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REMARKS ON OCCASIONALLY WEAKLY COMPATIBLE MAPS VERSUS OCCASIONALLY WEAKLY COMPATIBLE MAPS

Abstract. In this paper, we discuss some important and interesting remarks on the concept of occasionally weakly compatible (owc) mappings, which is an active and interesting area of research in the present era. Also, we discuss here the lapses of several authors in quoting the definition of owc maps and provide the corrected proof by including some additional conditions on their results.

1. Introduction

The study of common fixed points of mappings satisfying some contractive type condition has been at the center of several research activity and a number of interesting results have been obtained by various authors. In 1976, Jungck [21] initiated a study of common fixed points of commuting maps. On the other hand in 1982, Sessa [29] defined weak commutativity and proved common fixed point theorem for weak commuting mappings. In 1986, Jungck [22] defined the compatible maps which have been useful as a tool for obtaining the fixed point theorems. Al-Thagafi and Shahzad [15] defined the concept of occasionally weakly compatible (owc) maps.

In this paper, we discuss some important remarks on the concept of occasionally weakly compatible maps. We point out the lapses of several authors ([1]–[13], [17], [19], [20], [30], [31]) in quoting the definition of owc maps.

2. Preliminaries

Throughout this paper, (X, d) denotes a metric space. For $x \in X$ and $A \subseteq X$,

$$d(x, A) = \inf\{d(x, y) : y \in A\}.$$

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Let $f : X \longrightarrow X$ and $g : X \longrightarrow X$. A point $x \in X$ is a fixed point of f if $x = fx$. The set of all fixed points of f is denoted by $F(f)$. A point $x \in X$ is a coincidence point of f and g if $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g. The set of all coincidence points and points of coincidence of f and g are denoted by $C(f, g)$ and $PC(f, g)$, respectively. A point $x \in X$ is a common fixed point of f and g if $x = fx = gx$. The set of all common fixed points of f and g is denoted by $F(f, g)$. The pair $\{f, g\}$ is called

1. commuting [21], if $fgx = gfx$, $\forall x \in X$,
2. weakly commuting [29], if $d(fgx, gfx) \leq d(fx, gx)$, for each $x \in X$,
3. compatible [22], if $\lim_n d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $lim_n fx_n = lim_n gx_n = t$, for some $t \in X$,
4. weakly compatible [24], if they commute at the coincidence points i.e. $fgx = gfx$, whenever $fx = gx$, for $x \in X$,
5. weakly f-biased [23], iff $d(fgx, fx) \leq d(gfx, gx)$, whenever $fx = gx$,
6. satisfied the (E.A) property [14], if there exists a sequence $\{x_n\}$ such that $\lim_{n \longrightarrow \infty} fx_n = \lim_{n \longrightarrow \infty} gx_n = t$, for some $t \in X$.

Definition 2.1. Let X be a non-empty set and d be a function $d : X \times X \longrightarrow [0, \infty)$ such that

$$d(x, y) = 0 \;\text{ iff }\; x = y, \quad \forall x, y \in X. \tag{1}$$

For a space (X, d) satisfying (1) and $A \subseteq X$, the diameter of A is defined by

$$\text{diam}(A) = sup\{\max\{d(x, y), d(y, x)\} : x, y \in A\}.$$

Definition 2.2. A symmetric on a set X is a mapping $d : X \times X \longrightarrow [0, \infty)$ such that

1. $d(x, y) = 0$ if and only if $x = y$, and
2. $d(x, y) = d(y, x)$.

A set X, together with a symmetric d, is called a symmetric space.

Lemma 2.1. [25] *If f and g have a unique coincidence point $w = fx = gx$, then w is the unique common fixed point of f and g.*

3. Occasionally weakly compatible mappings

In 2006, Jungck and Rhoades proved some common fixed point theorems for owc maps, which was introduced by Al-Thagafi and Shahzad [15], they wrote:

Definition (JR). Two selfmaps f and g of a set X are occasionally weakly compatible (owc) iff there is a point x in X which is a coincidence point of f and g at which f and g commute.

In 2008, Al-Thagafi and Shahzad [15] define the following concept of owc maps

Definition (AS). Let f and g be selfmaps of a subset D of a metric space X. Then f and g are called occasionally weakly compatible (owc) if $fgx = gfx$, for some $x \in C(f, g)$.

