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HOPFIAN AND CO-HOPFIAN SUBSEMIGROUPS AND EXTENSIONS

Abstract.   This paper investigates the preservation of hopficity and co-hopficity on passing to finite-index subsemigroups and extensions. It was already known that hopficity is not preserved on passing to finite Rees index subsemigroups, even in the finitely generated case. We give a stronger example to show that it is not preserved even in the finitely presented case. It was also known that hopficity is not preserved in general on passing to finite Rees index extensions, but that it is preserved in the finitely generated case. We show that, in contrast, hopficity is not preserved on passing to finite Green index extensions, even within the class of finitely presented semigroups. Turning to co-hopficity, we prove that within the class of finitely generated semigroups, co-hopficity is preserved on passing to finite Rees index extensions, but is not preserved on passing to finite Rees index subsemigroups, even in the finitely presented case. Finally, by linking co-hopficity for graphs to co-hopficity for semigroups, we show that without the hypothesis of finite generation, co-hopficity is not preserved on passing to finite Rees index extensions.

1. Introduction

An algebraic or relational structure is *hopfian* if it is not isomorphic to any proper quotient of itself, or, equivalently, if any surjective endomorphism of the structure is an automorphism. An algebraic or relational structure is *co-hopfian* if it is not isomorphic to any proper substructure of itself, or, equivalently, if any injective endomorphism of the structure is an automorphism.

Hopficity was first introduced by Hopf, who asked if all finitely generated groups were hopfian [8]. The celebrated Baumslag–Solitar groups $\langle x, y \mid x^m y = y x^n \rangle$ provide the easiest counterexample: $\langle x, y \mid x^2 y = y x^3 \rangle$ is finitely generated, and indeed finitely presented, and non-hopfian; see [2, Theorem 1]. Furthermore, $\langle x, y \mid x^{12} y = y x^{18} \rangle$ is hopfian but contains a non-hopfian subgroup of finite index [2, Theorem 2]. Hence, hopficity is

	Preserved on passing to finite index	
Property	Subgroups	Extensions
Hopficity	N [by f.p. case]	? Qu. 5.3
Hopficity & f.g.	N [by f.p. case]	Y [7, Co. 2]
Hopficity & f.p.	N [2, Th. 2]	Y [by f.g. result]
Co-hopficity	? Qu. 5.4	? Qu. 5.4
Co-hopficity & f.g.	? Qu. 5.4	? Qu. 5.4
Co-hopficity & f.p.	? Qu. 5.4	? Qu. 5.4

Table 1. Summary, for groups, of the preservation of hopficity and co-hopficity on passing to finite index subgroups and extensions. [*Key:* f.g. = finite generation; f.p. = finite presentation; Y = property is preserved; N = property is not preserved in general; ? = open question.]

not preserved under passing to finite-index subgroups. On the other hand, a finite extension of a finitely generated hopfian group is also hopfian [7, Corollary 2]. There seems to have been no study of whether co-hopficity for groups is preserved on passing to finite-index subgroups or extensions, and it seems that the questions of the preservation of co-hopficity in each direction are both open. Table 1 summarizes the state of knowledge about the preservation of hopficity and co-hopficity on passing to finite-index subgroups and extensions.

There are two useful notions of index for semigroups. For a semigroup S with a subsemigroup T, the *Rees index* of T in S is $|S-T|+1$, and the *Green index* of T in S is the number of T-relative $\mathcal{H}$-classes in $S - T$. Rees index is more established, and many finiteness properties, such as finite generation and finite presentability, are known to be preserved on passing to or from subsemigroups of finite Rees index; see the brief summary in [15, § 11] or the comprehensive survey [5]. Green index is newer, but has the advantage that finite Green index is a common generalization of finite Rees index and finite group index, and some progress has been made in proving the preservation of finiteness properties on passing to or from subsemigroups of finite Green index; see [4, 6].

The second author and Ruškuc proved that a finite Rees index extension of a finitely generated hopfian semigroup is itself hopfian [13, Theorem 3.1], and gave an example to show that this no longer holds without the hypothesis of finite generation [13, § 2]. They also gave an example showing that hopficity is not preserved on passing to finite Rees index subsemigroups, even in the finitely generated case [13, § 5]. In this paper, we give an example showing that it is not preserved even in the finitely presented case (Example 3.1). We also give an example showing that, again even in the finitely presented case,

