choosing $m = 0$, we will obtain the direct sums between countable groups and $p^{\omega+n}$-projective groups, but choosing $n = 0$, we will obtain strongly ω_1-$p^{\omega+m}$-projectives.

Definition 1.5. Suppose $m \geq 0$ and $n \geq 0$ are integers. The group G is called *nicely m-$p^{\omega+n}$-projective* if there is a p^m-bounded nice subgroup Y of G such that G/Y is $p^{\omega+n}$-projective.

Putting $m = 0$, we get $p^{\omega+n}$-projectives, and putting $n = 0$, we get $p^{\omega+m}$-projectives. Likewise, nicely m-$p^{\omega+n}$-projective groups are both nice decomposably m-ω_1-$p^{\omega+n}$-projective and $p^{\omega+m+n}$-projective. Actually, $p^{\omega+m+n}$-projectives are groups for which there is (not necessarily nice) a p^m-bounded subgroup M and, respectively, a p^n-bounded subgroup N, such that G/M is $p^{\omega+n}$-projective, respectively, G/N is $p^{\omega+m}$-projective.

Obviously, Definition 1.5 yields Definition 1.4 in its "nicely" variant. However, if length$(G) \leq \omega + n$, then the converse holds as well.

Generally, the following self containments are fulfilled (this visualizes some immediate relationships between the new group classes):

- {decomposably m-ω_1-$p^{\omega+n}$-projective groups} $\subseteq$ {m-ω_1-$p^{\omega+n}$-projective groups}.

- {nice decomposably m-ω_1-$p^{\omega+n}$-projective groups} $\subseteq$ { strongly m-ω_1-$p^{\omega+n}$-projective groups}.

- {nicely m-ω_1-$p^{\omega+n}$-projective groups} $\subseteq$ { nice decomposably m-ω_1-$p^{\omega+n}$-projective groups}.

2. Some more relationships

In this section, we will prove certain basic relation properties of the groups from the above definitions. Throughout the rest of the paper, m and n are arbitrary fixed naturals.

First of all, we shall prove that without the condition on "niceness" of the subgroup A in Definition 1.2, the defined groups are hardly new; in fact they coincide with ω_1-$p^{\omega+m+n}$-projectives. Specifically, the following holds:

Theorem 2.1. *For any group G there exists a p^m-bounded subgroup K such that G/K is ω_1-$p^{\omega+n}$-projective if and only if G is ω_1-$p^{\omega+m+n}$-projective.*

Proof. We shall first show that G is as in the necessity of the theorem $\iff$ there exists $C \leq G$ such that $p^m C$ is countable with $p^m C \subseteq p^\omega G$ and G/C

is $p^{\omega+n}$-projective $\iff$ there exists $L \leq G$ with $p^{m+n}L$ countable and G/L a direct sum of cyclics.

Since the second equivalence follows directly by Nunke's criterion from [16], we will be concentrated on the first one.

"$\Rightarrow$". Suppose by assumption that there is a p^m-bounded subgroup $K \leq G$ such that G/K is ω_1-$p^{\omega+n}$-projective. Owing to Proposition 1.4 of [15], there exists a countable (nice) subgroup C/K of G/K such that $(G/K)/(C/K) \cong G/C$ is $p^{\omega+n}$-projective and $C/K \subseteq p^\omega(G/K) = [\cap_{i<\omega}(p^iG + K)]/K$. Therefore, $C \leq G$, $C = K + L$ for some countable $L \leq C$ and $C \subseteq \cap_{i<\omega}(p^iG + K)$. These conditions together imply that $p^mC \subseteq L$ is countable and $p^mC \subseteq \cap_{i<\omega}p^{i+m}G = p^\omega G$, as required.

"$\Leftarrow$". Write $C = X \oplus V$ where X is countable and V is p^m-bounded. Hence $G/C = G/(X \oplus V) \cong (G/V)/(X \oplus V)/V$ is $p^{\omega+n}$-projective, where $(X \oplus V)/V \cong X$ is countable. Thus, in accordance with Theorem 1.2 of [15], G/V is ω_1-$p^{\omega+n}$-projective, as desired. Moreover, $(X \oplus V)/V$ can be chosen so that $p^m[(X \oplus V)/V] = (p^mX \oplus V)/V = (p^mC \oplus V)/V \subseteq (p^\omega G + V)/V \subseteq p^\omega(G/V)$. This proves the preliminary claim.

Now, we have all the information necessary to prove the full assertion. To that aim, we just show that G is ω_1-$p^{\omega+m+n}$-projective $\iff$ there is $S \leq G$ such that $p^{m+n}S$ is countable and G/S is a direct sum of cyclics, which is precisely the stated above equivalence.

"**Necessity**". Appealing to [15], G is ω_1-$p^{\omega+m+n}$-projective if there is a countable subgroup K with G/K being $p^{\omega+m+n}$-projective. Thus, in view of [16], there exists $S \leq G$ containing K such that G/S is a direct sum of cyclic groups and $p^{m+n}S \subseteq K$. The last yields that $p^{m+n}S$ is countable, as required.

"**Sufficiency**". Suppose now that there exists $S \leq G$ such that $p^{m+n}S$ is countable and G/S is a direct sum of cyclics. Therefore, the quotient $G/S \cong (G/p^{m+n}S)/(S/p^{m+n}S)$ being a direct sum of cyclics implies, with the aid of [16], that $G/p^{m+n}S$ is $p^{\omega+m+n}$-projective. And since $p^{m+n}S$ is countable, again the application of [15] leads to G beeing ω_1-$p^{\omega+m+n}$-projective, as desired. ∎

Remark 1. Note that the condition $p^mC \subseteq p^\omega G$ stated in the proof of Theorem 2.1 was at all redundant and therefore not further used.

Imitating Theorem 2.1, it is quite natural to ask whether or not strongly m-ω_1-$p^{\omega+n}$-projective groups are exactly the strongly ω_1-$p^{\omega+m+n}$-projective ones. Referring to the following statement, this seems to be true.

Proposition 2.2. *If G is a strongly m-ω_1-$p^{\omega+n}$-projective group, then G is strongly ω_1-$p^{\omega+m+n}$-projective.*

Proof. Assume that there exists a p^m-bounded nice subgroup T such that G/T is strongly ω_1-$p^{\omega+n}$-projective. Thus there is a nice subgroup A/T of G/T with the property that $p^nA \subseteq T$ and G/A is the direct sum of a countable group and a direct sum of cyclic groups. Hence $p^{n+m}A = \{0\}$ and A is nice in G, which conditions ensure that G is strongly ω_1-$p^{\omega+m+n}$-projective, as claimed. ∎

A question of some majority is of whether or not the converse holds, that is, whether or not every strongly ω_1-$p^{\omega+m+n}$-projective group is strongly m-ω_1-$p^{\omega+n}$-projective.

