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ON LACUNARY p-ABSOLUTELY SUMMABLE FUZZY REAL-VALUED DOUBLE SEQUENCE SPACE

Abstract. In this article, we introduce the class of p-absolutely summable fuzzy real valued double sequence $({}_2\ell_p)_\theta^F$. We have studied some algebraic properties like solid, symmetric, convergence free, sequence algebra. Further, we establish some relation with the class of p-Cesàro summable double sequences and some other important inclusion results.

1. Introduction

A lacunary is an increasing sequence $\theta = (k_r)(r = 0, 1, 2, 3, ..)$ of positive integers such that $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$ with $k_0 = 0$. The interval determined by θ is given by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ is denoted by q_r.

By a double lacunary we mean an increasing sequence $\theta_{r,s} = \{(k_r, \ell_s)\}$ of positive integers such that

$$k_r - k_{r-1}(= h_r) \to \infty \text{ as } r \to \infty \text{ with } k_0 = 0,$$

and

$$\ell_s - \ell_{s-1}(= \acute{h}_s) \to \infty \text{ as } s \to \infty \text{ with } \ell_0 = 0.$$

The interval determined by $\theta_{r,s}$ is represented by $I_{r,s} = \{(k, \ell) : k_{r-1} < k \leq k_r; \ell_{s-1} < \ell \leq \ell_s\}$ and $k_{r,s} = k_r\ell_s$, $h_{r,s} = h_r\acute{h}_s$. The ratios $\frac{k_r}{k_{r-1}}$, $\frac{\ell_s}{\ell_{s-1}}$ are denoted by $q_r, \acute{q}_s$, respectively and $q_r\acute{q}_s = q_{r,s}$.

Different classes of lacunary sequences have been studied by some renowned researchers in the recent past. Altin [1], Gokhan et. al. [4], Savas ([7], [8]) and Savas and Patterson [9], Savas and Mursaleen [10], Subramanian and Esi [11], Esi [3], Tripathy and Dutta [16], Tripathy and Baruah [18], Tripathy and Mahanta [17] are some of them.

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A fuzzy real number X is a fuzzy set on R, more precisely a mapping $X : R \to I(= [0,1])$, associating each real number t, with its grade of membership $X(t)$ which satisfies the following properties.

(i) X is normal i.e. there exists $t_0 \in R$ such that $X(t_0) = 1$.
(ii) X is upper-semi-continuous i.e. for each $\varepsilon > 0$ and for all $a \in I$, $X^{-1}([0, a+\varepsilon))$ is open in the usual topology of R.
(iii) X is convex i.e. $X(t) \geq X(s) \wedge X(r) = min(X(s), X(r))$, where $s < t < r$.
(iv) The closure of $\{t \in R : X(t) > 0\}$ is compact.

The class of all *upper-semi-continuous*, *normal*, *convex* fuzzy real numbers is denoted by $R(I)$.

The absolute value of $X \in R(I)$ is defined by

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{for } t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The set R of all real numbers can be embedded in $R(I)$. For $r \in R$, $\bar{r} \in R(I)$ is defined by

$$\bar{r}(t) = \begin{cases} 1, & \text{for } t = r, \\ 0, & \text{for } t \neq r. \end{cases}$$

We denote the additive identity and multiplicative identity of $R(I)$ by $\bar{0}$ and $\bar{1}$, respectively.

For any $X, Y, Z \in R(I)$, the linear structure of $R(I)$ induces addition $X + Y$ and scalar multiplication $\lambda X, \lambda \in R$ in terms of α-level set, defined as $[X+Y]^\alpha = [X]^\alpha + [Y]^\alpha$ and $[\lambda X]^\alpha = \lambda [X]^\alpha$, for each $\alpha \in [0,1]$. A subset E of $R(I)$ is said to be bounded above if there exists a fuzzy real number μ such that $X \leq \mu$ for every $X \in E$. We call μ an upper bound of E and it is called the least upper bounds if $\mu \leq \mu^*$ for all upper bound μ^* of E. A lower bound and greatest lower bound can be defined similarly. The set E is said to be bounded if it is both bounded above and bounded below.

