DE GRUYTER
OPEN

DEMONSTRATIO MATHEMATICA
Vol. XLVII No 3 2014

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SOLUTIONS TO THE QUASISTATIC PROBLEM FROM THE THEORY OF INELASTIC DEFORMATIONS WITH LINEAR GROWTH CONDITION

Abstract. This paper refers to standard models in the theory of inelastic deformations. We assume that non-linear inelastic constitutive function is of monotone type, that the growth condition holds and that the model is quasistatic. Initial, generic problem is transformed into an evolution equation in a maximal monotone field. Then we find solutions with very low regularity requirements of the forces acting on a body.

1. Problem formulation

Suppose we have a body represented by $U \subset \mathbf{R^n}$, made of a solid material, and some forces act on the body. For any point at any time, the material is of the same density $\rho > 0$. Let $u(t,x)$ denote the displacement of the point $x \in U$ of the body at the time $t \geq 0$. We focus on the symmetric part of the gradient of u: $\epsilon = \epsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$, which is called the linear strain tensor in the case of the small deformations. Moreover ϵ^p is a plastic strain tensor which is also symmetric. We assume the the body is made of a viscoplastic material. It implies that a deformation of the body is decomposed to an elastic part and an inelastic part i.e. ϵ^p. The inelastic part is controlled by the following constitutive equation:

$$\epsilon^p_t = M(\epsilon(u), \epsilon^p),$$

where M is the non-linear operator with the domain $Dom\ M \subset \mathbf{R^{n\times n}} \times \mathbf{R^{n\times n}}$. The function M is called the constitutive function or the inelastic constitutive function. The value of the function M is a parameter of a given material and it is established through engineering experiments. For details on the constitutive functions and models in the theory of inelastic deformations we refer to [1].

2010 *Mathematics Subject Classification*: 74C10.
Key words and phrases: inelastic deformation theory, Yosida approximation.

DOI: 10.2478/dema-2014-0061

We define $T(t,x)$, stress tensor, using the elastic constitutive equation:

$$T = \mathcal{D}(\epsilon(u) - \epsilon^p),$$

where $\mathcal{D}$ is a linear operator with constant coefficients, $\mathcal{D}$ is symmetric and positive definite.

The law of conservation of momentum holds:

$$\rho\, u_{tt} = \operatorname{div} T + f,$$

where f represents the exterior forces acting on the body.

Coupling it together with the initial and boundary conditions, we obtain the initial model:

$$\begin{cases} \rho\, u_{tt} = \operatorname{div} T + f, \\ T = \mathcal{D}(\epsilon(u) - \epsilon^p), \\ \epsilon^p_t = M(\epsilon(u), \epsilon^p), \\ u|_{\partial U} = 0, \\ u(0,x) = u^{(1)}(x), \\ u_t(0,x) = u^{(2)}(x), \\ \epsilon^p(0,x) = \epsilon^{p(0)}(x), \end{cases}$$

where $u^{(1)}$, $u^{(2)}$, $\epsilon^{p(0)}$ are the given initial data.

In engineering applications, the dimension $n = 1, 2, 3$ are considered, where the notions used in the above model have physical meaning. The methods used in this paper do not require such assumption, hence we assume just $n \geq 1$.

We consider this as a quasistatic problem i.e. we assume that ρu_{tt} is very small and can be neglected. Consequently, the law of conservation of momentum is replaced by the balance of forces (the sign is ignored):

$$\operatorname{div} T = f.$$

We assume that $\mathcal{D}$ is an identity. By combining the balance of forces with the elastic constitutive equation, we get the simple elliptic equation:

$$\operatorname{div}(\epsilon(u) - \epsilon^p) = f.$$

We consider the model to be of the monotone type. We say that model is of the monotone type if the inelastic constitutive function M is a maximal monotone operator and depends only on the stress tensor T. For details on the monotone type of models, we refer to [1].

Having the above assumptions, we can state the main problem:

$$(MP) \qquad \begin{cases} \operatorname{div}(\epsilon(u)(t,x) - \epsilon^p(t,x)) = f(t,x), \\ \epsilon^p_t(t,x) = M(\epsilon(u)(t,x) - \epsilon^p(t,x)), \\ u(t,x) = 0 \quad \text{if} \quad x \in \partial U, \\ \epsilon^p(x,0) = \epsilon^{p(0)}(x), \end{cases}$$

$U \subset \mathbf{R^n}$ is the open bounded set with smooth boundary,

$M : \mathbf{R^{n\times n}} \to \mathbf{R^{n\times n}}$ is a maximal monotone vector field, additionally we assume that $M(0) = 0$,

$\epsilon^{p(0)} : U \to \mathbf{R^{n\times n}}$ is the given initial value,

$(0,T)$ is the time interval, T is given,

$f : (0,T) \times U \to \mathbf{R^n}$ is given,

$u : (0,T) \times U \to \mathbf{R^n}$ and $\epsilon^p : (0,T) \times U \to \mathbf{R^{n\times n}}$ are unknown.

