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UNIQUE COMMON FIXED POINT THEOREMS FOR PAIRS OF HYBRID MAPS UNDER A NEW CONDITION IN PARTIAL METRIC SPACES

Abstract. In this paper, we introduce a new condition namely, 'condition (W.C.C)' and obtain two unique common fixed point theorems for pairs of hybrid mappings on a partial Hausdorff metric space without using any continuity and commutativity of the mappings.

1. Introduction and preliminaries

In 1969, Nadler [20] initiated the development of the geometric fixed point theory for multivalued mappings. He used the concept of the Hausdorff metric to establish the multivalued contraction principle containing the Banach contraction principle as a special case. Indeed, the fixed point theorems for multivalued mappings are quite useful in control theory and have been frequently used in solving many problems of economics, game theory, convex optimization and differential equations.

Here, we recall that a Hausdorff metric H induced by a metric d on a set X is given by

$$H(A,B) = \max\Big\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\Big\},$$

for every $A, B \in CB(X)$, where $d(x,B) = \inf\{d(x,y) : y \in B\}$ and $CB(X)$ is the collection of the closed and bounded subsets of X.

Theorem 1.1. [20] *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a mapping satisfying $H(Tx,Ty) \leq kd(x,y)$, where $k \in [0,1)$ then there exists $x \in X$ such that $x \in Tx$.*

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In the last decades, a number of fixed point results (see, for example, [1, 2, 8, 14, 15, 17, 18, 19]) have been obtained in attempts to generalize Theorem 1.1.

The other basic notion for the development of our work is the concept of the partial metric space, that was introduced by Matthews [21] as a part of the study of denotational semantics of data flow networks. He presented a modified version of the Banach contraction principle, more suitable in this context, see also [3, 6]. In fact, the partial metric spaces constitute a suitable framework to model several distinguished examples of the theory of computation and also to model metric spaces via domain theory, see [4, 5, 7, 10, 11, 12, 13, 16, 21, 22]. In this direction, Aydi et al. [9] introduced the concept of a partial Hausdorff metric and extended Nadler's fixed point theorem in the setting of partial metric spaces.

In view of the above considerations, the aim of this paper is to introduce a new condition namely, 'condition (W.C.C)' and obtain unique common fixed point theorems for pairs of hybrid mappings in a partial Hausdorff metric space without using any continuity and commutativity of the mappings. The presented results extend and unify some recently obtained comparable results for multivalued mappings (see [9] and the references therein).

Consistent with [9, 10, 21], the following definitions and results will be needed in the sequel.

Definition 1.2. [21] A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(p_2) $p(x,x) \le p(x,y)$,
(p_3) $p(x,y) = p(y,x)$,
(p_4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

In this case (X,p) is called a partial metric space.

It is clear that $|p(x,y) - p(y,z)| \le p(x,z)$ $\forall x,y,z \in X$. It is also clear that $p(x,y) = 0$ implies $x = y$ from (p_1) and (p_2). But if $x = y$, $p(x,y)$ may not be zero. A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x,y \in \mathbb{R}^+$. Each partial metric p on X generates τ_0 topology τ_p on X which has as a base the family of open p - balls $\{B_p(x,\epsilon) \mid x \in X,\ \epsilon > 0\}$ for all $x \in X$ and $\epsilon > 0$, where $B_p(x,\epsilon) = \{y \in X \mid p(x,y) < p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$. If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$, given by $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$, is a metric on X.

Definition 1.3. [21] Let (X,p) be a partial metric space.