We continue with

Lemma 2.3. *If G is strongly n,m-simply presented with $p^{\omega+m+n}G = \{0\}$, then G is nicely m-$p^{\omega+n}$-projective.*

Proof. Suppose that G/N is strongly n-simply presented for some p^m-bounded nice subgroup N of G. Moreover, according to [8], we have that $(G/N)/p^{\omega+n}(G/N) \cong G/(N + p^{\omega+n}G)$ is $p^{\omega+n}$-projective. Since $N + p^{\omega+n}G$ remains p^m-bounded and nice in G, it follows that G is nicely m-$p^{\omega+n}$-projective, as asserted. ∎

As a consequence, we derive:

Proposition 2.4. *Let G be a group with countable $p^{\omega+m+n}G$. Then G is strongly n,m-simply presented if and only if G is strongly m-ω_1-$p^{\omega+n}$-projective.*

Proof. The sufficiency being elementary, we concentrate on the necessity. Applying [5] (see [6], too), it follows that the quotient $G/p^{\omega+m+n}G$ must be strongly n,m-simply presented as well, so that Lemma 2.3 is applicable to deduce that $G/p^{\omega+m+n}G$ is nicely m-$p^{\omega+n}$-projective, whence it is strongly m-ω_1-$p^{\omega+n}$-projective. Hereafter Theorem 3.12 presented below can be applied to conclude that G is strongly m-ω_1-$p^{\omega+n}$-projective. ∎

In [10] was proved that if G is an m,n-simply presented group with $p^{\omega+m}G = \{0\}$, then G is $p^{\omega+m+n}$-projective. This can be strengthened to the following (note that when $m = 0$ this was established in [9]):

Proposition 2.5. *If G is an m,n-simply presented group with countable $p^{\omega+m}G$, then G is ω_1-$p^{\omega+m+n}$-projective.*

Proof. Let $H \leq G$ with $p^nH = \{0\}$ and suppose G/H is strongly m-simply presented. Hence, in view of [8], $(G/H)/p^{\omega+m}(G/H)$ is $p^{\omega+m}$-projective. However,

$$(G/H)/p^{\omega+m}(G/H) \cong G/[p^m(\cap_{i<\omega}(p^iG + H)) + H]$$

and $p^{m+n}(\cap_{i<\omega}(p^iG+H)) \subseteq p^{\omega+m}G$ is countable. Thus $p^m(\cap_{i<\omega}(p^iG+H))+H$ is p^n-countable which means that $p^m(\cap_{i<\omega}(p^iG + H)) + H = K \oplus L$, where

K is countable and L is p^n-bounded. Furthermore, $G/(K \oplus L)$ is $p^{\omega+m}$-projective and applying Nunke's criterion from [16] there is $A \leq G$ with $p^m A \subseteq K \oplus L$ and G/A is a direct sum of cyclic groups. These together imply that $p^{m+n}A$ is countable, whence $A = M \oplus P$ where M is countable and P is p^{m+n}-bounded, and that $(G/P)/(M \oplus P)/P \cong G/(M \oplus P) = G/A$ is a direct sum of cyclic groups. The last assures with the aid of the classical Charles' lemma from [1] (see [7] as well) that G/P is the direct sum of a countable group and a direct sum of cyclic groups, because $(M \oplus P)/P \cong M$ is countable. And finally, we employ [15] to get the wanted claim that G is ω_1-$p^{\omega+m+n}$-projective. ∎

As a direct consequence, we obtain:

Corollary 2.6. *If G is weakly m,n-simply presented with countable $p^{\omega+m}G$, then G is ω_1-$p^{\omega+m+n}$-projective.*

Proof. Since in virtue of [10] the group G is m,n-simply presented, we utilize Proposition 2.5 to conclude the statement. ∎

Under some additional conditions on $p^\omega G$, Proposition 2.5 can be somewhat refined thus:

Proposition 2.7. *Let $m \geq n$. If G is m,n-simply presented with countable $p^{\omega+m-n}G$, then G is n-ω_1-$p^{\omega+m}$-projective.*

Proof. Assume that $P \leq G[p^n]$ and that G/P is strongly m-simply presented. However, we subsequently have that $p^{\omega+m}(G/P) = p^m(p^\omega(G/P)) = p^m(\cap_{i<\omega}(p^iG + P)/P) = p^{m-n}(p^n(\cap_{i<\omega}(p^iG + P)/P)) = p^{m-n}((p^n(\cap_{i<\omega}(p^iG+P))+P)/P) \subseteq p^{m-n}((p^\omega G+P)/P) = (p^{\omega+m-n}G+P)/P \cong p^{\omega+m-n}G/(p^{\omega+m-n}G \cap P)$ is countable. Finally, applying [3], the quotient G/P will be strongly ω_1-$p^{\omega+m}$-projective, as required. ∎

As an immediate consequence, we list:

Corollary 2.8. *Let $m \geq n$. If G is weakly m,n-simply presented (in particular, if G is weakly m-ω_1-$p^{\omega+n}$-projective) with countable $p^{\omega+m-n}G$, then G is n-ω_1-$p^{\omega+m}$-projective.*

Proof. It follows from [10] that G is m,n-simply presented. Hence Proposition 2.7 may be applied to derive the desired assertion. ∎

The next assertion gives a transversal between two of the defined above classes.

Corollary 2.9. *Let $m \geq n$ and G is a group with countable $p^{\omega+m-n}G$. If G is weakly n-ω_1-$p^{\omega+m}$-projective, then G is n-ω_1-$p^{\omega+m}$-projective.*

Proof. Utilizing [10], it follows that G is m,n-simply presented. Hereafter, we employ Proposition 2.7 to get the claim. ∎

An other question of some interest, which immediately arises, is also whether or not $p^{\omega+m+n}$-projective groups are strongly m-ω_1-$p^{\omega+n}$-projective (and, in particular, weakly m-ω_1-$p^{\omega+n}$-projective). This is inspired by the fact that, taking $m = 0$, $p^{\omega+n}$-projective groups are themselves strongly ω_1-$p^{\omega+n}$-projective (cf. [3]).

In this way, we have the following relationship:

Proposition 2.10. *If G is a $p^{\omega+m+n}$-projective group, then G is a decomposably m-ω_1-$p^{\omega+n}$-projective group.*

Proof. Let $P \leq G$ such that G/P is a direct sum of cyclic groups and $p^{m+n}P = \{0\}$. Since $G/P \cong (G/p^nP)/(P/p^nP)$, we deduce that G/p^nP is $p^{\omega+n}$-projective and hence it is a direct sum of a countable group and a $p^{\omega+n}$-projective group. But $p^m(p^nP) = \{0\}$ and so G is decomposably m-ω_1-$p^{\omega+n}$-projective, as promised. ∎

Remark 2. The converse implication is, however, not true as simple examples show. Nevertheless, decomposably m-ω_1-$p^{\omega+n}$-projectives are intermediate situated between $p^{\omega+m+n}$-projectives and m-ω_1-$p^{\omega+n}$-projectives.

Likewise, we ask if almost m-ω_1-$p^{\omega+n}$-projective groups coincide with m-ω_1-$p^{\omega+n}$-projective groups. However, the following one way containment is valid:

Proposition 2.11. *If the group G is weakly m-ω_1-$p^{\omega+n}$-projective, then G is almost m-ω_1-$p^{\omega+n}$-projective.*

Proof. By virtue of Definition 1.2 listed above, G/X is ω_1-$p^{\omega+n}$-projective for some $X \leq G[p^m]$ which is nice in G. Furthermore, using Proposition 1.4 in [15], we have that $(G/X)/(V/X) \cong G/V$ is $p^{\omega+n}$-projective for some countable V/X which is nice in G/X and so that $V/X \subseteq p^\omega(G/X)$. But one can deduce that p^mV is countable, because $p^mX = \{0\}$. Moreover, since X is nice in G, we conclude that V is nice in G and $V/X \subseteq (p^\omega G + X)/X$. Thus $V \subseteq p^\omega G + X$ whence $p^mV \subseteq p^{\omega+m}G$. ∎

For separable groups (i.e., groups without elements of infinite height) all of the above notions are tantamount; we do not consider here concrete examples to show that these concepts are independent for lengths beyond ω, but we refer the interested reader to [8] or [10] for more details when the group length is $> \omega$.