Although the resulting model is quite simplified, its behaviour and contained problems are very similar to the initial model. Understanding the problems in the resulting model, helps to understand the problems in the initial model. Methods presented in this paper can be used to find solutions to similar models in the theory of inelastic deformations.

We need a growth condition also known as the sublinear growth condition.

Definition. M fulfils a growth condition if there exists a constant C such that for all $x \in \mathbf{R^n}$, $|M(x)| \leq C(1 + |x|)$ holds.

The aim is to prove the following theorem.

Theorem 1. *Assume M satisfies the growth condition i.e. there exists the constant $C > 0$ such that*

$$|M(x)| \leq C(1 + |x|)$$

holds for every $x \in \mathbb{R}^n$. Then for any

$$f \in L^2(0,T;L^2(U)),\ \epsilon^{p(0)} \in L^2(U)$$

there exists the unique solution to the problem (MP) such that

$$\epsilon^p \in H^1(0,T;L^2(U)),\ u \in L^2(0,T;H^1_0(U)).$$

Remark. We do not need the initial value of u. It could be derived from $\epsilon^{p(0)}$ and f if the function f would have better regularity with respect to time.

The growth condition seems to be quite strong assumption. In the literature, it is common to consider weaker conditions. In [5] and [6], a polynomial growth condition was used. [8] assumes the gradient structure of the inelastic constitutive function. In [7] and [9], the constitutive function satisfies a coercive condition is controlled by some $\mathcal{N}$-function and the solutions are found using Orlicz spaces. Methods used in this paper differ from those presented in the above works and are comparable to the manners presented in [2], where classical Yosida approximation is used. Weaker assumptions for the constitutive function cause more assumptions on data. Usually, a higher regularity is assumed with respect to time. The problem becomes even simpler if data functions, representing external forces and boundary conditions, (some or all) are equal to zero. This approach was presented in [6] and [7]. [8] and [9] introduce the save load condition as a way to control troublesome time-dependent data functions. Even though we do assume homogeneous boundary values, similar results are expected in the non-homogeneous cases.

Thanks to the growth condition, many problems considered in the works mentioned above do not apply here. In addition, we can operate easily in L^2 Hilbert space and we do not assume existence of time derivatives of data functions of any order, as it is assumed in [5], [8] and [9] or as in [6] and [7], where static process is considered. This way, the process of solving the model can take a form of a functional operator acting on the proper spaces, as shown in Theorem 3. This operator is not only continuous, but Theorem 3 states that the growth condition property of the inelastic constitutive function is carried over and dependency of the solutions on the data functions has also the growth condition property.

2. Solution to abstract problem

The Theorem 1 is not proved directly. Instead, we solve a derived problem and show how to obtain solutions to the problem (MP) having solutions to the derived problem.

The equations above are transformed into a more useful differential equation in a Hilbert space. This points out the functional structure of the problem.

Let $\cdot$ denote the scalar product in $L^2(U)$ throughout this work. Moreover, let $\|\_\|$ denote the norm in $L^2(U)$ and $\|\_\|_{L^p(L^q)}$ denote the norm in $L^p(0,T;L^q(U))$.

Let us consider $\epsilon^p(t) = \epsilon^p(t,\_) \in L^2(U)$ and $u(t) = u(t,\_) \in H^1_0(U)$ spaces. We use ϵ^p and u instead of $\epsilon^p(t)$ and $u(t)$ unless it is needed.

Let us define the subspace L_0:

$$L^2(U) \supset L_0 = \{\epsilon(v)\colon \in H^1_0(U)\}.$$

We can see that L_0 is closed in the norm topology of $L^2(U)$.

Now we can state that $\epsilon^p = z^0 + z^1$, where $z^0 \in L_0$ and $z^1 \in L_0^\perp$. Obviously there exists $\overline{z} \in H^1_0(U)$ such that $z^0 = \epsilon(\overline{z})$.