(i) A sequence $\{x_n\}$ in (X,p) is said to converge to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.
(ii) A sequence $\{x_n\}$ in (X,p) is said to be Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.
(iii) (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Lemma 1.4. [21] *Let (X,p) be a partial metric space.*

(a) *$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).*
(b) *(X,p) is complete iff the metric space (X,p^s) is complete. Further more, $\lim_{n\to\infty} p^s(x_n,x) = 0$ if and only if*

$$p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n,m\to\infty} p(x_n,x_m).$$

Lemma 1.5. [10] *Let (X,p) be a partial metric space and A any nonempty set in (X,p), then $a \in \overline{A}$ if and only if $p(a,A) = p(a,a)$, where $\overline{A}$ denotes the closure of A with respect to the partial metric p.*

Consistent with [9], let (X,p) be a partial metric space and let $CB^p(X)$ be the family of all non-empty, closed and bounded subsets of the partial metric space (X,p), induced by the partial metric p. Note that the closedness is taken from (X,τ_p) (τ_p is the topology induced by p) and the boundedness is given as follows: A is a bounded subset in (X,p) if there exist $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0,M)$, that is, $p(x_0,a) < p(x_0,x_0) + M$. For $A,B \in CB^p(X)$, $x \in X$, $\delta_p : CB^p(X) \times CB^p(X) \to \mathbb{R}^+$ define

$$p(x,A) = \inf\{p(x,a), a \in A\}, \qquad \delta_p(A,B) = \sup\{p(a,B) : a \in A\},$$
$$\delta_p(B,A) = \sup\{p(b,A) : b \in B\}, \qquad H_p(A,B) = \max\{\delta_p(A,B), \delta_p(B,A)\}.$$

The mapping $H_p : CB^p(X) \times CB^p(X) \to \mathbb{R}^+$ is called the partial Hausdorff metric induced by partial metric p. Every Hausdorff metric is a partial Hausdorff metric but the converse is not true, see Example 2.6 in [9].

Lemma 1.6. [9] *Let (X,p) be a partial metric space. For any $A,B,C \in CB^p(X)$, we have*

(i) $\delta_p(A,A) = sup\ \{p(a,a) : a \in A\}$,
(ii) $\delta_p(A,A) \leq \delta_p(A,B)$,
(iii) $\delta_p(A,B) = 0$ *implies that* $A \subseteq B$,
(iv) $\delta_p(A,B) \leq \delta_p(A,C) + \delta_p(C,B) - \inf_{c\in C} p(c,c)$.

Lemma 1.7. [9] *Let (X,p) be a partial metric space. For any $A, B, C \in CB^p(X)$, we have*

(i) $H_p(A,A) \leq H_p(A,B)$,
(ii) $H_p(A,B) = H_p(B,A)$,
(iii) $H_p(A,B) \leq H_p(A,C) + H_p(C,B)$ - $\inf_{c \in C} p(c,c)$.

Lemma 1.8. [9] *Let (X,p) be a partial metric space. For any $A, B \in CB^p(X)$ the following holds: $H_p(A,B) = 0$ implies that $A = B$.*

In [9], they also show that by an example, $H_p(A,A)$ need not be zero.

Lemma 1.9. [9] *Let (X,p) be a partial metric space, $A, B \in CB^p(X)$ and $h > 1$. For any $a \in A$, there exists $b \in B$ such that $p(a,b) \leq hH_p(A,B)$.*

Theorem 1.10. [9] *Let (X,p) be a complete partial metric space and $T : X \to CB^p(X)$ is a multi-valued mapping such that for all $x, y \in X$, we have $H_p(Tx,Ty) \leq k\ p(x,y)$, where $k \in (0,1)$ then T has a fixed point.*

We state and prove our main results.

2. Main results

Lemma 2.1. *Let $x_n \to x$ as $n \to \infty$ in a partial metric space (X,p) such that $p(x,x) = 0$ then $\lim_{n\to\infty} p(x_n,B) = p(x,B)$ for any $B \in CB^p(X)$.*

Proof. Since $x_n \to x$, we have $\lim_{n\to\infty} p(x_n,x) = p(x,x) = 0$.

By using triangular inequality for $x_n \in X$ and $y \in B$, we have

$$p(x_n,y) \leq p(x_n,x) + p(x,y) - p(x,x),$$

which implies that

$$p(x_n,B) \leq p(x_n,x) + p(x,B).$$

Therefore, we get $\lim_{n\to\infty} p(x_n,B) \leq p(x,B)$.......(i).