| | Preserved on passing to | | | |
| | Finite Rees index | | Finite Green index | |
Property	Subsemigroups	Extensions	Subsemigroups	Extensions
Hopficity	N [by f.g. case]	N [13, § 2]	N [by Rees index case]	N [by Rees index case]
Hopficity & f.g.	N [13, § 5]	Y [13, Th. 3.1]	N [by Rees index case]	N [by f.p. case]
Hopficity & f.p.	N Ex. 3.1	Y [by f.g. result]	N [by Rees index case]	N Ex. 3.2
Co-hopficity	N [by f.p. case]	N Ex. 4.7	N [by Rees index case]	N [by Rees index case]
Co-hopficity & f.g.	N [by f.p. case]	Y Th. 4.2	N [by Rees index case]	? Qu. 5.1
Co-hopficity & f.p.	N Ex. 4.1	Y [by f.g. result]	N [by Rees index case]	? Qu. 5.1

Table 2. Summary, for semigroups, of the preservation of hopficity and co-hopficity on passing to finite index subsemigroups and extensions. [*Key:* f.g. = finite generation; f.p. = finite presentation; Y = property is preserved; N = property is not preserved in general; ? = open question.]

a finite Green index extension of a hopfian semigroup need not be hopfian (Example 3.2), showing that the result of the second author and Ruškuc does not generalize to finite Green index.

Next we turn to co-hopficity. We prove that a finite Rees index extension of a finitely generated co-hopfian semigroup is itself co-hopfian (Theorem 4.2), and construct an example showing that this does not hold without the hypothesis of finite generation (Example 4.7). We also give an example of a non-co-hopfian finite Rees index subsemigroup of a finitely presented co-hopfian semigroup (Example 4.1).

Table 2 summarizes the state of knowledge for semigroups about the preservation of hopficity and co-hopficity on passing to finite Rees and Green index subsemigroups and extensions.

2. Preliminaries

2.1. Presentations and rewriting systems. The group presentation with (group) generators A and defining relations $\mathcal{R}$ (which may involve inverses of elements of A) is denoted $\mathrm{Gp}\langle A \mid \mathcal{R}\rangle$. The semigroup presentation with (semigroup) generators from A and defining relations $\mathcal{R}$ is denoted $\mathrm{Sg}\langle A \mid \mathcal{R}\rangle$. For a semigroup S presented by $\mathrm{Sg}\langle A \mid \mathcal{R}\rangle$ and words $u, v \in A^+$, write $u = v$ to indicate that u and v are equal as words, and write $u =_S v$ to indicate they represent the same element of S.

A *string rewriting system*, or simply a *rewriting system*, is a pair $(A, \mathcal{R})$, where A is a finite alphabet and $\mathcal{R}$ is a set of pairs (ℓ, r), often written $\ell \to r$, known as *rewriting rules*, drawn from $A^* \times A^*$. The single reduction relation $\to$ is defined as follows: $u \to v$ (where $u, v \in A^*$) if there exists a rewriting rule $(\ell, r) \in \mathcal{R}$ and words $x, y \in A^*$ such that $u = x\ell y$ and $v = xry$. The reduction relation $\to^*$ is the reflexive and transitive closure of $\to$. A word $w \in A^*$ is *reducible* if it contains a subword ℓ that forms the left-hand side of a rewriting rule in $\mathcal{R}$; it is otherwise called *irreducible*.

The string rewriting system $(A, \mathcal{R})$ is *Noetherian* if there is no infinite sequence $u_1, u_2, \ldots \in A^*$ such that $u_i \to u_{i+1}$ for all $i \in \mathbb{N}$. The rewriting system $(A, \mathcal{R})$ is *confluent* if, for any words $u, u', u'' \in A^*$ with $u \to^* u'$ and $u \to^* u''$, there exists a word $v \in A^*$ such that $u' \to^* v$ and $u'' \to^* v$. A rewriting system is *complete* if it is both confluent and Noetherian.

Let $(A, \mathcal{R})$ be a complete rewriting system. Then for any word $u \in A^*$, there is a unique irreducible word $v \in A^*$ with $u \to^* v$ [3, Theorem 1.1.12]. The irreducible words are said to be in *normal form*. The semigroup presented by $\mathrm{Sg}\langle A \mid \mathcal{R}\rangle$ may be identified with the set of normal form words under the operation of 'concatenation plus reduction to normal form'.

2.2. Indices. Let S be a semigroup and let T be a subsemigroup of S. The *Rees index* of T in S is defined to be $|S - T| + 1$. If T is an ideal of S, then the Rees index of T in S is cardinality of the Rees factor semigroup $S/T = (S - T) \cup \{0\}$.