Theorem 2.12. *Suppose G is a group such that $p^\omega G = \{0\}$. Then all of the next points are equivalent:*

(a) *G is ω_1-$p^{\omega+m+n}$-projective;*
(b) *G is m-ω_1-$p^{\omega+n}$-projective;*
(c) *G is strongly m-ω_1-$p^{\omega+n}$-projective;*

(d) *G is weakly m-ω_1-$p^{\omega+n}$-projective;*
(e) *G is almost m-ω_1-$p^{\omega+n}$-projective;*
(f) *G is decomposably m-ω_1-$p^{\omega+n}$-projective;*
(g) *G is nice decomposably m-ω_1-$p^{\omega+n}$-projective;*
(h) *G is nicely m-$p^{\omega+n}$-projective;*
(i) *G is $p^{\omega+m+n}$-projective.*

Proof. Apparently, all of the points (i)–(b) imply (a) and, in virtue of [15], we obtain that point (i) holds provided (a) is fulfilled. Moreover, it is easy to see that clause (h) implies all other ones. So, what remains to show is the implication (i) $\Rightarrow$ (h). To this purpose, [16] helps us to write that G/Z is a direct sum of cyclic groups for some subgroup $Z \leq G$ which is bounded by p^{m+n}. Thus $(G/Z[p^m])/(Z/Z[p^m]) \cong G/Z$ being a direct sum of cyclics guarantees again by [16] that $G/Z[p^m]$ is $p^{\omega+n}$-projective since $Z/Z[p^m] \cong p^m Z$ is obviously bounded by p^n. But $Z[p^m] = Z \cap G[p^m]$ and both Z and $G[p^m]$ are nice in G because G/Z is p^ω-bounded and $G/G[p^m] \cong p^m G \subseteq G$ is p^ω-bounded, too. So, resulting, $Z[p^m]$ must be nice in G (see, e.g., [11]), and since $Z[p^m]$ is p^m-bounded, we consequently get the desired fact that G is nicely m-$p^{\omega+n}$-projective. ∎

We now proceed with three useful necessary and sufficient conditions which are needed for applicable purposes in the next section.

Proposition 2.13. *The group G is strongly m-ω_1-$p^{\omega+n}$-projective if and only if there exists a p^m-bounded nice subgroup T of G such that $G/(T + p^{\omega+n}G)$ is $p^{\omega+n}$-projective and $p^{\omega+n}(G/T)$ is countable.*

Proof. It follows directly from ([3], Corollaries 4.4 and 4.5) because the isomorphism

$$(G/T)/p^{\omega+n}(G/T) \cong G/(T + p^{\omega+n}G)$$

is fulfilled. ∎

Proposition 2.14. *The group G is weakly m-ω_1-$p^{\omega+n}$-projective if and only if there exists a p^m-bounded nice subgroup X of G such that $G/(X + p^{\omega+n}G)$ is ω_1-$p^{\omega+n}$-projective and $p^{\omega+n}(G/X)$ is countable.*

Proof. It follows immediately from ([15], Corollary 2.1) since the isomorphism

$$(G/X)/p^{\omega+n}(G/X) \cong G/(X + p^{\omega+n}G)$$

holds. ∎

Proposition 2.15. *The group G is nice decomposably m-ω_1-$p^{\omega+1}$-projective if and only if there is a nice p^m-bounded subgroup H of G such that $G/(H + p^{\omega+1}G)$ is $p^{\omega+1}$-projective and $p^{\omega+1}(G/H)$ is countable.*

Proof. To treat the necessity, write $G/H = B \oplus R$, where B is countable and R is $p^{\omega+1}$-projective for some nice $H \leq G$ with $p^m H = \{0\}$. Since $p^{\omega+1}R = \{0\}$, we deduce that $p^{\omega+1}(G/H) = p^{\omega+1}B$ is countable. Moreover, $(B/p^{\omega+1}B) \oplus R \cong (G/H)/p^{\omega+1}(G/H) = (G/H)/(p^{\omega+1}G + H)/H \cong G/(p^{\omega+1}G + H)$ is obviously $p^{\omega+1}$-projective, as claimed.

As for the sufficiency, the isomorphism

$$G/(p^{\omega+1}G + H) \cong (G/H)/p^{\omega+1}(G/H)$$

accomplished with ([7], Corollary 2.11) allow us to conclude that G/H is the direct sum of a countable group and a $p^{\omega+1}$-projective group, as required. ∎

Remark 3. Since the used Corollary 2.11 from [7] is not longer true for $p^{\omega+n}$-projectives when $n > 1$ – see the Example in [7] after this corollary – perhaps Proposition 2.15 is also not generally valid for nice decomposably m-ω_1-$p^{\omega+n}$-projectives provided $n \geq 2$.

3. Ulm subgroups and Ulm factors

In [3], it was proved that if the group G is strongly ω_1-$p^{\omega+n}$-projective, then so is $G/p^{\alpha}G$ for any ordinal α. In [15], Keef does not established the same for ω_1-$p^{\omega+n}$-projectives, so that we will do that in the next statement. Actually, he proves the next claim only for $\alpha = \omega + n$ but his proof is rather indirect; our one presented below is rather more conceptual.

Proposition 3.1. *If G is an ω_1-$p^{\omega+n}$-projective group, then $G/p^{\alpha}G$ is an ω_1-$p^{\omega+n}$-projective group for every ordinal α.*

Proof. For finite ordinals α, the assertion is self-evident. So, we will assume that α is infinite. By virtue of ([15], Theorem 1.2 (a1)), let G/A be the direct sum of a countable group and a direct sum of cyclic groups for some $A \leq G$ with $p^n A = \{0\}$. Thus $p^{\alpha}(G/A)$, being contained in the countable direct summand of G/A, remains countable and $(G/A)/p^{\alpha}(G/A)$ is again a direct sum of a countable group and a direct sum of cyclics. If $T \subseteq p^{\alpha}(G/A)$, the same is still true for $(G/A)/T$. We will specially take $T = (p^{\alpha}G + A)/A$.

But the following isomorphisms hold:

$$(G/A)/(p^{\alpha}G + A)/A \cong G/(p^{\alpha}G + A) \cong (G/p^{\alpha}G)/(p^{\alpha}G + A)/p^{\alpha}G.$$

Observing that $p^n((p^{\alpha}G + A)/p^{\alpha}G) = \{0\}$, we are done. ∎

Remark 4. Reciprocally, Keef showed in [15] that a group G is ω_1-$p^{\omega+n}$-projective if and only if $p^{\omega+n}G$ is countable and $G/p^{\omega+n}G$ is ω_1-$p^{\omega+n}$-projective.