On the other hand, we have

$$p(x,y) \leq p(x,x_n) + p(x_n,y) - p(x_n,x_n).$$

Thus

$$p(x,y) \leq p(x,x_n) + p(x_n,y).$$

By taking infinimum over $y \in B$, we get

$$p(x,B) \leq p(x,x_n) + p(x_n,B).$$

Therefore, we get $p(x,B) \leq \lim_{n\to\infty} p(x_n,B)$.......(ii).

From (i) and (ii), we have $\lim_{n\to\infty} p(x_n,B) = p(x,B)$. ∎

Now we introduce the following new condition, namely, the condition (W.C.C) on mappings which are not necessarily continuous and commutative.

Definition 2.2. Let (X,p) be a partial metric space. Let $f,g : X \to X$ and $S : X \to CB^p(X)$ be mappings. Then

(i) the triplet $(f,g;S)$ is said to satisfy the condition (W.C.C) if
$$p(fx,gy) \leq p(y,Sx),\ \forall\ x,y \in X,$$

(ii) the pair $(f;S)$ is said to satisfy the condition (W.C.C) if
$$p(fx,fy) \leq p(y,Sx),\ \forall\ x,y \in X.$$

The following example illustrates the condition (W.C.C).

Example 2.3. Let $X = [0,1]$ and $p(x,y) = \max\{x,y\}$, $\forall\ x,y \in X$. Let $f,g : X \to X$ and $S : X \to CB^p(X)$ be defined by $fx = 0$, $\forall\ x,y \in X$,
$$gx = \begin{cases} 0, & \text{if } x \in [0,\frac{1}{2}], \\ \frac{x}{32}, & \text{if } x \in (\frac{1}{2},1], \end{cases}$$
and $Sx = [0,\frac{1}{4}]$, $\forall\ x,y \in X$. We consider the following two cases.

Case (a): $x \in X$ and $y \in [0,\frac{1}{2}]$. Then $p(fx,gy) = 0 = p(y,Sx)$.
Case (b): $x \in X$ and $y \in (\frac{1}{2},1]$. Then $p(fx,gy) = \frac{y}{32} < y = p(y,Sx)$.

Thus $(f,g;S)$ satisfies the condition (W.C.C).

The following example shows that the triplet $(f,g;S)$, satisfying the condition (W.C.C), need not be continuous even when S is a single-valued mapping.

Example 2.4. Let $X = [0,1]$ and $p(x,y) = \max\{x,y\}$, $\forall x,y \in X$. Let $f,g,S : X \to X$ be defined by
$$fx = \begin{cases} \frac{x}{12}, & \text{if } x \neq 1, \\ \frac{1}{24}, & \text{if } x = 1, \end{cases} \qquad x = \begin{cases} \frac{x}{6} & \text{if } x \neq 1, \\ \frac{1}{12} & \text{if } x = 1, \end{cases}$$
and
$$Sx = \begin{cases} \frac{x}{2}, & \text{if } x \neq 1, \\ \frac{1}{4}, & \text{if } x = 1. \end{cases}$$

Clearly, all the mappings f,g and S are discontinuous.

Now, we distinguish the following cases to show that $(f,g;S)$ satisfies the condition (W.C.C).

Case (i): $x \neq 1$ and $y \neq 1$.
$$p(fx,gy) = \max\left\{\frac{x}{12},\frac{y}{6}\right\} = \frac{1}{6}\max\left\{\frac{x}{2},y\right\} = \frac{1}{6}p(y,Sx).$$

Case (ii): $x \neq 1$ and $y = 1$.
$$p(fx,gy) = \max\left\{\frac{x}{12},\frac{1}{12}\right\} < \frac{1}{6}\max\left\{1,\frac{x}{2}\right\} = \frac{1}{6}p(y,Sx).$$

Case (iii): $x = 1$ and $y \neq 1$.

$$p(fx, gy) = \max\left\{\frac{1}{24}, \frac{y}{6}\right\} = \frac{1}{6}p(y, Sx).$$

Case (iv): $x = 1$ and $y = 1$.

$$p(fx, gy) = \max\left\{\frac{1}{24}, \frac{1}{12}\right\} = \frac{1}{12} < \max\left\{1, \frac{1}{4}\right\} = p(y, Sx).$$

Thus, $(f, g; S)$ satisfies the condition (W.C.C).