Let D be the set of all closed and bounded intervals $X = [X^L, X^R]$. We define a metric on D by

$$d(X, Y) = \max(|X^L - Y^L|, |X^R - Y^R|).$$

It is straightforward that (D, d) is a complete metric space.

Define $\bar{d} : R(I) \times R(I) \to R$ by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha), \text{ for } X, Y \in R(I).$$

It is well established that $(R(I), \bar{d})$ is a complete metric space.

The aim of this article is to introduce the concept of lacunary p-absolutely summable double sequence of fuzzy real numbers and make an effort to study some algebraic properties as well as some inclusion relation.

2. Preliminaries and background

In this part, we recall some fundamental notions, which are closely related to the article.

A fuzzy real-valued sequence is denoted by (X_k), where $X_k \in R(I)$, for all $k \in N$.

A sequence (X_k) of fuzzy real numbers is said to be convergent to the fuzzy real number X_0, if for every $\varepsilon > 0$, there exists $k_0 \in N$ such that $\bar{d}(X_k, X_0) < \varepsilon$, for all $k \geq k_0$.

Let E^F be the class of sequences of fuzzy real numbers, the linearity of E^F can be understood as follows.

For $(X_k), (Y_k) \in E^F$ and $r \in R$,

(i) $$(X_k) + (Y_k) = (X_k + Y_k) \in E^F,$$

(ii) $$r(X_k) = (rX_k) \in E^F,$$

where

$$r(X_k)(t) = \begin{cases} X_k(r^{-1}t), & \text{if } r \neq 0, \\ \bar{0}, & \text{if } r = 0. \end{cases}$$

A fuzzy real-valued double sequence is a double infinite array of fuzzy real numbers. We denote it by (X_{nk}), where X_{nk} are fuzzy real numbers for each $n, k \in N$. We denote the class of all fuzzy real-valued double sequences by ${}_2w_F$.

Definition 1. A fuzzy real-valued double sequence (X_{nk}) is said to be convergent in Pringsheim's sense to the fuzzy real number X, if for every $\varepsilon > 0$, there exist $n_0 = n_0(\varepsilon), k_0 = k_0(\varepsilon)$, such that $\bar{d}(X_{nk}, X) < \varepsilon$ for all $n \geq n_0$ and $k \geq k_0$.

Definition 2. A fuzzy real-valued double sequence (X_{nk}) is said to be bounded if $\sup_{n,k} \bar{d}(X_{nk}, \bar{0}) < \infty$, equivalently, if there exist $R^*(I)$ such that $|X_{nk}| \leq \mu$ for all $n, k \in N$, where $R^*(I)$ denotes the set of all positive fuzzy real numbers.

Definition 3. A fuzzy real valued double sequence space ${}_2w_F$ is said to be solid if $(Y_{nk}) \in {}_2w_F$, whenever $(X_{nk}) \in {}_2w_F$ and $|Y_{nk}| \leq |X_{nk}|$, for all $n, k \in N$.

Definition 4. A fuzzy real valued double sequence space ${}_2w_F$ is said to be symmetric if $(X_{\pi(nk)}) \in {}_2w_F$, whenever $(X_{nk}) \in {}_2w_F$, where π is a permutation on $N \times N$.

Definition 5. A fuzzy real valued double sequence space ${}_2w_F$ is said to be convergence free if $(Y_{nk}) \in {}_2w_F$, whenever $(X_{nk}) \in {}_2w_F$ and $X_{nk} = \bar{0}$ implies $Y_{nk} = \bar{0}$.

Definition 6. A fuzzy real valued double sequence space ${}_2w_F$ is said to be sequence algebra if $(X_{nk}) \bigotimes (Y_{nk}) \in {}_2w_F$, whenever $(X_{nk}), (Y_{nk}) \in {}_2w_F$.