This can be somewhat generalized for any ordinal λ to the following fact: The group G is ω_1-$p^{\omega+n}$-projective if and only if $G/p^{\lambda}G$ is ω_1-$p^{\omega+n}$-projective, provided $p^{\lambda}G$ is countable. In fact, the necessity follows utilizing directly

Proposition 3.1. As for the sufficiency, let $A/p^\lambda G$ be a countable subgroup of $G/p^\lambda G$ with $A \leq G$ such that $(G/p^\lambda G)/(A/p^\lambda G) \cong G/A$ is $p^{\omega+n}$-projective. Since A is obviously countable because $p^\lambda G$ is, we are done.

Our further work in this section will be focussed on the behavior of the new group classes about Ulm subgroups and Ulm factors. Our main results presented below settle this in some aspect.

To that aim, we need a modified variant of the background material from [8], which will be used without a concrete referring.

• If δ is an ordinal and t is a natural, then for any group A with $p^\delta A = \{0\}$ and its subgroup H with $p^t H = \{0\}$, we have $p^{\delta+t}(A/H) = \{0\}$.

• Let G be a group with $P \leq G[p^n]$ and set $X/P = p^{\lambda+n}(G/P)$ for some $X \leq G$ and some ordinal λ. Since $p^\lambda(G/p^\lambda G) = \{0\}$ and $p^n((p^\lambda G + P)/p^\lambda G) = \{0\}$, utilizing the preceding point the isomorphism

$$(G/p^\lambda G)/(p^\lambda G + P)/p^\lambda G \cong G/(p^\lambda G + P)$$

gives that $p^{\lambda+n}(G/(p^\lambda G + P)) = \{0\}$. Moreover, the isomorphism

$$(G/P)/(p^\lambda G + P)/P \cong G/(p^\lambda G + P)$$

guarantees that

$$[p^{\lambda+n}(G/P) + (p^\lambda G + P)/P]/(p^\lambda G + P)/P \subseteq p^{\lambda+n}(G/(p^\lambda G + P)) = \{0\},$$

so that $p^{\lambda+n}(G/P) \subseteq (p^\lambda G + P)/P$. Thus $X/P \subseteq (p^\lambda G + P)/P$ whence $X \subseteq p^\lambda G + P$.

On the other hand, $(p^{\lambda+n}G + P)/P \subseteq p^{\lambda+n}(G/P) = X/P$ and hence $p^{\lambda+n}G + P \subseteq X$. Finally, the group X satisfies the inequalities

$$p^{\lambda+n}G + P \subseteq X \subseteq p^\lambda G + P.$$

Likewise, the following isomorphism sequence holds:

$$(A/p^{\lambda+n}A)/T \cong (G/P)/(p^\lambda G + P)/P \cong G/(p^\lambda G + P),$$

where $A = G/P$ and $T = (p^\lambda G + P)/P/(X/P) \subseteq p^\lambda((G/P)/p^{\lambda+n}(G/P)) = p^\lambda(G/P)/p^{\lambda+n}(G/P)$ and the latter factor-group is bounded by p^n. This finishes our preliminary discussion.

The following claim on niceness is pivotal. Its proof, although not difficult, is rather technical, so that we leave it to the readers.

Lemma 3.2. *Suppose N is a nice subgroup of a group A and $M \subseteq p^\lambda A$ for some infinite ordinal λ where $p^\lambda A$ is bounded. Then $(N + M)/M$ is nice in A/M.*

We now proceed by proving the next crucial statement, needed for our further application.

Proposition 3.3. *Let A be a group and $\lambda \geq \omega$ an ordinal.*

(i) *If A is strongly ω_1-$p^{\omega+n}$-projective and $Z \subseteq p^\lambda A$, where $p^\lambda A$ is bounded, then A/Z is strongly ω_1-$p^{\omega+n}$-projective.*
(ii) *If $X \subseteq p^{\omega+n}A$, $p^{\omega+n}A$ is countable and A/X is strongly ω_1-$p^{\omega+n}$-projective, then A is also strongly ω_1-$p^{\omega+n}$-projective.*

Proof. (i) Let Q be a nice subgroup of A with $p^nQ = \{0\}$ and suppose A/Q is a direct sum of a countable group and a direct sum of cyclic groups, say $A/Q = K \oplus S$. It is easily seen that $Q' = (Q + Z)/Z$ is p^n-bounded and in accordance with Lemma 3.2 it is nice in $A' = A/Z$, as well. In addition, $A'/Q' \cong A/(Q + Z) \cong (A/Q)/(Q + Z)/Q$ and $(Q + Z)/Q \subseteq (Q + p^\lambda A)/Q = p^\lambda(A/Q)$. Since $p^\lambda(A/Q) = p^\lambda K$ is countable, this means that $(A/Q)/(Q+Z)/Q \cong (K/[(Q+Z)/Q]) \oplus S$ is again a direct sum of a countable group and a direct sum of cyclic groups, as required.

(ii) With the aid of [3], we observe that $(A/X)/p^{\omega+n}(A/X) = (A/X)/(p^{\omega+n}A/X) \cong A/p^{\omega+n}A$ is $p^{\omega+n}$-projective. We next again employ [3] to derive that A is strongly ω_1-$p^{\omega+n}$-projective, as asserted. ∎

The next statement is pivotal.

Lemma 3.4. *Suppose that A is a group with a subgroup B such that A/B is bounded. Then*

(i) *A is $p^{\omega+n}$-projective if and only if B is $p^{\omega+n}$-projective.*
(ii) *A is strongly ω_1-$p^{\omega+n}$-projective if and only if B is strongly ω_1-$p^{\omega+n}$-projective.*
(iii) *A is m-ω_1-$p^{\omega+n}$-projective if and only if B is m-ω_1-$p^{\omega+n}$-projective.*

Proof. (i) It is straightforward.

(ii) Since $p^tA \subseteq B$ for some $t \in \mathbb{N}$, we obtain that $p^\omega A - p^\omega B$ and thus $p^{\omega+n}A = p^{\omega+n}B$. Moreover, in virtue of (i), $B/p^{\omega+n}B = B/p^{\omega+n}A$ is $p^{\omega+n}$-projective uniquely when $A/p^{\omega+n}A$ is ω_1-$p^{\omega+n}$-projective, because the factor-group $(A/p^{\omega+n}A)/(B/p^{\omega+n}A) \cong A/B$ remains bounded. We finally apply the First Reduction Criterion from [3] to conclude the claim.

(iii) "$\Rightarrow$". Let A/H be strongly ω_1-$p^{\omega+n}$-projective for some $H \leq A[p^m]$. Since $(A/H)/(B + H)/H \cong A/(B + H)$ remains bounded as an epimorphic image of A/B, we deduce with the help of (ii) that $(B+H)/H \cong B/(B \cap H)$ is strongly ω_1-$p^{\omega+n}$-projective. In addition, $B \cap H \leq B[p^m]$, and we are finished.

"$\Leftarrow$". Let B/L be strongly ω_1-$p^{\omega+n}$-projective factor-group for some $L \leq B[p^m]$. Since $(A/L)/(B/L) \cong A/B$ is bounded, point (ii) is applicable to infer that A/L is strongly ω_1-$p^{\omega+n}$-projective. But $L \leq A[p^m]$, and we are done. ∎

We have now at our disposal all the ingredients needed to prove the following.