Considering the equation $\mathrm{div}(\epsilon(u) - \epsilon^p) = 0$, we have for all $v \in H_0^1(U)$:

$$0 = \epsilon(v) \cdot (\epsilon(u) - \epsilon^p) = \epsilon(v) \cdot (\epsilon(u) - \epsilon(\overline{z}) - z^1) = \epsilon(v) \cdot \epsilon(u - \overline{z}).$$

Then $u = \overline{z}$ since $\epsilon(v) \cdot \epsilon(u - \overline{z}) = 0$ for $v = u - \overline{z}$ gives $\epsilon(u - \overline{z}) \cdot \epsilon(u - \overline{z}) = 0$, what implies $\epsilon(u) = \epsilon(\overline{z})$, and the assumption $u|_{\partial U} = \overline{z}|_{\partial U} = 0$ holds. Therefore

$$\epsilon(u) - \epsilon^p = \epsilon(\overline{z}) - z^0 - z^1 = z^0 - z^0 - z^1 = -z^1 = -\pi_{L_0^\perp} \epsilon^p,$$

where π is the usual orthogonal projection onto a linear subspace in a Hilbert space.

In the general case $\mathrm{div}(\epsilon(u) - \epsilon^p) = f$ we proceed as follows. The operator $\mathrm{div}\, \epsilon(u)$ is an elliptic operator. Let $\overline{u} \in H_0^1(U)$ be a solution to $\mathrm{div}\, \epsilon(\overline{u}) = f$, then:

$$\begin{aligned} \mathrm{div}(\epsilon(u) - \epsilon^p) &= f = \mathrm{div}\, \epsilon(\overline{u}), \\ \mathrm{div}(\epsilon(u - \overline{u}) - \epsilon^p) &= 0. \end{aligned}$$

It follows that $\overline{z} = u - \overline{u} \in H_0^1(U)$ and $u = \overline{z} + \overline{u}$.

We introduce two denominations:

1. Let us define $w : [0, T] \to L^2(U)$ and $g : [0, T] \to L^2(U)$ as $w(t) = -\pi_{L_0^\perp} \epsilon^p(t) \in L_0^\perp$ and $g(t) = \epsilon(\overline{u})(t) \in L_0$, respectively. Then the expression $\epsilon(u) - \epsilon^p$ takes the following form:

$$\epsilon(u) - \epsilon^p = \epsilon(\overline{z}) + \epsilon(\overline{u}) - z^0 - z^1 = -z^1 + \epsilon(\overline{u}) = w + g.$$

2. Let us define $A : L^2(U) \to L^2(U)$, where $DomA \subset L^2(U)$ provided by $A(v)(x) = M(v(x))$ for all $x \in \mathbf{R^n}$. A is simply the pointwise operator M raised to the $L^2(U)$ level.

Remark. If the growth condition holds for M then it also holds for the operator A:

$$\begin{aligned} \|A(v)\| &= \sqrt{\int_U |M(v(x))|^2 dx} \leq \sqrt{\int_U C^2 (1 + |v(x)|)^2 dx} \\ &\leq C\sqrt{\int_U 1^2 dx} + C\sqrt{\int_U |v(x)|^2 dx} = C(\sqrt{|U|} + \|v\|). \end{aligned}$$

Remark. It is important that if the growth condition holds then $DomA = L^2(U)$.

The operator A is obviously monotone and from the above Remark it follows that the operator A is a maximal monotone operator.

Now we can transform the main problem as follows:

$$\begin{aligned} \epsilon^p_t &= M(\epsilon(u) - \epsilon^p), \\ -\pi_{L_0^\perp}\epsilon^p_t &= -\pi_{L_0^\perp} M(\epsilon(u) - \epsilon^p), \\ w_t &= -\pi_{L_0^\perp} A(w + g). \end{aligned}$$

With $\epsilon^{p(0)}$ we can calculate the initial value $w_0 = w(0)$.

Thus our final problem takes the form:

$$(FP) \qquad \begin{cases} w_t = -\pi_{L_0^\perp} A(w + g), \\ w(0) = w_0. \end{cases}$$

We are going to solve the problem in the form stated above.

Before proceeding, let us present the following lemma. It is required to provide the approximation step in the Yosida approximation. Lemma is stated in an arbitrary Hilbert space.