The following example shows that the triplet $(f, g; S)$ satisfying the condition (W.C.C), need not be commuting even when S is a single-valued mapping.

Example 2.5. Let a and b be non-negative real numbers such that $b < a$. Let $X = \{a, b\}$ and $p(x, y) = \max\{x, y\}$, $\forall x, y \in X$. Let $f, g, S : X \to X$ be defined by $fa = fb = b$, $ga = b, gb = a$ and $Sa = Sb = a$.

Clearly the triplet $(f, g; S)$ satisfies the condition (W.C.C) and the pairs (f, S), (g, S) and (f, g) are not commuting.

Now, we state and prove our main results.

Theorem 2.6. *Let (X, p) be a complete partial metric space and let $S, T : X \to CB^p(X)$ and $f, g : X \to X$ be mappings satisfying*

(2.6.1) $$H_p(Sx, Ty) \leq \alpha \max\left\{\begin{matrix} p(fx, gy), & \frac{1}{2}[p(fx, Sx) + p(gy, Ty)], \\ & \frac{1}{2}[p(fx, Ty) + p(gy, Sx)] \end{matrix}\right\}$$

for all $x, y \in X$ and $\alpha \in (0, 1)$,

(2.6.2) $\bigcup_{x \in X} Sx \subseteq g(X)$ *and* $\bigcup_{x \in X} Tx \subseteq f(X)$,

(2.6.3) *the triplet $(f, g; S)$ or the triplet $(f, g; T)$ satisfies the condition* (W.C.C).

Then f, g, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$. From (2.6.2), there exist $x_1,\ y_1 \in X$ such that $y_1 = gx_1 \in Sx_0$.

From (2.6.2) and Lemma 1.9 with $h = \frac{1}{\sqrt{\alpha}}$, there exist $x_2,\ y_2 \in X$ such that $y_2 = fx_2 \in Tx_1$ and

$$p(y_1, y_2) \leq \frac{1}{\sqrt{\alpha}} H_p(Sx_0, Tx_1).$$

Again from (2.6.2) and Lemma 1.9, there exist $x_3,\ y_3 \in X$ such that $y_3 = gx_3 \in Sx_2$ and

$$p(y_2, y_3) \leq \frac{1}{\sqrt{\alpha}} H_p(Sx_2, Tx_1).$$

Continuing in this way, we get the sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n+1} = gx_{2n+1} \in Sx_{2n}$, $y_{2n+2} = fx_{2n+2} \in Tx_{2n+1}$, $n = 0, 1, 2, 3, \ldots$ and

$$p(y_{2n+1}, y_{2n}) \leq \frac{1}{\sqrt{\alpha}} H_p(Sx_{2n}, Tx_{2n-1}), n = 1, 2, 3, \ldots$$
$$p(y_{2n+1}, y_{2n+2}) \leq \frac{1}{\sqrt{\alpha}} H_p(Sx_{2n}, Tx_{2n+1}), n = 0, 1, 2, 3, \ldots$$

Now from (2.6.1), we have

$$\begin{aligned}
p(y_{2n+1}, y_{2n+2}) &\leq \frac{1}{\sqrt{\alpha}} H_p(Sx_{2n}, Tx_{2n+1}), \\
&\leq \sqrt{\alpha} \max \left\{ \begin{array}{c} p(fx_{2n}, gx_{2n+1}), \frac{1}{2}[p(fx_{2n}, Sx_{2n}) + p(gx_{2n+1}, Tx_{2n+1})], \\ \frac{1}{2}[p(fx_{2n}, Tx_{2n+1}) + p(gx_{2n+1}, Sx_{2n})] \end{array} \right\} \\
&\leq \sqrt{\alpha} \max \left\{ \begin{array}{c} p(y_{2n}, y_{2n+1}), \frac{1}{2}[p(y_{2n}, y_{2n+1}) + p(y_{2n+1}, y_{2n+2})], \\ \frac{1}{2}[p(y_{2n}, y_{2n+2}) + p(y_{2n+1}, y_{2n+1})] \end{array} \right\} \\
&\leq \sqrt{\alpha} \max \left\{ p(y_{2n}, y_{2n+1}), \frac{1}{2}[p(y_{2n}, y_{2n+1}) + p(y_{2n+1}, y_{2n+2})] \right\} \\
&\qquad \text{from } (p_4).
\end{aligned}$$