Proposition 3.5. *If the group G is m-ω_1-$p^{\omega+n}$-projective, then so are both $p^\alpha G$ and $G/p^\alpha G$ for all ordinals α.*

Proof. Let $A \leq G$ with $p^m A = \{0\}$ and suppose that G/A is strongly ω_1-$p^{\omega+n}$-projective. In conjunction with the discussion initiated above, accomplished with Lemma 3.1 (b) from [8], the isomorphism

$$p^{\lambda+m}(G/A)/(p^{\lambda+m}G + A)/A \cong Y/(p^{\lambda+m}G + A)$$

holds, where Y is a subgroup of G containing A such that $Y/A = p^{\lambda+m}(G/A)$. Besides, $p^m Y \subseteq p^{\lambda+m}G + A$ so that $Y/(p^{\lambda+m}G + A)$ is bounded. We therefore subsequently apply [3] and Lemma 3.4 (ii) to infer that both $p^{\lambda+m}(G/A)$ and $(p^{\lambda+m}G + A)/A \cong p^{\lambda+m}G/(p^{\lambda+m}G \cap A)$ are ω_1-$p^{\omega+n}$-projective. Since $p^{\lambda+m}G \cap A \leq (p^{\lambda+m}G)[p^m]$, we deduce that $p^{\lambda+m}G$ is m-ω_1-$p^{\omega+n}$-projective. But the quotient $p^\lambda G/p^{\lambda+m}G$ is bounded and thus Lemma 3.4 (iii) finally applies to conclude that $p^\lambda G$ is m-ω_1-$p^{\omega+n}$-projective, as stated.

Furthermore, referring to Corollary 4.4 from [3], we deduce that the quotient $(G/A)/p^{\lambda+n}(G/A)$ is strongly ω_1-$p^{\omega+n}$-projective, too. An appeal to Proposition 3.3 (i) implies that $G/(p^\lambda G + A)$ is also strongly ω_1-$p^{\omega+n}$-projective. But the isomorphism $G/(p^\lambda G + A) \cong (G/p^\lambda G)/(p^\lambda G + A)/p^\lambda G$ holds, where $(p^\lambda G + A)/p^\lambda G \cong A/(A \cap p^\lambda G)$ is obviously p^m-bounded. This insures that $G/p^\lambda G$ is an m-ω_1-$p^{\omega+n}$-projective group, as formulated. ∎

So, we are ready to establish our next basic assertion on both Ulm subgroups and Ulm factors pertaining to the other remaining group classes.

Proposition 3.6. *If the group G is either*

(a) *strongly m-ω_1-$p^{\omega+n}$-projective*

or

(b) *weakly m-ω_1-$p^{\omega+n}$-projective*

or

(c) *nice decomposably m-ω_1-$p^{\omega+n}$-projective*

or

(d) *nicely m-$p^{\omega+n}$-projective,*

then the same are both $p^\alpha G$ and $G/p^\alpha G$ for any ordinal α.

Proof. (a) Suppose that G/T is strongly ω_1-$p^{\omega+n}$-projective for some nice p^m-bounded subgroup T of G. Thus $p^\alpha G/(p^\alpha G \cap T) \cong (p^\alpha G + T)/T = p^\alpha(G/T)$ is also strongly ω_1-$p^{\omega+n}$-projective in view of (Corollary 4.3, [3]), with $p^\alpha G \cap T$ being p^m-bounded and nice in $p^\alpha G$ (cf. [11]). Hence $p^\alpha G$ is strongly m-ω_1-$p^{\omega+n}$-projective.

To show the second part, we consequently apply Corollary 4.4 from [3] to infer that

$$(G/T)/p^\alpha(G/T) = (G/T)/(p^\alpha G + T)/T \cong G/(p^\alpha G + T) \\ \cong (G/p^\alpha G)/(p^\alpha G + T)/p^\alpha G$$

is strongly ω_1-$p^{\omega+n}$-projective. Moreover, it is plainly observed that $(p^\alpha G + T)/p^\alpha G$ is bounded by p^m because so is T, and that $(p^\alpha G + T)/p^\alpha G$ is nice in $G/p^\alpha G$ since $p^\alpha G + T$ is nice in G - see, for example, [11].

(b) Suppose G/X is ω_1-$p^{\omega+n}$-projective for some nice $X \leq G$ with $p^m X = \{0\}$. Observe that the following relations are fulfilled:

$$p^\alpha G/(p^\alpha G \cap X) \cong (p^\alpha G + X)/X \subseteq G/X.$$

But a subgroup of an ω_1-$p^{\omega+n}$-projective group is again ω_1-$p^{\omega+n}$-projective (cf. [15], Corollary 1.6). Thus $p^\alpha G/(p^\alpha G \cap X)$ is ω_1-$p^{\omega+n}$-projective as well. Moreover, $p^\alpha G \cap X$ is obviously p^m-bounded and also, in accordance with [11], it is nice in $p^\alpha G$. So $p^\alpha G$ is weakly m-ω_1-$p^{\omega+n}$-projective.

Furthermore,

$$(G/X)/p^\alpha(G/X) = (G/X)/(p^\alpha G + X)/X \cong G/(p^\alpha G + X) \\ \cong (G/p^\alpha G)/(p^\alpha G + X)/p^\alpha G$$

is ω_1-$p^{\omega+n}$-projective too, owing to Proposition 3.1.

Besides, it is obviously seen that $p^m((p^\alpha G + X)/p^\alpha G) = (p^{\alpha+m} G + p^\alpha G)/p^\alpha G = \{0\}$, and in the case of niceness that $(p^\alpha G + X)/p^\alpha G$ is nice in $G/p^\alpha G$ because it is well known that $p^\alpha G + X$ is nice in G - see, for instance, [11].

(c) Accordingly, write $G/H = B \oplus R$, where B is countable and R is $p^{\omega+n}$-projective for some p^m-bounded nice subgroup H of G. But

$$p^\alpha G/(p^\alpha G \cap H) \cong (p^\alpha G + H)/H \\ = p^\alpha(G/H) = p^\alpha B \oplus p^\alpha R,$$

where $p^\alpha B$ is countable and $p^\alpha R$ is $p^{\omega+n}$-projective. Since $p^\alpha G \cap H$ is p^m-bounded and nice in $p^\alpha G$ (see [11]), we derive that $p^\alpha G$ is nice decomposably m-ω_1-$p^{\omega+m}$-projective, as stated.

Concerning the other part, the direct sum

$$(B/p^\alpha B) \oplus (R/p^\alpha R) \cong (G/H)/p^\alpha(G/H) \\ \cong G/(p^\alpha G + H) \cong (G/p^\alpha G)/(p^\alpha G + H)/p^\alpha G$$

is again a direct sum of a countable group and a $p^{\omega+n}$-projective group, because of the obvious facts that $B/p^\alpha B$ is countable and $R/p^\alpha R$ is $p^{\omega+n}$-projective. In this vein, it is self-evident that $(p^\alpha G + H)/p^\alpha G$ is bounded by

p^m and, in conjunction with [11], that $(p^\alpha G + H)/p^\alpha G$ is nice in $G/p^\alpha G$, as required.

(d) Given a p^m-bounded nice subgroup Y of G such that G/Y is $p^{\omega+n}$-projective. Hence $p^\alpha G/(p^\alpha G \cap Y) \cong (p^\alpha G + Y)/Y \subseteq G/Y$ is $p^{\omega+n}$-projective as well, with $p^\alpha G \cap Y$ being p^m-bounded and nice in $p^\alpha G$ (cf. [11]).