Lemma. *Let H be a Hilbert space with the norm $\|\_\|_H$, $B : H \to H$ a Lipschitzean operator with constant γ, such that $B(0) = 0$, $[0, T]$ be the given time interval, $x_0 \in H$ be the initial point, $h \in L^2(0, T; H)$ be the perturbation. Then there exists $x \in H^1(0, T; H) \cap L^\infty(0, T; H)$ satisfying:*

$$\begin{cases} x_t(t) = B(x(t) + h(t)), \\ x(0) = x_0. \end{cases}$$

This lemma is a result from classical theory of differential equations. Proofs of similar results can be found in [10].

Now we start to analyse the solvability of the problem (FP).

Theorem 2. *Assume $g \in L^2(0, T; L_0)$, $w_0 \in L_0^\perp$ and a maximal monotone operator $A : DomA \to L^2$ fulfilling the growth condition i.e. there exists the constant $C > 0$ such that*

$$\|A\| \le C + C\|v\|$$

holds for every $v \in L^2(U)$. Then there exists the unique solution

$$w \in H^1(0, T; L_0^\perp) \cap L^\infty(0, T; L_0^\perp)$$

to the problem:

$$\begin{cases} w_t(t) = -\pi_{L_0^\perp} A(w(t) + g(t)), \\ w(0) = w_0. \end{cases}$$

Proof. a) We prove that $w_0 \mapsto w(\_)$ is a contraction.

Let w^1 and w^2 be solutions to the problem with initial values w_0^1 and w_0^2, respectively. It is enough to calculate:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w^1 - w^2\|^2 &= (w^1 - w^2)\cdot(w^1_t - w^2_t)\\ &= -(w^1 + g - w^2 - g)\cdot \pi_{L_0^\perp}(A(w^1+g) - A(w^2+g))\\ &= -(w^1 + g - w^2 - g)\cdot (A(w^1+g) - A(w^2+g))\\ &\le 0.\end{aligned}$$

Therefore:

$$\|w^1 - w^2\| \le \|w^1_0 - w^2_0\|$$

and

$$\|w^1 - w^2\|_{L^\infty(L^2)} \le \|w^1_0 - w^2_0\|$$

implies the uniqueness.

b) We prove that the solution is consisted in the desired spaces: w and w_t are bounded and the bound depends only on g, w_0 and the constant C from the growth condition.

$$\begin{aligned}\frac{d}{dt}\|w\| = \frac{d}{dt}\sqrt{\|w\|^2} = \frac{1}{2}\frac{\frac{d}{dt}\|w\|^2}{\sqrt{\|w\|^2}} = \frac{w\cdot w_t}{\|w\|} \le \frac{\|w\|\cdot\|w_t\|}{\|w\|}\\ \le \|\pi_{L_0^\perp}A(w+g)\| \le \|A(w+g)\| \le C(1 + \|w\| + \|g\|).\end{aligned}$$

Thus

$$\|w\| \le \|w_0\|e^{Ct} + e^{Ct} + e^{Ct}\int_0^t \|g\| \le \|w_0\|e^{CT} + e^{CT} + e^{CT}\|g\|_{L^1(L^2)}$$

and $w \in L^\infty(0,T;L^2(U))$. The calculation below shows that w_t is bounded in $L^2(0,T;L^2(U))$ since w is bounded:

$$\|w_t\| \le \|A(w+g)\| \le C(1 + \|w\|_{L^\infty(L^2)} + \|g\|).$$

c) We prove that there exist solutions w^λ to approximated problem.

We denote A_λ the Yosida approximation of A and we consider the approximated problem:

$$\begin{cases} w^\lambda_t = -\pi_{L_0^\perp}A_\lambda(w^\lambda + g),\\ w^\lambda(0) = w_0.\end{cases}$$

An operator $L_0^\perp \ni v \mapsto -\pi_{L_0^\perp}A_\lambda(v)$ is Lipschitzean with the constant $\frac{1}{\lambda}$ since A_λ is a Lipschitzean operator with the constant $\frac{1}{\lambda}$. There exists the unique solution w^λ to the approximated problem according to the lemma presented above.