In 2009, Al-Thagafi and Shahzad [16] give a brief note on owc maps. They write

“We note that the concept of owc maps was first introduced by Al-Thagafi and Shahzad [15]. This was explicitly acknowledged by Jungck and Rhoades in [25]."

We now furnish an example to show that the conditions of Definition (JR) are stronger then conditions of Definition (AS).

Example 3.1. Let $X = \mathbb{C}$, define $f, g : X \longrightarrow X$ by

$$f(z) = \begin{cases} 1 + \sin(\log\ z), & \text{if } z \neq 0, \\ 2, & \text{if } z = 0, \end{cases} \qquad g(z) = e^{|z-1|},\ \forall z \in \mathbb{C}.$$

Then $C(f, g) = \phi$. Thus, the pair (f, g) is owc map according to Definition (AS) (since, condition of owc vacuously satisfied), but not an owc map according to Definition (JR).

Thus, Definition (JR) implies Definition (AS) of an owc maps, but the converse need not be true.

It is evident from Example 3.1 above that the two definitions (JR) and (AS) have different characteristic properties and are not similar in their meaning. The critical difference between the definition (JR) and the definition(AS) may be stated as follows:

Definition (AS) does not ensure $C(f, g) \neq \phi$, because a pair of mappings without any point of coincidence can also be realized as a owc pair of Definition (AS) (as the requirement of the definition is vacuously satisfied), whereas Definition (JR) ensure that $C(f, g) \neq \phi$.

Notice that there are dozens of papers involving the Definition (AS) or Definition (JR) with some misleading citations or uncorrected proof of their results, which we shall discuss in our next section. So, this matter opens a wider platform for discussion on owc maps. It is convenient to divide these papers into four categories $(C_1) - (C_4)$ as below:

- (C_1) There are those who give the definition of (JR) attributing it to (AS) and then use the definition they have given. (For example [3], [4], [6]–[9], [11], [13], [17], [20]).

(C_2) There are those who give the definition of (AS) attributing it to (JR) and prove $C(f,g) \neq \phi$ before using this definition.

(C_3) There are those who give the definition of (AS) attributing it to (JR) and then use the definition of (JR) without demonstrating that the set $C(f,g) \neq \phi$. (For example [1], [2], [30], [31]).

(C_4) There are those who give the definition of (JR) referring to [25], use this definition, cite or not [15] in the text and only cite [15] in the bibliography. (For example [5], [10], [12], [19]).

It is clear that the papers of (C_1) does not contain, as such, mathematical errors, only an error of attribution of a definition, items (C_3) commit an error of reasoning and must be corrected and those of (C_2) and (C_4) commit no error.

4. Main results

In this section, we see some corrected version of the results of several authors using the concept of owc maps. In 2010, Bhatt et al. [3] proved some common fixed point theorems for owc maps belonging to (C_1) category under relaxed conditions.

Theorem 4.1. (Theorem 2.2, [3]) *Let X be a nonempty set and $d : X \times X \longrightarrow [0,\infty)$ be a function satisfying (1). If f and g owc self mappings of X and*

$$(2)\qquad d(fx,fy) < \max\{d(gx,gy), d(gx,fy), d(gy,fx), d(gy,fy)\},$$

for all $x,y \in X, x \neq y$ then f and g have a unique fixed point.

That paper [3] contains only an error of attribution of the definition of owc maps.

In 2008, Aliouche and Popa [1] prove the following result under the assumption that F_6 be the family of all functions $F(t_1,t_2,t_3,t_4,t_5,t_6) : R_+^6 \longrightarrow \mathbb{R}$ with $t_3 + t_4 \neq 0$.

Theorem 4.2. (Theorem 3.1, [1]) *Let f, g, S and T be selfmappings of a symmetric space (X,d) satisfying the following conditions:*

$$(3)\quad F(d(Sx,Ty), d(fx,gy), d(fx,Sx), d(gy,Ty), d(fx,Ty), d(Sx,gy)) \leq 0,$$

for all $x,y \in X$, if $d(fx,Sx) - d(gy,Ty) \neq 0$, where $F \in F_6$, or

$$(4)\qquad d(Sx,Ty) = 0 \quad \text{if} \quad d(fx,Sx) + d(gy,Ty) = 0.$$

Suppose that the pairs (S,f) and (T,g) are occasionally weakly compatible. Then f, g, S and T have a unique common fixed point in X.