On the other hand, $(G/p^\alpha G)/(Y + p^\alpha G)/p^\alpha G \cong G/(Y + p^\alpha G) \cong (G/Y)/(Y + p^\alpha G)/Y = (G/Y)/p^\alpha(G/Y)$ is $p^{\omega+n}$-projective. Since $(Y + p^\alpha G)/p^\alpha G \cong Y/(Y \cap p^\alpha G)$ is p^m-bounded and nice in $G/p^\alpha G$ (see [11]), the assertion follows. ∎

Under some extra restrictions on α, we can say even more:

Proposition 3.7. *If G is a nice decomposably m-ω_1-$p^{\omega+n}$-projective group, then $G/p^{\alpha+m}G$ is nicely m-$p^{\omega+n}$-projective for every ordinal $\alpha \leq \omega + n$. In particular, $G/p^{\omega+m+n}G$ is nicely m-$p^{\omega+n}$-projective.*

Proof. By Definition 1.5, we write $G/H = B \oplus R$ where B is countable and R is $p^{\omega+n}$-projective for some p^m-bounded nice subgroup H of G. An appeal to the proof of Proposition 3.6 (e) gives that

$$(G/H)/p^\alpha(G/H) \cong G/(p^\alpha G + H) \cong (G/p^{\alpha+m}G)/(p^\alpha G + H)/p^{\alpha+m}G$$

is $p^{\omega+n}$-projective with $p^m((p^\alpha G + S)/p^{\alpha+m}G) = p^{\alpha+m}G/p^{\alpha+m}G = \{0\}$, so that the claim follows. The final part is an immediate consequence by taking $\alpha = \omega + n$. ∎

Proposition 3.8. *If G is a decomposably m-ω_1-$p^{\omega+n}$-projective group, then $p^\alpha G$ is decomposably m-ω_1-$p^{\omega+n}$-projective for all ordinals α. In particular, if $\alpha \geq \omega$, then $p^\alpha G$ is ω_1-$p^{\omega+m}$-projective.*

In addition, if G is a nice decomposably m-ω_1-$p^{\omega+n}$-projective group and $\alpha \geq \omega$, then $p^\alpha G$ is strongly ω_1-$p^{\omega+m}$-projective.

Proof. Using Definition 1.4, let $S \leq G[p^m]$ such that $G/S = B \oplus R$, where B is countable and R is $p^{\omega+n}$-projective. If $\alpha \geq \omega$, then one sees that $p^\alpha G/(p^\alpha G \cap S) \cong (p^\alpha G + S)/S \subseteq p^\alpha(G/S) = K \oplus P$, where K is countable and P is p^n-bounded. Hence $p^\alpha G/(p^\alpha G \cap S)$ is also such a direct sum of a countable group and a p^n-bounded group (which itself is a direct sum of cyclics) with p^m-bounded intersection $S \cap p^\alpha G$, so that $p^\alpha G$ is ω_1-$p^{\omega+m}$-projective.

If now $\alpha < \omega$ is finite, then $p^\alpha G/(p^\alpha G \cap S) \cong (p^\alpha G + S)/S = p^\alpha(G/S) = p^\alpha B \oplus p^\alpha R$ is again a direct sum of the countable group $p^\alpha B$ and the $p^{\omega+n}$-projective group $p^\alpha R$, as needed. That is why, in both cases, $p^\alpha G$ is decomposably m-ω_1-$p^{\omega+n}$-projective, too.

The final part follows easily since S being nice in G yields that $S \cap p^\alpha G$ is nice in $p^\alpha G$ (cf. [11]). ∎

We now strengthen the idea in the proof of Proposition 3.1 via the following statement; however we cannot yet establish that for all ordinals α, the Ulm factor $G/p^\alpha G$ possesses the decomposable m-ω_1-$p^{\omega+n}$-projectivity property provided the same for G.

Proposition 3.9. *If G is a decomposably m-ω_1-$p^{\omega+n}$-projective group, then $G/p^\alpha G$ is decomposably m-ω_1-$p^{\omega+n}$-projective for all ordinals $\alpha \geq \omega+n$.*

Proof. Utilizing Definition 1.4, write that $G/S = B \oplus R$, where B is countable and R is $p^{\omega+n}$-projective for some p^m-bounded subgroup S of G.

Standardly, the following isomorphisms are true:

$$(G/p^\alpha G)/(S + p^\alpha G)/p^\alpha G \cong G/(S + p^\alpha G) \cong (G/S)/(S + p^\alpha G)/S.$$

Moreover, $(S + p^\alpha G)/S \subseteq p^\alpha(G/S) = p^\alpha B$. Therefore, setting $T = (S + p^\alpha G)/S$, we deduce that

$$(G/S)/T = (B \oplus R)/T \cong (B/T) \oplus R$$

is again a direct sum of a countable group and a $p^{\omega+n}$-projective group. And since $p^m((p^\alpha G + S)/p^\alpha G) = \{0\}$, we are finished. ∎

Remark 5. When $\alpha = \omega$, we know in view of ([15], Corollary 2.2) that $G/p^\omega G$ is $p^{\omega+m+n}$-projective and thus by Proposition 2.10 it is decomposably m-ω_1-$p^{\omega+n}$-projective. The unsettled situation is when $\omega < \alpha < \omega + n$.

We are now in a position to establish our first central result which reduces the study of m-ω_1-$p^{\omega+n}$-projectives to these of length at most $\omega + m + n$.

Theorem 3.10. (First Reduction Criterion) *The group G is m-ω_1-$p^{\omega+n}$-projective if and only if the following two conditions are fulfilled:*

(1) *$p^{\omega+m+n}G$ is countable;*
(2) *$G/p^{\omega+m+n}G$ is m-ω_1-$p^{\omega+n}$-projective.*

Proof. "⇒". Since, as observed before, G is ω_1-$p^{\omega+m+n}$-projective, point (1) follows automatically appealing to [15]. Concerning point (2), it follows directly from Proposition 3.5.

"⇐". Assume now that clauses (1) and (2) are valid. For convenience put $k = m + n$. By definition, let $L/p^{\omega+k}G \leq G/p^{\omega+k}G$ be a p^m-bounded subgroup such that $(G/p^{\omega+k}G)/(L/p^{\omega+k}G) \cong G/L$ is strongly ω_1-$p^{\omega+n}$-projective. Thus $p^m L \subseteq p^{\omega+k}G$. Since G/L is $p^{\omega+k+m}$-bounded, we see that $p^{\omega+n}(G/L)$ is bounded (by p^{2m}), and applying Proposition 3.3 (ii) to G/L, we deduce that

$$(G/L)/(p^{\omega+n}G + L)/L \cong G/(p^{\omega+n}G + L)$$

is strongly ω_1-$p^{\omega+n}$-projective, because $(p^{\omega+n}G + L)/L \subseteq p^{\omega+n}(G/L)$. Putting $M = p^{\omega+n}G + L$, it is obvious that $p^{\omega+n}G \subseteq M$ and $p^m M = p^{\omega+k}G$.

That is why, G/M is strongly ω_1-$p^{\omega+n}$-projective with $M \leq G$ satisfying the above two relations.