We have $w_0 \in L_0^\perp$, $w^\lambda_t \in L_0^\perp$ and

$$w^\lambda(t) = w_0 + \int_0^t w^\lambda_t(\tau)\,d\tau,$$

thus $w^\lambda \in L_0^\perp$.

d) We show that w^λ is a Cauchy sequence. According to the growth condition, all solutions w^λ are uniformly bounded in $L^\infty(0,T;L^2(U))$:

$$\begin{aligned}\frac{d}{dt}\|w^\lambda\| &= \frac{d}{dt}\sqrt{\|w^\lambda\|^2} = \frac{1}{2}\frac{\frac{d}{dt}\|w^\lambda\|^2}{\sqrt{\|w^\lambda\|^2}} \\ &= \frac{w^\lambda\cdot w^\lambda_t}{\|w^\lambda\|} \leq \frac{\|w^\lambda\|\cdot\|w^\lambda_t\|}{\|w^\lambda\|} \leq \|A_\lambda(w^\lambda+g)\| \\ &\leq \|A(w^\lambda+g)\| \leq C(1+\|w^\lambda\|+\|g\|),\end{aligned}$$

as seen above $\|w^\lambda\| \leq \|w_0\|e^{CT} + e^{CT} + e^{CT}\|g\|_{L^1(L^2)}$ and so w^λ is bounded in $L^\infty(0,T;L^2(U))$ and the bound does not depend on λ. An inequality $\|A_\lambda(w^\lambda+g)\| \leq \|A(w^\lambda+g)\|$ comes from the Yosida theorem.

The functions w^λ_t are uniformly bounded in $L^2(0,T;L^2(U))$ since $g \in L^2(0,T;L^2(U))$ and w^λ are uniformly bounded:

$$\|w^\lambda_t\| = \|-\pi_{L_0^\perp}A_\lambda(w+g)\| \leq C(1+\|w^\lambda\|+\|g\|).$$

In the same way, we show the following estimate:

$$\|A_\lambda(w^\lambda+g)\| \leq C(1+\|w^\lambda\|+\|g\|),$$

which yields that $A_\lambda(w^\lambda+g)$ are uniformly bounded in $L^2(0,T;L^2(U))$ and the bound does not depend on λ.

With properties of the Yosida approximations, the convergence of w^λ in $L^\infty(0,T;L^2(U))$ is achieved by the calculation:

$$\begin{aligned}&\frac{1}{2}\|w^\lambda - w^\mu\|^2 \\ &= \int_0^t \frac{1}{2}\frac{d}{dt}\|w^\lambda - w^\mu\|^2 d\tau \\ &= \int_0^t (w^\lambda - w^\mu)\cdot(w^\lambda_t - w^\mu_t)d\tau \\ &= -\int_0^t (w^\lambda - w^\mu)\cdot\pi_{L_0^\perp}(A_\lambda(w^\lambda+g) - A_\mu(w^\mu+g))d\tau \\ &= -\int_0^t (w^\lambda - w^\mu)\cdot(A_\lambda(w^\lambda+g) - A_\mu(w^\mu+g))d\tau \\ &= -\int_0^t (w^\lambda + g - w^\mu - g)\cdot(A_\lambda(w^\lambda+g) - A_\mu(w^\mu+g))d\tau \\ &= -\int_0^t (\lambda A_\lambda(w^\lambda+g) - \mu A_\mu(w^\mu+g))\cdot(A_\lambda(w^\lambda+g) - A_\mu(w^\mu+g))d\tau \\ &\quad -\int_0^t (J_\lambda(w^\lambda+g) - J_\mu(w^\mu+g))\cdot(A_\lambda(w^\lambda+g) - A_\mu(w^\mu+g))d\tau\end{aligned}$$

$$\begin{aligned}
&\le -\int_0^t (\lambda A_\lambda(w^\lambda+g) - \mu A_\mu(w^\mu+g)) \cdot (A_\lambda(w^\lambda+g) - A_\mu(w^\mu+g))d\tau \\
&= \int_0^t \lambda A_\lambda(w^\lambda+g)\cdot A_\mu(w^\mu+g)) + \mu A_\mu(w^\mu+g))\cdot A_\lambda(w^\lambda+g) \\
&\quad - \lambda\|A_\lambda(w^\lambda+g)\|^2 - \mu\|A_\mu(w^\mu+g)\|^2 d\tau \\
&\le \int_0^t \lambda\|A_\lambda(w^\lambda+g)\|^2 + \frac{\lambda}{4}\|A_\mu(w^\mu+g))\|^2 + \mu\|A_\mu(w^\mu+g))\|^2 \\
&\quad + \frac{\mu}{4}\|A_\lambda(w^\lambda+g)\|^2 - \lambda\|A_\lambda(w^\lambda+g)\|^2 - \mu\|A_\mu(w^\mu+g)\|^2 d\tau \\
&\le \frac{1}{4}\int_0^t \lambda\|A_\mu(w^\mu+g)\|^2 + \mu\|A_\lambda(w^\lambda+g)\|^2 d\tau \\
&\le \frac{1}{4}(\mu\|A_\lambda(w^\lambda+g)\|^2_{L^2(L^2)} + \lambda\|A_\mu(w^\mu+g)\|^2_{L^2(L^2)}).
\end{aligned}$$

This yields the convergence in $L^\infty(0,T;L^2(U))$ since $\|A_\lambda(w^\lambda+g)\|_{L^2(L^2)}$ are uniformly bounded. Let $w = \lim w^\lambda$ with $\lambda \to 0^+$.