To define the Green index of T in S, we must first define the T-relative Green's relations on S. As usual, S^1 denotes the semigroup S with an identity element adjoined. Extend this notation to subsets of S: that is, $X^1 = X \cup \{1\}$ for $X \subseteq S$. Define the *T-relative Green's relations* $\mathcal{R}^T$, $\mathcal{L}^T$, and $\mathcal{H}^T$ on the semigroup S by

$$x \, \mathcal{R}^T \, y \Leftrightarrow xT^1 = yT^1; \qquad x \, \mathcal{L}^T \, y \Leftrightarrow T^1 x = T^1 y; \qquad \mathcal{H}^T = \mathcal{R}^T \cap \mathcal{L}^T \ .$$

Each of these relations is an equivalence relation on S. When $T = S$, they coincide with the standard Green's relations on S. Furthermore, these relations respect T, in the sense that each $\mathcal{R}^T$-, $\mathcal{L}^T$-, and $\mathcal{H}^T$-class lies either wholly in T or wholly in $S - T$. Following [6], define the *Green index* of T in S to be one more than the number of $\mathcal{H}^T$-classes in $S - T$. If S and T are groups, then T has finite group index in S if and only if it has finite Green index in S [6, Proposition 8].

3. Hopficity

It is known that the hopficity is not preserved on passing to finite Rees index subsemigroups, even for finitely generated semigroups [13, § 5]. The following example shows that within the class of finitely *presented* semigroups, and even within the class of semigroups presented by finite complete rewriting systems, hopficity is not preserved on passing to finite Rees subsemigroups. This example has already appeared in the second author's Ph.D. thesis [12, Examples 5.6.1 & 5.6.2].

EXAMPLE 3.1. Let

$$(3.1) \qquad\qquad T = \mathrm{Sg}\langle a, b \mid abab^2 ab = b \rangle.$$

Notice that

$$abab^3 =_T abab^2 (abab^2 ab) = (abab^2 ab)ab^2 ab =_T bab^2 ab.$$

It is easy to check that the rewriting system $(\{a, b\}, \{abab^2 ab \to b, abab^3 \to bab^2 ab\})$ is confluent and Noetherian. Clearly, T is also presented by

$$\mathrm{Sg}\langle a, b \mid (abab^2 ab, b), (abab^3, bab^2 ab) \rangle.$$

Define an endomorphism

$$\phi : T \to T; \qquad\qquad a \mapsto a, \qquad b \mapsto bab.$$

This endomorphism is well defined since the words on the two sides of the defining relation in the presentation (3.1) for T are mapped by ϕ to the same

element of T:

$$(abab^2ab)\phi =_T a\ bab\ a\ (bab)^2\ a\ bab$$
$$= abab\ abab^2ab\ abab$$
$$\rightarrow abab^2ab\ ab$$
$$\rightarrow bab$$
$$= b\phi.$$

Since $a\phi = a$ and

(3.2) $$(ab^2)\phi =_T a(bab)^2 =_T abab^2ab \rightarrow b,$$

the endomorphism ϕ is surjective. Furthermore, applying (3.2) shows that

$$(ab^2a^2b^2)\phi =_T (ab^2\ a\ ab^2)\phi =_T bab =_T b\phi.$$

But both $ab^2a^2b^2$ and b are irreducible and so $ab^2a^2b^2 \neq_T b$. Hence ϕ is not bijective and so not an automorphism. This proves that T is not hopfian.

Let

$$S = \mathrm{Sg}\langle a, b, f \mid abab^2ab = b, fa = ba, af = ab, fb = bf = f^2 = b^2\rangle.$$

Notice that $S = T \cup \{f\}$ since all products of two or more generators (regardless of whether they include generators f) must lie in T. So T is a finite Rees index subsemigroup of S. Notice further that since T is presented by a finite complete rewriting system, so is S [16, Theorem 1].

Let $\psi : S \rightarrow S$ be a surjective endomorphism. Since a and f are the only indecomposable elements of S, we have $\{a, f\}\psi = \{a, f\}$. Let $\vartheta = \psi^2$; then ϑ is a surjective endomorphism of S with $a\vartheta = a$ and $f\vartheta = f$.

If $b\vartheta = f$ then $f =_S b\vartheta =_S (abab^2ab)\vartheta =_S afaf^2af =_S abab^2ab =_S b$, which is a contradiction. Hence $b\vartheta = w \in T$. Then

$$ab =_S af = (a\vartheta)(f\vartheta) =_S (af)\vartheta =_S (ab)\vartheta = (a\vartheta)(b\vartheta) = aw.$$

Now, ab and aw lie in the subsemigroup T and so $ab =_T aw$. But T is left-cancellative by Adjan's theorem [1]; hence $b =_T w$ and so $b =_S w$. That is, $b\vartheta = w =_S b$. Since $a\vartheta = a$ and $f\vartheta = f$, the endomorphism ϑ must be the identity mapping on S and so bijective. Hence ψ is bijective and so an automorphism. This proves that S is hopfian.