In the proof part of the above theorem the authors [1] consider that the occasionally weakly compatibility of (f,g) implies that $C(f,g) \neq \phi$, which is not correct due to the Definition (AS), therefore the results which include owc maps are required to be sharpened utilizing some additional condition like assuming that $C(f,g) \neq \phi$ and other part of the proof follows from [1], as below:

Theorem 4.3. (Corrected version of Theorem 4.2) *Let f, g, S and T be selfmappings of a symmetric space (X,d) satisfying the following conditions:*

$$(5)\quad F(d(Sx,Ty),d(fx,gy),d(fx,Sx),d(gy,Ty),d(fx,Ty),d(Sx,gy)) \leq 0,$$

for all $x,y \in X$, if $d(fx,Sx) - d(gy,Ty) \neq 0$, where $F \in F_6$, or

$$(6)\qquad d(Sx,Ty) = 0 \text{ if } d(fx,Sx) + d(gy,Ty) = 0.$$

Suppose that the pairs (S,f) and (T,g) are occasionally weakly compatible with $C(f,g) \neq \phi$. Then f, g, S and T have a unique common fixed point in X.

Proof. With the assumption that $C(f,g) \neq \phi$, there exist $u,v \in X$ such that $fu = Su$ and $gv = Tv$. As $d(fu,Su) + d(gv,Tv) = 0$, it follows from (6) that $Su = Tv$ and so $fu = Su = gv = Tv$. Moreover, if there is another point u_0 such that $fu_0 = Su_0$, using (6) it follows that $fu_0 = Su_0 = gv = Tv$. Therefore, $z = fu = Su$ is the unique point of coincidence of f and S. By Lemma 2.1, z is the unique common fixed point of f and S. Similarly, z_0 is the unique common fixed point of g and T. On the other hand, $d(fz,Sz) + d(gz_0,Tz_0) = 0$ implies that $d(Sz,Tz_0) = 0$, hence $z = fz = Sz = gz_0 = Tz_0 = z_0$. Therefore, z is the unique common fixed point of f,g,S and T. ■

Remark 4.4. Similarly to the above result, one can correct Theorem 3.4, Corollary 3.5, Theorem 3.6 and Corollaries (3.7 − 3.11) of Aliouche and Popa [1].

In 2010, Sumitra et al. [30], prove the following results for owc maps as in Definiton (AS).

Theorem 4.5. (Lemma 3.1, [30]) *Let M be a nonempty subset of a metric space (X,d). Suppose that f and T are occasionally weakly compatible self mappings of M, T is a generalized f-contraction mapping satisfying*

$$(7)\quad d(Tx,Ty) \leq k \max\left\{d(fx,fy),d(fx,Tx),d(fy,Ty),\frac{1}{2}[d(fx,Ty)+d(fy,Tx)]\right\},$$

for all $x,y \in M$ and some $k \in [0,1)$ then there exists a unique common fixed point of f and T in M.

In the proof of the above results, the authors claim that, f and T beeing owc maps of type (AS), there exists $x \in X$, such that $fx = Tx$, which is not correct. The corrected form of Theorem 4.5 is as below:

Theorem 4.6. (Corrected Theorem 4.5) *Let M be a nonempty subset of a metric space (X, d). Suppose that f and T are occasionally weakly compatible self mappings of M, T is a generalized f-contraction mapping satisfying*

$$(8) \quad d(Tx, Ty) \leq k \max\left\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{1}{2}[d(fx, Ty) + d(fy, Tx)]\right\},$$

for all $x, y \in M$ and some $k \in [0, 1)$. Suppose that $TX \subseteq fX$, (f, T) satisfy property (E.A.) *and fX is a closed subspace of X, then f and T has a unique common fixed point in M.*

Proof. Since the pair (f, T) satisfies property (E.A.), there exists a sequence $\{x_n\}$ in X such that

$$(9) \quad \lim_{n \longrightarrow \infty} fx_n = \lim_{n \longrightarrow \infty} Tx_n = z, \text{ for some } z \in X.$$