For the next step, we need to show also that:

$$J_\lambda(w^\lambda+g) \to_{L^2(L^2)} w+g.$$

The above convergence follows by:

$$\|w^\lambda + g - J_\lambda(w^\lambda+g)\| = \lambda\|A_\lambda(w^\lambda+g)\|.$$

e) Finally, we prove that w is the solution. Since w_t^λ and $A_\lambda(w^\lambda+g)$ are bounded in $L^2(0,T;L^2(U))$ then there exists a sequence λ_n such that:

$$w^{\lambda_n} \to_{L^2(L^2)} w,\ w_t^{\lambda_n} \rightharpoonup_{L^2(L^2)} w_t,\ A_{\lambda_n}(w^{\lambda_n}+g) \rightharpoonup_{L^2(L^2)} u.$$

Therefore, we have $w_t = -\pi_{L_0^\perp} u$.

Let $\mathcal{A}$ be the operator on $L^2(0,T;L^2(U))$ defined as:

$$\mathcal{A}(v)(t) = A(v(t)).$$

$\mathcal{A}$ is a maximal monotone operator as proved in [2]. Moreover, we have:

$$\mathcal{A}_\lambda(v)(t) = A_\lambda(v(t)), \quad \mathcal{J}_\lambda(v)(t) = J_\lambda(v(t)),$$

where $\mathcal{A}_\lambda$, A_λ, $\mathcal{J}_\lambda$, J_λ are respectively the Yosida approximations and resolvents of $\mathcal{A}$ and A.

Since $J_\lambda(w^\lambda+g) \to_{L^2(L^2)} w+g$ and $\mathcal{A}(\mathcal{J}_{\lambda_n}(w^{\lambda_n}+g)) \ni \mathcal{A}_{\lambda_n}(w^{\lambda_n}+g)$ then $(w+g,u) \in Graph\mathcal{A}$. Hence $w_t = -\pi_{L_0^\perp} A(w+g)$. ∎

As stated in Theorem 2, to a given data g we can assign the solution w. We treat this assignment as the solving operator. The next theorem says that properties of the operator A are carried over onto this solving operator.

Theorem 3. *The solving operator, which assigns a solution*

$$w \in L^{\infty}(0,T;L^2(U))$$

to a given data

$$g \in L^2(0,T;L^2(U)),$$

is the continuous operator $L^2(0,T;L^2(U)) \to L^{\infty}(0,T;L^2(U))$ *satisfying the growth condition i.e. there exists the constant* $C > 0$ *that*

$$\|w\|_{L^{\infty}(L^2)} \le C + C\|g\|_{L^2(L^2)}.$$

Proof. Firstly, we prove that the growth condition holds:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|^2 = w \cdot w_t = -\pi_{L_0^{\perp}} A(w+g)\cdot w &\le C(1+\|w\|+\|g\|)\cdot\|w\| \\ &\le C\|w\| + C\|w\|^2 + \frac{C}{2}\|w\|^2 + \frac{C}{2}\|g\|^2 \\ &\le C + C\|w\|^2 + C\|w\|^2 + \frac{C}{2}\|w\|^2 + \frac{C}{2}\|g\|^2.\end{aligned}$$

Hence, using the Gronwall inequality:

$$\|w\|^2 \le e^{5CT}\|w_0\|^2 + e^{5CT}\frac{2}{5} + Ce^{5CT}\|g\|^2_{L^2(L^2)}.$$

Thus the growth condition holds.