Furthermore, suppose that Y is a maximal p^m-bounded summand of $p^{\omega+n}G$, so there is a direct decomposition $p^{\omega+n}G = X \oplus Y$ and, by what we have just shown above, the inclusions $X \subseteq p^{\omega+n}G \subseteq M$ are true. We can without loss of generality assume that X is countable because of the following reasons: Since $p^{\omega+k}G = p^m X$ is countable, it follows that $X = K \oplus Z$, where K is countable and Z is p^m-bounded. Therefore, $p^{\omega+n}G = K \oplus Z \oplus Y = K \oplus Y'$, where $Y' = Z \oplus Y$. We next routinely verify that $X[p] = (p^{\omega+k}G)[p]$ and thus $Y \cap p^{\omega+k}G = \{0\}$. So, suppose H is a $p^{\omega+k}$-high subgroup of G such that $H \supseteq Y$. Now, $G[p] = (p^{\omega+k}G)[p] \oplus H[p] = X[p] \oplus H[p]$ together with H being pure in G (cf. [11]) readily force that $G[p^m] = X[p^m] \oplus H[p^m]$, whenever $m \geq 1$.

Given $g \in G$ with $p^m g \in p^{\omega+k}G$, we write $p^m g = p^m a$, where $a \in p^{\omega+n}G = X \oplus Y$. Then $p^m g = p^m x$ for some $x \in X$, whence $g \in x + G[p^m] \subseteq X + H[p^m]$. Besides, $X \cap H[p^m] \subseteq X \cap H = \{0\}$ and consequently $(G/p^{\omega+k}G)[p^m] = (X \oplus H[p^m])/p^{\omega+k}G$ because $p^{\omega+k}G = p^m X \subseteq X$. Since $M/p^{\omega+k}G \subseteq (G/p^{\omega+k}G)[p^m]$, it follows that $M \subseteq X \oplus H[p^m]$ and hence $M = (X \oplus H[p^m]) \cap M = X + H[p^m] \cap M$ by virtue of the modular law. Substituting $P = H[p^m] \cap M$, we derive that $p^m P = \{0\}$ and that $M = X + P$. In addition, $M = M + p^{\omega+n}G = P + p^{\omega+n}G$ and so $G/(p^{\omega+n}G + P) \cong G/P/(p^{\omega+n}G + P)/P$ is strongly ω_1-$p^{\omega+n}$-projective.

We next claim that $p^{\omega+n}(G/P)$ is countable. In fact, $p^{\omega+n}(G/M)$ is countable because G/M is strongly ω_1-$p^{\omega+n}$-projective (see [3]). But we subsequently have that $p^{\omega+n}(G/M) = p^n(p^\omega(G/M)) = p^n(\cap_{i<\omega}(p^i G + M)/M) = p^n(\cap_{i<\omega}(p^i G + P)/M) \cong p^n(\cap_{i<\omega}(p^i G + P)/P/(M/P)) = p^n(p^\omega(G/P)/(M/P)) = [p^{\omega+n}(G/P) + (M/P)]/(M/P) = p^{\omega+n}(G/P)/(M/P)$ since $M/P = (p^{\omega+n}G + P)/P \subseteq p^{\omega+n}(G/P)$. Moreover, $M/P = M/(M \cap H[p^m]) \cong (M + H[p^m])/H[p^m] = (X + H[p^m])/H[p^m] \cong X/(X \cap H[p^m]) \cong X$ is countable. Finally, $p^{\omega+n}(G/P)$ is countable as well, as claimed.

Also, because $(p^{\omega+n}G + P)/P \leq p^{\omega+n}(G/P)$, Proposition 3.3 (ii) applied to G/P shows that G/P is strongly ω_1-$p^{\omega+n}$-projective with $p^m P = \{0\}$, as required. ∎

Imitating the method illustrated above, we have (compare with Remark 4 alluded to above):

Corollary 3.11. *Suppose that $\lambda \geq \omega$ is an ordinal and $p^\lambda G$ is countable. Then the group G is m-ω_1-$p^{\omega+n}$-projective if and only if $G/p^\lambda G$ is.*

Utilizing the same ideas as in the preceding First Reduction Criterion and its consequence, with Proposition 2.13 at hand, we deduce:

Theorem 3.12. (Second Reduction Criterion) *The group G is strongly m-ω_1-$p^{\omega+n}$-projective if and only if the following two conditions are fulfilled:*

(1) *$p^{\omega+m+n}G$ is countable;*
(2) *$G/p^{\omega+m+n}G$ is strongly m-ω_1-$p^{\omega+n}$-projective.*

Corollary 3.13. *Suppose that $p^{\lambda}G$ is countable for some ordinal $\lambda \geq \omega$. Then the group G is strongly m-ω_1-$p^{\omega+n}$-projective if and only if $G/p^{\lambda}G$ is.*

We furthermore have all the ingredients to prove our next basic result.

Theorem 3.14. (Third Reduction Criterion) *The group G is weakly m-ω_1-$p^{\omega+n}$-projective if and only if*

(1) *$p^{\omega+m+n}G$ is countable;*
(2) *$G/p^{\omega+m+n}G$ is weakly m-ω_1-$p^{\omega+n}$-projective.*

Proof. "$\Rightarrow$". It follows directly from [15] together with Propositions 3.6.

"$\Leftarrow$". For our convenience, set $k = m + n$. By definition, let $T/p^{\omega+k}G \leq G/p^{\omega+k}G$ be a p^m-bounded nice subgroup such that $(G/p^{\omega+k}G)/(T/p^{\omega+k}G) \cong G/T$ is ω_1-$p^{\omega+n}$-projective. Thus T is nice in G (see, e.g., [11]), and $p^mT \subseteq p^{\omega+k}G$. Applying Proposition 3.1 or Corollary 2.1 in [15], $G/(T + p^{\omega+n}G) \cong (G/T)/(T + p^{\omega+n}G)/T = (G/T)/p^{\omega+n}(G/T)$ is also ω_1-$p^{\omega+n}$-projective. Putting $T' = T + p^{\omega+n}G$, we see that G/T' is ω_1-$p^{\omega+n}$-projective and that $T' \supseteq p^{\omega+n}G$ remains nice in G and $p^mT' = p^mT + p^{\omega+k}G = p^{\omega+k}G$. So, replacing hereafter T' with T, we may assume that $p^{\omega+n}G \leq T$.

Suppose now that Y is a maximal p^m-bounded summand of $p^{\omega+n}G$; so there exists a decomposition $p^{\omega+n}G = X \oplus Y$ and thus the inclusions $X \subseteq p^{\omega+n}G \subseteq T$ hold. We may assume with no harm of generality that X is countable; in fact, $p^{\omega+k}G = p^mX$ is countable and therefore, we can decompose $X = K \oplus Z$, where K is countable and Z is p^m-bounded (whence Z is a p^m-bounded summand of $p^{\omega+n}G$ and so $Z \subseteq Y$). Consequently, it is readily checked that $p^{\omega+n}G = K \oplus Y$ with countable summand K, as wanted.

Next, a straightforward check shows that $X[p] = (p^{\omega+k}G)[p] = (p^mX)[p]$ and thus $Y \cap p^{\omega+k}G = \{0\}$ because $(Y \cap p^{\omega+k}G)[p] = Y \cap (p^{\omega+k}G)[p] = Y \cap X[p] = \{0\}$. Let now H be a $p^{\omega+k}$-high subgroup of G containing Y (thus H is maximal with respect to $H \cap p^{\omega+k}G = \{0\}$ with $H \supseteq Y$). We now assert that

$$(G/p^{\omega+k}G)[p^m] = (X \oplus H[p^m])/p^{\omega+k}G.$$

In fact, as noted above $X[p] = (p^{\omega+k}G)[p]$ and thereby $X \cap H = \{0\}$ because $(X \cap H)[p] = X[p] \cap H = (p^{\omega+k}G)[p] \cap H = \{0\}$.