Therefore S is a hopfian semigroup, finitely presented by a complete rewriting system, with a non-hopfian subsemigroup T of finite Rees index, which is also finitely presented by a finite complete rewriting system.

We now give an example to show that a finite Green index extension of a finitely generated (and, indeed, finitely presented) hopfian semigroup is not necessarily hopfian, in contrast to the situation for finite Rees index [13, Theorem 3.1].

EXAMPLE 3.2. Let G and H be the groups presented by

$$G = \mathrm{Gp}\langle a, b, c \mid a^{-1}ba = b^2, bc = cb \rangle,$$
$$H = \mathrm{Gp}\langle a', b', c' \mid a'^{-1}b'a' = b'^2, b'c' = c'b',$$
$$a'b'^{-1}a'^{-1}c'^{-1}a'b'a'^{-1}c'a'b'^{-1}a'^{-1}c'^{-1}a'b'a'^{-1}c' = 1 \rangle.$$

These groups were defined by Neumann [14, p. 543–4], except that he used redundant generators $b_1' = a'b'a'^{-1}$ and $d' = b_1'^{-1}c'^{-1}b_1'c'$ to shorten the presentation of H; we have removed the redundant generators to clarify the reasoning that follows. Let

$$\lambda : G \to H; \qquad a\lambda = a', \qquad b\lambda = b', \qquad c\lambda = c';$$
$$\mu : H \to G; \qquad a'\mu = a, \qquad b'\mu = a^{-1}ba, \qquad c'\mu = c.$$

The map λ is obviously a well-defined surjective homomorphism; Neumann [14, p. 544] showed that μ is also a well-defined surjective homomorphism, and that neither λ nor μ is injective. That is, G and H are proper homomorphic images of each other under the surjective homomorphisms λ and μ. [Neumann defined $b'\mu = b^2$, but since $a^{-1}ba = b^2$ by the defining relations of G, our modified definition is equivalent.] Furthermore, G and H are non-isomorphic [14, Theorem on p. 544].

Let $\vartheta = \lambda \circ \mu$. Then $\vartheta : G \to G$ is a surjective endomorphism of G that is not an isomorphism. Notice that

$$a\vartheta = a, \qquad b\vartheta = a^{-1}ba, \qquad c\vartheta = c.$$

Let F be the free group with basis $\{x, y, z\}$. Define a homomorphism

$$\phi : F \to G; \qquad x\phi = a, \qquad y\phi = b, \qquad z\phi = c.$$

Partially order $\{F, G\}$ by $F > G$. Let S be the Clifford semigroup formed from the groups F and G with the order $\geq$ and the homomorphism ϕ. [See [9, § 4.2] for the definition of Clifford semigroups.]

Clearly F is a subsemigroup of S. Since the homomorphism ϕ is surjective, any element of G can be right-multiplied (in S) by an element of F to give any other element of G; thus all elements of G are related by $\mathcal{R}^F$. Similarly, all elements of G are $\mathcal{L}^F$-related and so $\mathcal{H}^F$-related. Therefore G is the unique $\mathcal{H}^F$-class in $S - F$ and so F has finite Green index in S.

Define an endomorphism

$$\psi : S \to S; \qquad x\psi = x, \qquad y\psi = x^{-1}yx, \qquad z\psi = z,$$
$$a\psi = a, \qquad b\psi = a^{-1}ba, \qquad c\psi = c.$$

It is easy to see that ψ is a homomorphism as a consequence of $\psi|_G = \vartheta : G \to G$ being a homomorphism. Since $\psi|_G = \vartheta$ is surjective, we have $G \subseteq \mathrm{im}\,\psi$. Since $\{x, y, z\}\psi = \{x, x^{-1}yx, z\}$ generates F (as a group), we see

that $F \subseteq \operatorname{im} \psi$. So ψ is surjective. However, since $\psi|_G = \vartheta$ is not injective, ψ is not injective. Hence S is not hopfian.

Finally, note that the finitely generated free group F is hopfian [11, Proposition I.3.5], and that S is finitely presented [10, Theorem 5.1]. Therefore, F is a finitely presented hopfian semigroup with a finitely presented non-hopfian extension S of finite Green index.