Now we show continuity of the solving operator. Let w^1, w^2 be solutions for given g^1, g^2.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w^1 - w^2\|^2 &= (w^1 - w^2)\cdot(w^1_t - w^2_t) \\ &\le -\pi_{L_0^{\perp}}(A(w^1+g^1) - A(w^2+g^2))\cdot(w^1 - w^2) \\ &= -(A(w^1+g^1) - A(w^2+g^2))\cdot(w^1 - w^2) \\ &= -(A(w^1+g^1) - A(w^2+g^2))\cdot(w^1 + g^1 - w^2 - g^2 - g^1 + g^2) \\ &\le (A(w^1+g^1) - A(w^2+g^2))\cdot(g^1 - g^2).\end{aligned}$$

$$\frac{1}{2}\|w^1 - w^2\|^2 \le \int (A(w^1+g^1) - A(w^2+g^2))\cdot(g^1 - g^2)dt.$$

Let $g^n \to g$ converges in $L^2(0,T;L^2(U))$ and w^n, w be associated solutions. Thus g^n, g are bounded in $L^2(0,T;L^2(U))$ and $w^n + g^n, w + g$ are bounded in $L^2(0,T;L^2(U))$ provided by the just proved growth condition property. $A(w^n + g^n), A(w+g)$ are also bounded in $L^2(0,T;L^2(U))$ since A satisfies the growth condition. Finally:

$$\frac{1}{2}\|w^n - w\|^2 = \int (A(w^n+g^n) - A(w+g))\cdot(g^n - g)dt \to 0.$$

This completes the proof of continuity. ∎

3. Solution to the main problem

Proof of Theorem 1. To find solutions ϵ^p and u to the problem (MP), firstly we transform it to the problem (FP). We recall that $f \in L^2(0,T;L^2(U))$ if and only if $g \in L^2(0,T;L^2(U))$. The solution w to the problem (FP) is provided by Theorem 2. Having w, we can obtain ϵ^p and u. For ϵ^p holds:

$$\begin{aligned}\epsilon(u) - \epsilon^p &= w + g,\\ \epsilon^p_t &= M(\epsilon(u) - \epsilon^p) = A(w + g).\end{aligned}$$

Applying the growth condition, we have $\epsilon^p_t \in L^2(0,T;L^2(U))$. Integrating ϵ^p_t with respect to time and the initial value $\epsilon^{p(0)}$ implies $\epsilon^p \in H^1(0,T;L^2(U))$ and the value $\epsilon^p(0)$ is well defined.

For u holds:

$$\begin{aligned}\mathrm{div}(\epsilon(u) - \epsilon^p) &= f,\\ u|_{\partial U} &= 0,\end{aligned}$$

what implies that u is unique and $u \in L^2(0,T;H^1_0(U))$ since

$$f \in L^2(0,T;L^2(U)). \ \blacksquare$$

4. Regularity of solutions

Now we show how to improve the regularity of the solutions, both in time and space. If $f \in H^1(0,T;L^2(U))$ then $g \in H^1(0,T;L^2(U))$. It follows that $\epsilon(u) \in H^1(0,T;L^2(U))$. Moreover, the value of ϵ^p_t is bounded in $L^2(U)$ because

$$\|\epsilon^p_t\| = \|M(\epsilon(u) - \epsilon^p)\| \le C(1 + \|\epsilon(u) - \epsilon^p\|)$$

and $\epsilon(u) - \epsilon^p$ is bounded in $L^2(U)$. It follows that $\epsilon^p \in W^{1,\infty}(0,T;L^2(U))$.

Now we employ techniques of space regularity improvement. Firstly, we adopt an approach known in the theory of inelastic deformations, see [4] for instance.

Let W, V be open sets in R^n such that $U \supset \overline{W} \supset W \supset \overline{V}$. Let ξ be a smooth function defined as follows: $0 \le \xi \le 1$, $\xi = 1$ on V and $\xi = 0$ on $U \backslash W$. Finally, let us denote differential quotient as

$$D^h_k w = \frac{w(x + h\, e_k) - w(x)}{h}.$$

We have:

$$\frac{1}{2}\frac{d}{dt}\|\xi D^h_k(\epsilon(u) - \epsilon^p)\|^2 = \int (\xi D^h_k(\epsilon(u_t) - \epsilon^p_t)) \cdot (\xi D^h_k(\epsilon(u) - \epsilon^p))\, dx$$