Thus, we have

(1) $$p(y_{2n+1}, y_{2n+2}) \leq \beta p(y_{2n}, y_{2n+1}),$$

where $\beta = \max\left\{\sqrt{\alpha}, \frac{\sqrt{\alpha}/2}{1-\sqrt{\alpha}/2}\right\} < 1$.
Similarly, we can show that

(2) $$p(y_{2n+1}, y_{2n}) \leq \beta p(y_{2n}, y_{2n-1}).$$

From (1) and (2), we have

(3) $$p(y_{n+1}, y_n) \leq \beta p(y_n, y_{n-1}), \qquad \text{for all } n = 1, 2, 3, \ldots$$

By continuing in this way, we get

(4) $$p(y_{n+1}, y_n) \leq \beta^n p(y_1, y_0).$$

Since $\beta < 1$, which in turn yields that

(5) $$p(y_{n+1}, y_n) \to 0 \quad \text{as} \quad n \to \infty.$$

For $m > n$, we have

(6) $$\begin{aligned}
p(y_n, y_m) &\leq p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) + \cdots + p(y_{m-1}, y_m), \\
&\leq \left(\beta^n + \beta^{n+1} + \cdots + \beta^{m-1}\right) p(y_1, y_0) \quad \text{from} \quad (2) \\
&\leq \frac{\beta^n}{1-\beta} p(y_1, y_0) \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Thus $\{y_n\}$ is a Cauchy sequence in X. Hence from Lemma 1.4, $\{y_n\}$ is a Cauchy sequence in (X, p^s).

Since (X, p) is complete and from Lemma 1.4, it follows that (X, p^s) is complete. So $\{y_n\}$ converges to some $z \in X$. That is

$$\lim_{n\to\infty} p^s(y_n, z) = 0.$$

Now from Lemma 1.4 and (6), we have

$$p(z, z) = \lim_{n\to\infty} p(y_n, z) = \lim_{n\to\infty} p(y_n, y_m) = 0. \tag{7}$$

Suppose the triplet $(f, g; S)$ satisfies the condition (W.C.C) then

$$p(fx, gy) \le p(y, Sx) \quad \text{for all} \quad x, y \in X. \tag{8}$$

Let $x = x_{2n}$ and $y = z$ in (8), we have

$$p(fx_{2n}, gz) \le p(z, Sx_{2n}) \le p(z, gx_{2n+1}).$$

Letting $n \to \infty$, using Lemma 2.1 and (7), we can obtain

$$p(z, gz) \le 0 \quad \text{so that} \quad gz = z.$$

Now by using (2.6.1), we have

$$\begin{aligned} p(gx_{2n+1}, Tz) &\le H_p(Sx_{2n}, Tz), \\ &\le \alpha \max \left\{ \begin{array}{c} p(fx_{2n}, gz), \frac{1}{2}[p(fx_{2n}, Sx_{2n}) + p(gz, Tz)], \\ \frac{1}{2}[p(fx_{2n}, Tz) + p(gz, Sx_{2n})] \end{array} \right\} \\ &\le \alpha \max \left\{ \begin{array}{c} p(fx_{2n}, z), \frac{1}{2}[p(fx_{2n}, gx_{2n+1}) + p(z, Tz)], \\ \frac{1}{2}[p(fx_{2n}, Tz) + p(z, gx_{2n+1})] \end{array} \right\}. \end{aligned}$$