4. Co-hopficity

The following example exhibits a finitely generated co-hopfian semigroup S with a non-co-hopfian subsemigroup T of finite Rees index, showing that co-hopficity is not preserved on passing to finite Rees index subsemigroups, even in the finitely generated (and, indeed, finitely presented) case:

EXAMPLE 4.1. Let T be the free semigroup with basis x. Then any map $x \mapsto x^k$ extends to an injective endomorphism from T to itself; for $k \geq 2$ this endomorphism is not bijective and so not an automorphism. Thus T is not co-hopfian.

Let

$$S = \operatorname{Sg}\langle x, y \mid y^2 = xy = yx = x^2 \rangle.$$

Notice that $S = T \cup \{y\}$ since all products of two or more generators must lie in T. So T is a finite Rees index subsemigroup of S. It is easy to check that the rewriting system $(\{x, y\}, \{y^2 \to x^2, xy \to x^2, yx \to x^2\})$ is confluent and Noetherian. Identify S with the set of irreducible words with respect to this rewriting system. The Cayley graph of S with respect to $\{x, y\}$ is shown in Figure 1.
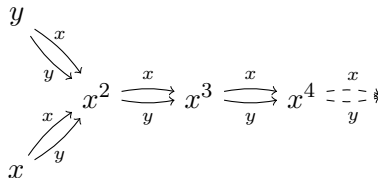

Fig. 1. The Cayley graph of the semigroup S from Example 4.1 with respect to the generating set $\{x, y\}$.

Let $\phi : S \to S$ be an injective endomorphism. Suppose for *reductio ad absurdum* that $x\phi = x^k$ with $k \geq 2$. Then $(y\phi)^2 = (y^2)\phi = (x^2)\phi = x^{2k}$, and so $y\phi = x^k = x\phi$ since the unique square root of x^{2k} in S is x^k, which contradicts the injectivity of ϕ. Hence either $x\phi = x$ or $x\phi = y$. In the former case, $x^\ell\phi = x^\ell$ for all $\ell \in \mathbb{N}$ and so $y\phi = y$ by the injectivity of ϕ; hence ϕ is surjective. In the latter case, $x^\ell\phi = y^\ell \to^* x^\ell$ for all $\ell \geq 2$ and so $y\phi = x$ by the injectivity of ϕ; hence ϕ is surjective. In either case, ϕ is a bijection and so an automorphism. Hence, S is co-hopfian.

Therefore, S is a co-hopfian semigroup presented by a finite complete rewriting system, with a non-co-hopfian subsemigroup T of finite Rees index, which is also finitely presented by a finite complete rewriting system (since it is free).

We have a positive result for passing to finite Rees index extensions in the finitely generated case:

THEOREM 4.2. *Let S be a semigroup and T a subsemigroup of S of finite Rees index. Suppose T is finitely generated and co-hopfian. Then S is co-hopfian.*

Notice that, in Theorem 4.2, S is also finitely generated.

Proof. Let X be a finite generating set for T and let $\phi : S \to S$ be an injective endomorphism. Let $t \in T$. Consider the images $t\phi$, $t\phi^2$, …. If $t\phi^i = t\phi^j$ for $i < j$, then the injectivity of ϕ forces $t\phi^{j-i} = t$ and so $t\phi^{\ell(j-i)} \in T$ for all $\ell \in \mathbb{N}$. On the other hand, if the elements $t\phi$, $t\phi^2$, … are all distinct, then since $S - T$ is finite, $t\phi^\ell \in T$ for all sufficiently large ℓ. In either case, there exist some $k_t, m_t \in \mathbb{N}$ such that $t\phi^{\ell m_t} \in T$ for all $\ell \geq k_t$. Let $k = \max\{k_t : t \in X\}$ and $m = \mathrm{lcm}\{m_t : t \in X\}$; both k and m exist because X is finite. Then $X\phi^{km} \subseteq T$, and so $T\phi^{km} \subseteq T$ since X generates T.

Since $\phi : S \to S$ is an injective endomorphism, so is $\phi^{km} : S \to S$. Hence $\phi^{km}|_T$ is an injective endomorphism from T to T. Since T is co-hopfian, $\phi^{km}|_T : T \to T$ is a bijection. Therefore, $\phi^{km}|_{S-T}$ must be an injective map from $S - T$ to $S - T$, and hence a bijection since $S - T$ is finite. Thus $\phi^{km} : S \to S$ is a bijection, and hence so is ϕ.

Therefore, any injective endomorphism from S to itself is bijective and so an automorphism. Thus S is co-hopfian. ∎

We will shortly exhibit an example showing that Theorem 4.2 does not hold without the hypothesis of finite generation. First, we need to define a construction that builds a semigroup from a simple graph and establish some of its properties.