$$
\begin{aligned}
&= \int (\xi D_k^h(\epsilon(u_t)))\cdot(\xi D_k^h(\epsilon(u)-\epsilon^p))\,dx - \int (\xi D_k^h(\epsilon_t^p))\cdot(\xi D_k^h(\epsilon(u)-\epsilon^p))\,dx\\
&\le \int (\epsilon(D_k^h u_t))\cdot(\xi^2 D_k^h(\epsilon(u)-\epsilon^p))\,dx\\
&= \int (\xi D_k^h u_t)\cdot(\xi D_k^h \operatorname{div}(\epsilon(u)-\epsilon^p))\,dx + 2\int (\nabla\xi\otimes D_k^h u_t)\cdot(\xi D_k^h(\epsilon(u)-\epsilon^p))\,dx\\
&= \int (\xi D_k^h u_t)\cdot(\xi D_k^h f)\,dx + 2\int (\nabla\xi\otimes D_k^h u_t)\cdot(\xi D_k^h(\epsilon(u)-\epsilon^p))\,dx\\
&\le \frac{1}{2}\|\xi D_k^h u_t\|^2 + \frac{1}{2}\|\xi D_k^h f\|^2 + \|\nabla\xi\otimes D_k^h u_t\|^2 + \|\xi D_k^h(\epsilon(u)-\epsilon^p)\|^2\\
&\le C\|\xi\epsilon(u_t)\|^2 + C\|\xi D f\|^2 + C\|\nabla\xi\otimes\epsilon(u_t)\|^2 + \|\xi D_k^h(\epsilon(u)-\epsilon^p)\|^2.
\end{aligned}
$$

We could use the Gronwall inequality and estimate $\|\xi D_k^h(\epsilon(u)-\epsilon^p)\|$, uniform bound independent of h would imply $\|\epsilon(u)-\epsilon^p\| \in H^1_{loc}(U)$. But we have to know that the right hand side of the above calculation is properly defined e.g. $\|\epsilon(u_t)\|$ and $\|Df\|$ exist. This leads us to the following result:

Theorem 4. *Let u and ϵ^p be the solution to the problem* (MP) *and $f \in H^1(0,T;H^1_{loc}(U))$ and $\epsilon^{p(0)} \in H^1_{loc}(U)$. Then*

$$\epsilon(u)-\epsilon^p \in L^2(0,T;H^1_{loc}(U)). \ \blacksquare$$

Note that the growth condition is not used in the above Theorem.

The model considered in this paper controls only $\epsilon(u)-\epsilon^p$, it does not control $\epsilon(u)$. The so called self-controlling models, which control $\epsilon(u)$, are considered in [4], [11]. We cannot employ techniques presented there to improve regularity of $\epsilon(u)$. Instead we use the growth condition.

Theorem 5. *Let u and ϵ^p be the solution to the problem* (MP) *and $f \in H^1(0,T;H^1_{loc}(U))$ and $\epsilon^{p(0)} \in H^1_{loc}(U)$. Assume that DM satisfies the growth condition i.e. there exists the constant $C>0$ such that*

$$|DM(x)| \le C(1+|x|)$$

holds for every $x \in \mathbb{R}^n$. Then

$$\epsilon^p \in H^1(0,T;H^1_{loc}(U)) \quad \textit{and} \quad \epsilon(u) \in L^2(0,T;H^1_{loc}(U)).$$

Proof. It suffices to show that $\epsilon^p \in H^1(0,T;H^1_{loc}(U))$. Actually it is enough to show that $\epsilon_t^p \in L^2(0,T;H^1_{loc}(U))$. Since $\epsilon_t^p = M(\epsilon(u)-\epsilon^p)$, we prove that $M(\epsilon(u)-\epsilon^p) \in L^2(0,T;H^1_{loc}(U))$.

Let W, V and ξ be given as previously. We have

$$
\begin{aligned}
D\xi\epsilon_t^p &= D\xi\otimes\epsilon_t^p + \xi DM(\epsilon(u)-\epsilon^p)\,D(\epsilon(u)-\epsilon^p),\\
\|D\xi\epsilon_t^p\| &= \|D\xi\otimes\epsilon_t^p\| + \|DM(\epsilon(u)-\epsilon^p)\|\cdot\|\xi D(\epsilon(u)-\epsilon^p)\|,\\
\|D\epsilon_t^p\|_{L^2(V)} &\le \|D\xi\|\cdot\|\epsilon_t^p\| + C(1+\|\epsilon(u)-\epsilon^p\|)\cdot\|\xi D(\epsilon(u)-\epsilon^p)\|.
\end{aligned}
$$

Since $\epsilon_t^p, \epsilon(u) - \epsilon^p \in L^\infty(0,T;L^2(U))$ and $\epsilon(u) - \epsilon^p \in L^2(0,T;H^1_{loc}(U))$, we conclude that $\epsilon_t^p \in L^2(0,T;H^1_{loc}(U))$. ∎

Notice that the assumption that DM satisfies the growth condition, as stated in the above Theorem, is weaker than the assumption that the function M is Lipschitz continuous.

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Received March 11, 2013, revised version September 2, 2013.

Communicated by H. D. Alber.