Letting $n \to \infty$, using Lemma 2.1, (7) and (5), we get

$$p(z, Tz) \le \frac{\alpha}{2} p(z, Tz).$$

Hence $p(z, Tz) = 0$, which in turn yields from Lemma 1.5 and (7) that $z \in \overline{Tz} = Tz$. Thus

$$gz = z \in Tz. \tag{9}$$

Now from (8), we have

$$p(fz, z) = p(fz, gz) \le p(z, Sz). \tag{10}$$

Using (2.6.1), we have

$$\begin{aligned} p(z,Sz) &\le H_p(Sz,Tz) \\ &\le \alpha \max \left\{ \begin{array}{c} p(fz,gz), \frac{1}{2}[p(fz,Sz)+p(gz,Tz)], \\ \frac{1}{2}[p(fz,Tz)+p(gz,Sz)] \end{array} \right\} \\ &\le \alpha \max \{p(z,Sz), p(z,Sz), p(z,Sz)\} \quad \text{from} \quad (p_4), (10), (9) \\ &= \alpha p(z,Sz), \end{aligned}$$

which in turn yields that $p(z,Sz)=0$. From Lemma 1.5 and (7), we have $z \in \overline{Sz} = Sz$.

Now from (8), we get $p(fz,z) \le 0$ so that $fz = z$. Thus

$$fz = z \in Sz. \tag{11}$$

From (9) and (11), it follows that z is a common fixed point of f, g, S and T.

Suppose z' is another common fixed point of f, g, S and T. From (8), we have

$$p(z,z') = p(fz,gz') \le p(z',Sz) \le H_p(Sz,Tz'). \tag{12}$$

Now

$$\begin{aligned} H_p(Sz,Tz') &\le \alpha \max \left\{ \begin{array}{c} p(fz,gz'), \frac{1}{2}[p(fz,Sz)+p(gz',Tz')], \\ \frac{1}{2}[p(fz,Tz')+p(gz',Sz)] \end{array} \right\} \\ &\le \alpha \max \left\{ \begin{array}{c} H_p(Sz,Tz'), \frac{1}{2}[H_p(Sz,Sz)+H_p(Tz',Tz')], \\ \frac{1}{2}[H_p(Sz,Tz')+H_p(Tz',Sz)] \end{array} \right\} \\ &\qquad\qquad \text{from} \quad (12) \\ &\le \alpha H_p(Sz,Tz') \qquad \text{from Lemma 1.7 (i)} . \end{aligned}$$

Thus $H_p(Sz,Tz') = 0$ so that from (12), $z = z'$. Hence z is the unique common fixed point of f, g, S and T.

Similarly we can prove the theorem when $(f,g;T)$ satisfies the condition (W.C.C). ∎

Proceeding as in Theorem 2.6, one can easily prove the following.

Theorem 2.7. *Let (X,p) be a complete partial metric space and let $S,T : X \to CB^p(X)$ and $f : X \to X$ be mappings satisfying*

(2.7.1) $H_p(Sx,Ty) \le \alpha \max \left\{ \begin{array}{c} p(fx,fy), p(fx,Sx), p(fy,Ty), \\ \frac{1}{2}[p(fx,Ty)+p(fy,Sx)] \end{array} \right\}$ *for all $x,y \in X$ where $0 \le \alpha < 1$,*

(2.7.2) $\bigcup_{x\in X} Sx \subseteq f(X)$ *and* $\bigcup_{x\in X} Tx \subseteq f(X)$,

(2.7.3) *the pair $(f;S)$ or the triplet $(f;T)$ satisfies the condition* (W.C.C).

Then f, S and T have a unique common fixed point in X.

Finally, we give the following.

THEOREM 2.8. *Let (X,p) be a complete partial metric space and let $S,T : X \to CB^p(X)$ be mappings satisfying*

$$(2.8.1)\quad H_p(Sx,Ty) \le \alpha \max \left\{ \begin{array}{c} p(x,y), p(x,Sx), p(y,Ty), \\ \frac{1}{2}[p(x,Ty)+p(y,Sx)] \end{array} \right\} \text{ for all } x,y \in X,$$

where $0 \le \alpha < 1$.

Then S and T have a common fixed point in X. Further, if we assume that $p(x,y) \le p(y,Sx)$ or $p(x,y) \le p(y,Tx)$ for all $x,y \in X$ then S and T have a unique common fixed point in X.

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