DEFINITION 4.3. Let Γ be a simple graph. Let V be the set of vertices of Γ. Let $S_\Gamma = V \cup \{e, n, 0\}$. Define a multiplication on S_Γ by

$$v_1 v_2 = \begin{cases} e & \text{if there is an edge between } v_1 \text{ and } v_2 \text{ in } \Gamma, \\ n & \text{if there is no edge between } v_1 \text{ and } v_2 \text{ in } \Gamma, \end{cases} \quad \text{for } v_1, v_2 \in V,$$

$$ve = ev = vn = nv = 0, \qquad\qquad\qquad\qquad\quad \text{for } v \in V,$$

$$en = ne = e^2 = n^2 = 0,$$

$$0x = x0 = 0, \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{for } x \in S_\Gamma.$$

Notice that all products of two elements of S_Γ lie in $\{e, n, 0\}$ and all products of three elements are equal to 0. Thus this multiplication is associative and S_Γ is a semigroup.

We emphasize that Definition 4.3 only applies to simple graphs.

LEMMA 4.4. *Let Γ be a graph and let Δ be an induced subgraph of Γ. Then the vertex set of Δ is cofinite in the vertex set of Γ if and only if S_Δ is a finite Rees index subsemigroup of S_Γ.*

Proof. Suppose Γ has the vertex set V and Δ has the vertex set W. The result is immediate from the fact that $S_\Gamma - S_\Delta = (V \cup \{e, n, 0\}) - (W \cup \{e, n, 0\}) = V - W$. ■

The following two lemmata relate the co-hopficity of a graph Γ and the semigroup S_Γ. A homomorphism of graphs $\phi : \Gamma \to \Gamma'$ is a mapping from the vertex set of Γ to the vertex set of Γ' that preserves edges: that is, for all vertices v_1 and v_2 of Γ, if (v_1, v_2) is an edge of Γ, then $(v_1\phi, v_2\phi)$ is an edge of Γ'. Note, however, that the converse is not required to hold: it is possible that (v_1, v_2) is *not* an edge of Γ, but $(v_1\phi, v_2\phi)$ *is* an edge of Γ'. As with other types of relational or algebraic structure, a graph is co-hopfian if every injective endomorphism is an automorphism.

LEMMA 4.5. *If the graph Γ is co-hopfian, the semigroup S_Γ is co-hopfian.*

Proof. Let V be the vertex set of Γ and let $X = \{e, n, 0\}$, so that $S_\Gamma = V \cup X$.

Suppose Γ is co-hopfian; the aim is to show that S_Γ is co-hopfian. Let $\phi : S_\Gamma \to S_\Gamma$ be an injective endomorphism. Since X is the unique three-element null subsemigroup of S_Γ, we have $X\phi = X$ and so $V\phi \subseteq V$ since ϕ is injective. Furthermore, $0\phi = e^2\phi = (e\phi)^2 = 0$. Let $v \in V$; note that $v\phi \in V\phi \subseteq V$. Since Γ is simple, there are no loops at v or $v\phi$, and so we have $v^2 = n$ and $(v\phi)^2 = n$. Hence $n\phi = v^2\phi = (v\phi)^2 = n$. Therefore, since $X\phi = X$, it follows that $e\phi = e$. Let $v_1, v_2 \in V$. Then

$$\text{there is an edge between } v_1 \text{ and } v_2 \text{ in } \Gamma$$
$$\iff v_1 v_2 = e$$
$$\iff (v_1 v_2)\phi = e\phi$$
$$\iff (v_1\phi)(v_2\phi) = e$$
$$\iff \text{there is an edge between } v_1\phi \text{ and } v_2\phi \text{ in } \Gamma.$$

Hence $\phi|_V : V \to V$ is an injective endomorphism of Γ. Since Γ is co-hopfian, $\phi|_V$ is a bijection. Since $\phi|_X$ is a bijection, it follows that $\phi : S_\Gamma \to S_\Gamma$ is a bijection. This proves that S_Γ is co-hopfian. ■

LEMMA 4.6. *If Γ is a tree and the semigroup S_Γ is co-hopfian, the graph Γ is co-hopfian.*

Proof. Let V be the vertex set of Γ and let $X = \{e, n, 0\}$, so that $S_\Gamma = V \cup X$.