Since $G[p] = (p^{\omega+k}G)[p] \oplus H[p] = X[p] \oplus H[p]$ and H is pure in G (see [11]), it plainly follows that $G[p^m] = X[p^m] \oplus H[p^m]$.

Furthermore, given $v \in G$ with $p^m v \in p^{\omega+k}G$, it suffices to show that $v \in X \oplus H[p^m]$. In fact, $p^m v = p^m d$ where $d \in p^{\omega+n}G = X \oplus Y$. Then $p^m d = p^m x$ for some $x \in X$ and so $p^m v = p^m x$. Therefore $v \in x + G[p^m] = x + X[p^m] + H[p^m] \subseteq X + H[p^m]$, as required. So, the assertion is sustained.

By what we have obtained above, $T/p^{\omega+k}G \subseteq (G/p^{\omega+k}G)[p^m] = (X \oplus H[p^m])/p^{\omega+k}G$ implies that $T \subseteq X \oplus H[p^m]$; note also that $X \subseteq T$. Put $L = T \cap H[p^m] \subseteq H$, so that it is clear that $L \cap p^{\omega+k}G = \{0\}$. Moreover, the modular law ensures that $T = (X \oplus H[p^m]) \cap T = X \oplus (T \cap H[p^m]) = X \oplus L$. We consequently conclude that $T = p^{\omega+n}G + T = p^{\omega+n}G + L$ and $G/T = G/(p^{\omega+n}G + L)$ is ω_1-$p^{\omega+n}$-projective. Observe also that L is p^m-bounded, and that L is nice in G. The first fact is trivial, as for the second one $L \cap p^{\omega+k}G = \{0\}$ easily forces that $L \cap p^{\omega+n}G$ is nice in $p^{\omega+n}G$ and thus nice in G. On the other hand, as noticed above, $p^{\omega+n}G + L = T$ is also nice in G. According to [11], these two conditions together imply that L is nice in G.

What remains to illustrate is that $p^{\omega+n}(G/L)$ is countable. Indeed, we have $p^{\omega+n}(G/L) = (p^{\omega+n}G + L)/L = T/L$. Also, $T/L = T/(T \cap H[p^m]) \cong (T + H[p^m])/H[p^m] = (p^{\omega+n}G + H[p^m])/H[p^m] \cong p^{\omega+n}G/(p^{\omega+n}G \cap H[p^m])$. But as obtained above, $p^{\omega+n}G = X \oplus Y$ and since $Y \subseteq H$, we have with the aid of the modular law that $p^{\omega+n}G \cap H = (X \oplus Y) \cap H = (X \cap H) \oplus Y = Y$, whence $p^{\omega+n}G \cap H[p^m] = Y[p^m]$. We therefore establish that $T/L \cong (X \oplus Y)/Y[p^m] \cong X \oplus (Y/Y[p^m]) \cong X \oplus p^m Y = X$. Since X is shown to be countable, so is $T/L = p^{\omega+n}(G/L)$. We finally apply Proposition 2.14 to get the desired claim. ∎

Mimicking the method demonstrated above, we state:

Corollary 3.15. *Let $\lambda \geq \omega$ be an ordinal such that $p^\lambda G$ is countable. Then the group G is weakly m-ω_1-$p^{\omega+n}$-projective if and only if $G/p^\lambda G$ is.*

Now, we are ready to state our next reduction theorem.

Theorem 3.16. (Fourth Reduction Criterion) *The group G is nicely m-ω_1-$p^{\omega+n}$-projective if and only if*

(1) *$p^{\omega+m+n}G$ is countable;*
(2) *$G/p^{\omega+m+n}G$ is nicely m-$p^{\omega+n}$-projective.*

Proof. "⇒". Clause (1) follows immediately from [15].

As for clause (2), it follows directly by Proposition 3.6.

"⇐". Assume that (1) and (2) are fulfilled, so that there exist a nice p^m-bounded subgroup $A/p^{\omega+m+n}G$ of $G/p^{\omega+m+n}G$ with $A \leq G$ such that G/A is $p^{\omega+n}$-projective. Thus, as we have seen before, $p^m A \subseteq p^{\omega+k}G$ for $k = m+n$, and A is nice in G. Imitating the same technique as in Theorems 3.10 and 3.14, we complete the arguments. ∎

Same as above, we derive:

Corollary 3.17. *Let $\lambda \geq \omega$ be an ordinal for which $p^\lambda G$ is countable. Then the group G is nicely m-ω_1-$p^{\omega+n}$-projective if and only if $G/p^\lambda G$ is.*

We will be now concentrated on nice decomposably m-ω_1-$p^{\omega+n}$-projectives, which are somewhat difficult to handle. So, we will restrict our attention on the ideal case $n = 1$ by showing that the investigation of nice decomposably m-ω_1-$p^{\omega+1}$-projectives can be reduced to these of length not exceeding $\omega + m + 1$. Specifically, the following holds:

Theorem 3.18. (Fifth Reduction Criterion) *The group G is nice decomposably m-ω_1-$p^{\omega+1}$-projective if and only if*

(1) *$p^{\omega+m+1}G$ is countable;*
(2) *$G/p^{\omega+m+1}G$ is nice decomposably m-ω_1-$p^{\omega+m+1}$-projective.*

Proof. The "and only if" part follows directly via the combination of [15] and Proposition 3.6, respectively.

Concerning the "if" part, we set for simpleness $k = m + 1$. Using the corresponding definition, suppose $T/p^{\omega+k}G \leq G/p^{\omega+k}G$ is a p^m-bounded nice subgroup such that $(G/p^{\omega+k}G)/(T/p^{\omega+k}G) \cong G/T$ is a direct sum of a countable group and a $p^{\omega+1}$-projective group. Hence T is nice in G (see, e.g., [11]), and $p^m T \subseteq p^{\omega+k}G$. Also, it is routinely checked that $(G/T)/p^{\omega+1}(G/T) \cong G/(T + p^{\omega+1}G)$ is $p^{\omega+1}$-projective. Henceforth, the proof goes on imitating the same scheme as that in Theorems 3.10 and 3.14. We finally employ Proposition 2.15 to infer the wanted statement. ∎

Remark 6. As observed in Proposition 3.6, the necessity is valid for any natural n. However, the sufficiency probably fails for each other $n > 1$.

4. Open questions

We close the work with certain challenging problems which are worthwhile for a further study.

Problem 1. Are n, m-simply presented groups G with countable $p^{\omega+m+n}G$ m-ω_1-$p^{\omega+n}$-projective?

Problem 2. Is it true that weakly n-ω_1-$p^{\omega+m}$-projective groups are m-ω_1-$p^{\omega+n}$-projective?

Problem 3. Does it follow that m-ω_1-$p^{\omega+n}$-projective groups are almost m-ω_1-$p^{\omega+n}$-projective?

Problem 4. Are m-ω_1-$p^{\omega+n}$-projective groups strongly ω_1-$p^{\omega+m+n}$-projective?

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