Since, $TX \subseteq fX$, there exists a sequence $\{y_n\}$ in X such that $Tx_n = fy_n$. Hence,

$$(10) \quad \lim_{n \longrightarrow \infty} fy_n = z.$$

First, we claim that $\lim_{n \longrightarrow \infty} Ty_n = z$. For this purpose, we consider

$$(11) \quad d(Ty_n, Tx_n) \leq k \max\left\{d(fy_n, fx_n), d(fy_n, Ty_n), d(fx_n, Tx_n), \frac{1}{2}[d(fy_n, Tx_n) + d(fx_n, Ty_n)]\right\}.$$

Taking the limit superior in (11), using (9) and (10), we get

$$\lim_{n \longrightarrow \infty} \sup[d(Ty_n, Tx_n)] \leq k \lim_{n \longrightarrow \infty} d(fy_n, Ty_n) = k \lim_{n \longrightarrow \infty} d(Tx_n, Ty_n).$$

Hence, $\lim_{n \longrightarrow \infty} \sup[d(Ty_n, Tx_n)] = 0$. So,

$$(12) \quad \lim_{n \longrightarrow \infty} Ty_n = z.$$

Since $f(X)$ is a closed subspace of X, by (9), we have

$$(13) \quad z = fv, \text{ for some } v \in X.$$

If $Tv \neq z$ then

$$(14) \quad d(Ty_n, Tv) \leq k \max\left\{d(fy_n, fv), d(fy_n, Ty_n), d(fv, Tv), \frac{1}{2}[d(fy_n, Tv) + d(fv, Ty_n)]\right\}.$$

On letting $n \longrightarrow \infty$ in (14), using (9), (10), (12) and (13), we have $d(z, Tv) \leq kd(z, Tv)$, a contradiction. Hence,

$$Tv = z. \tag{15}$$

Hence, from (13) and (15), we get $z = fv = Tv$. Hence, $C(f, T) \neq \phi$. Since the pair (f, T) is owc, the later part of the proof, similarly as above, follows from Theorem 2.2 of [27]. ∎

In 2011, Hussain et al. [18] defines the following concept of $\mathcal{JH}$-operator pair and occasionally weakly biased maps.

Definition 4.1. Let X be a set together with a symmetric d and $f : X \longrightarrow X$ and $g : X \longrightarrow X$. The pair $\{f, g\}$ is called $\mathcal{JH}$-operator pair [18], if there is a point u in X such that $u \in C(f, g)$ and $d(u, fu) \leq \text{diam } (PC(f, g))$, for some $u \in C(f, g)$.

Definition 4.2. Let (X, d) be a metric space. $f : X \longrightarrow X$ and $g : X \longrightarrow X$, then the pair (f, g) is said to be occasionally weakly f-biased([18]), if and only if there exists some $x \in X$ such that $fx = gx$ and $d(fgx, fx) \leq d(gfx, fx)$.

They show that the two new classes $\mathcal{JH}$-operator pair and occasionally weakly f-biased maps contains the owc maps and weakly biased selfmaps as proper subclasses. Since every owc maps of definition (AS) not ensure the existence of coincidence point, these new classes not necessarily contains owc maps and weakly biased maps as proper subclass, as we see in the following example:

Example 4.1. Let $X = [0, \infty)$ with the usual metric. Define $f, g : X \longrightarrow X$ by

$$f(x) = e^{x^2}, \text{ and } \ g(x) = \sin(2x^2), \ \forall \ x \in X.$$

Since, $C(f, g) = \phi$. Thus, the pair (f, g) is owc map according to Definition (AS) and also it is weakly biased map (since, condition vacuously satisfied), but it is neither $\mathcal{JH}$-operator pair nor occasionally weakly biased maps. Hence, $\mathcal{JH}$-operator pair does not contains owc maps and weakly biased maps as a proper subclass as shown in [18].

5. Conclusion

From above discussion, we conclude that

1. To avoid triviality of an owc maps (AS), Al-Thagafi and Shahzad [16], wrote that it is better to assume that $C(f, g) \neq \phi$.
2. In case of using owc maps (AS), we have to assume some additional conditions as we have seen in Theorem 4.3 and Theorem 4.6 or apply some other type of suitable conditions.

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