Let Γ be a tree and suppose that S_Γ is co-hopfian; the aim is to show that Γ is co-hopfian. Let $\phi : \Gamma \to \Gamma$ be an injective endomorphism. Extend ϕ to a map $\hat{\phi} : S_\Gamma \to S_\Gamma$ by defining $n\hat{\phi} = n$, $e\hat{\phi} = e$, and $0\hat{\phi} = 0$. Notice that $\hat{\phi}$ is injective since ϕ is injective. We now have to check the homomorphism condition for $\hat{\phi}$ in various cases. Let $v_1, v_2 \in V$. Then either $v_1 v_2 = e$ or $v_1 v_2 = n$; we consider these cases separately:

- If $v_1 v_2 = e$, then there is an edge between v_1 and v_2 in Γ, and so, since ϕ is an endomorphism of Γ, there is an edge between $v_1 \phi$ and $v_2 \phi$ in Γ, and thus $(v_1 \phi)(v_2 \phi) = e$. Therefore $v_1 v_2 = e$ implies $(v_1 v_2)\hat{\phi} = e\hat{\phi} = e = (v_1 \phi)(v_2 \phi) = (v_1 \hat{\phi})(v_2 \hat{\phi})$.
- If $v_1 v_2 = n$, then there is no edge between v_1 and v_2. Since Γ is a tree and thus connected, there is a path $\pi = (v_1 = x_1, x_2, \ldots, x_n = v_2)$ from v_1 to v_2. Since there is no edge between v_1 and v_2, we have $n \geq 3$. Since ϕ is an injective endomorphism of Γ, there is a path $\pi\phi = (v_1 \phi = x_1 \phi, x_2 \phi, \ldots, x_n \phi = v_2 \phi)$ from $v_1 \phi$ to $v_2 \phi$. In particular, injectivity means that $v_1 \phi$ and $v_2 \phi$ are not among the intermediate vertices $x_2 \phi, \ldots, x_{n-1} \phi$. Now, if there were an edge between $v_1 \phi$ and $v_2 \phi$, then this edge and the path $\pi\phi$ would form a non-trivial cycle, contradicting the fact that Γ is a tree. Hence there is no edge between $v_1 \phi$ and $v_2 \phi$. Therefore $v_1 v_2 = n$ implies $(v_1 v_2)\hat{\phi} = n\hat{\phi} = n = (v_1 \phi)(v_2 \phi) = (v_1 \hat{\phi})(v_2 \hat{\phi})$.

Since any product where at least one of the element is not from V is equal to 0, it is easy to see that the endomorphism condition holds in these cases. Hence $\hat{\phi} : S_\Gamma \to S_\Gamma$ is an injective endomorphism. Since S_Γ is co-hopfian, $\hat{\phi}$ is a bijection, and so $\phi = \hat{\phi}|_V$ is a bijection. This proves that Γ is co-hopfian. ∎

We can now present the example showing that Theorem 4.2 no longer holds without the hypothesis of finite generation:
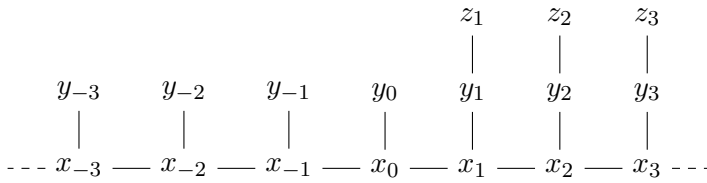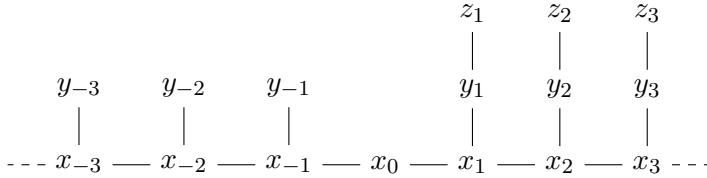
EXAMPLE 4.7. Define a graph Γ as follows. The vertex set is

$$V = \{x_i, y_i : i \in \mathbb{Z}\} \cup \{z_j : j \in \mathbb{N}\},$$

and there are edges between x_i and y_i for all $i \in \mathbb{Z}$, between y_j and z_j for all $j \in \mathbb{N}$, and between x_i and x_{i+1} for all $i \in \mathbb{Z}$. The graph Γ is as shown in Figure 2. Let Δ be the subgraph induced by $W = V - \{y_0\}$; the graph Δ is as shown in Figure 3. Note that Γ and Δ are trees and in particular simple.

Define a map

$$\phi : V \to V; \qquad \begin{aligned} x_i &\mapsto x_{i+1}, & \text{for all } i \in \mathbb{Z}, \\ y_i &\mapsto y_{i+1}, & \text{for all } i \in \mathbb{Z}, \\ z_i &\mapsto z_{i+1}, & \text{for all } i \in \mathbb{N}. \end{aligned}$$

$$
\begin{array}{ccccccc}
 & & & & z_1 & z_2 & z_3 \\
 & & & & | & | & | \\
y_{-3} & y_{-2} & y_{-1} & y_0 & y_1 & y_2 & y_3 \\
| & | & | & | & | & | & | \\
\text{---}\; x_{-3} \;\text{---}\; & x_{-2} \;\text{---}\; & x_{-1} \;\text{---}\; & x_0 \;\text{---}\; & x_1 \;\text{---}\; & x_2 \;\text{---}\; & x_3 \;\text{---}
\end{array}
$$

Fig. 2. The graph Γ from Example 4.7.

$$
\begin{array}{ccccccc}
 & & & & z_1 & z_2 & z_3 \\
 & & & & | & | & | \\
y_{-3} & y_{-2} & y_{-1} & & y_1 & y_2 & y_3 \\
| & | & | & & | & | & | \\
\text{---}\; x_{-3} \;\text{---}\; & x_{-2} \;\text{---}\; & x_{-1} \;\text{---}\; & x_0 \;\text{---}\; & x_1 \;\text{---}\; & x_2 \;\text{---}\; & x_3 \;\text{---}
\end{array}
$$

Fig. 3. The cofinite subgraph Δ of the graph Γ from Example 4.7.

It is easy to see that ϕ is an injective endomorphism of Γ. However, ϕ is not a bijection since $z_1 \notin \operatorname{im} \phi$. Thus, the graph Γ is not co-hopfian.

Suppose $\psi : W \to W$ is an injective endomorphism of Δ. Clearly, ψ must preserve adjacency in Δ. So the bi-infinite path through the vertices x_i must be mapped into itself. The preservation of adjacency requires that this path is mapped *onto* itself. All vertices on this path have degree 3 except x_0. Hence $x_0\psi = x_0$. The preservation of adjacency requires that either $x_i\psi = x_{-i}$ or $x_i\psi = x_i$ for all $i \in \mathbb{Z}$. The former case is impossible since it would force $y_i\psi = y_{-i}$ for all $i \in \mathbb{Z}$, but y_1 has degree 2 and y_{-1} has degree 1. Hence, the latter case holds, which forces $y_i\psi = y_i$ for all $i \in \mathbb{Z}$, and then $z_j\psi = z_j$ for all $j \in \mathbb{N}$. Hence ψ is the identity map and so bijective. Thus the subgraph Δ is co-hopfian.

By Lemma 4.4, S_Δ is a finite Rees index subsemigroup of S_Γ. By Lemma 4.5, S_Δ is co-hopfian. By Lemma 4.6, S_Γ is not co-hopfian.

5. Open questions

For semigroups, the main open problem in this area seem to be whether Theorem 4.2 generalizes to finite Green index extensions:

QUESTION 5.1. Let S be a semigroup and T a subsemigroup of finite Green index. Suppose T is finitely generated (or even finitely presented), so that S is finitely generated [4, Theorem 4.3]. If T is co-hopfian, must S be co-hopfian?

Notice that because finite Green index generalizes finite group index, this question subsumes the corresponding question for group extensions.

Since relative finiteness and finite presentability are not preserved on passing to finite Green index extensions unless the relative Schützenberger

groups of the relative $\mathcal{H}$-classes in the complement have the relevant property (see [6, Theorem 20] and [4, Example 6.5]), it is natural to ask the following question:

QUESTION 5.2. Let S be a semigroup and T a subsemigroup of finite Green index. Suppose T is finitely generated, so that S is finitely generated [4, Theorem 4.3] and the T-relative Schützenberger groups of the $\mathcal{H}^T$-classes in $S - T$ are finitely generated [4, Theorem 5.1]. If T is hopfian, and the T-relative Schützenberger groups of the $\mathcal{H}^T$-classes in $S - T$ are hopfian, must S be hopfian?

If the answer to Question 5.1 is 'no', then the question from the previous paragraph should be asked for co-hopficity: if T is co-hopfian, and the T-relative Schützenberger groups of the $\mathcal{H}^T$-classes in $S - T$ are co-hopfian, must S be co-hopfian?

For groups, the following question still seems to be open:

QUESTION 5.3. Is hopficity for groups preserved under passing to finite index extensions? (That is, does Hirshon's result [7, Corollary 2] hold without the hypothesis of finite generation?)

Finally, none of the relevant questions on co-hopficity for groups have been studied:

QUESTION 5.4. Is co-hopficity for groups preserved under passing to finite index subgroups and extensions? What about within the classes of finitely generated or finitely presented groups?

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