Tripathy and Dutta [12] introduced the notion of Cesàro summable and strongly p-Cesàro summable double sequences of fuzzy real numbers as follows:

Definition 7. A fuzzy real valued double sequence (X_{nk}) is said to be Cesàro summable to a fuzzy real number L if

$$\bar{d}\left(\frac{1}{uv}\sum_{n=1}^{u}\sum_{k=1}^{v} X_{nk}, L\right) \to 0 \text{ as } u, v \to \infty.$$

Definition 8. A fuzzy real-valued double sequence (X_{nk}) is said to be strongly p-Cesàro summable to a fuzzy real number L, if

$$\frac{1}{uv}\left(\sum_{n=1}^{u}\sum_{k=1}^{v}[\bar{d}(X_{nk}, L)]^p\right) \to 0 \text{ as } u, v \to \infty.$$

We denote it by $Ces_2(p)$.

Some important classes of sequences of fuzzy real numbers have been introduced and studied by Kwon [5], Savas and Patterson [6], Talo and Basar [12], Tripathy and Dutta ([13], [15]) and some others.

The class of fuzzy real-valued double sequences ${}_2\ell_F^p$ was introduced by Tripathy and Dutta [14] as follows:

$${}_2\ell_F^p = \left\{X = (X_{nk}) : \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left[\bar{d}(X_{nk}, \bar{0})\right]^p < \infty\right\},\ 1 \le p < \infty.$$

Dutta [2] introduced the class of lacunary p-absolutely summable fuzzy real-valued sequences $(\ell_p)_\theta^F$ as follows:

$$(\ell_p)_\theta^F = \left\{X = (X_n) : \sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{n\in I_r}\bar{d}(X_n, \bar{0})\right)^p < \infty\right\},\ 1 \le p < \infty.$$

We introduce the class of fuzzy real-valued double sequences $({}_2\ell_p)_\theta^F$ as follows:

$$({}_2\ell_p)_\theta^p = \left\{X = (X_{nk}) : \sum_{r,s=1}^{\infty,\infty}\left(\frac{1}{h_{rs}}\sum_{n,k\in I_{r,s}}\bar{d}(X_{nk}, \bar{0})\right)^p < \infty\right\},\ 1 \le p < \infty.$$

3. Main results

Theorem 1. *The class of sequences $({}_2\ell_p)_\theta^F$ is closed under addition and scalar multiplication.*

Proof. Let $\theta_{r,s} = \{k_r, \ell_s\}$ be a double lacunary sequence and $(X_{nk}) \in ({}_2\ell_p)_\theta^F$. Since $\bar{d}(cX_k^\alpha, cY_k^\alpha) = |c|\bar{d}(X_k^\alpha, Y_k^\alpha)$, we have $\bar{d}(cX, cY) = |c|\bar{d}(X, Y)$, for any

$c \in R$. This gives

$$\sum_{r,s=1,1}^{\infty,\infty}\left(\frac{1}{h_{r,s}}\sum_{n,k\in I_{r,s}}\bar{d}(cX_{nk},\bar{0})\right)^p = \sum_{r,s=1,1}^{\infty,\infty}\left(\frac{1}{h_{r,s}}\sum_{n,k\in I_{r,s}}|c|\bar{d}(X_{nk},\bar{0})\right)^p$$
$$= |c|^p\sum_{r,s=1,1}^{\infty,\infty}\left(\frac{1}{h_{r,s}}\sum_{n,k\in I_{r,s}}\bar{d}(X_{nk},\bar{0})\right)^p < \infty.$$

This shows that $(cX_{nk}) \in ({}_2\ell_p)_\theta^F$.

Next we suppose that $(X_{nk}), (Y_{nk}) \in ({}_2\ell_p)_\theta^F$ and we notice that

$$d(X_k^\alpha + Y_k^\alpha, X_0^\alpha + Y_0^\alpha) \le d(X_k^\alpha, X_0^\alpha) + d(Y_k^\alpha, Y_0^\alpha).$$

Thus we have

$$\sum_{r,s=1,1}^{\infty,\infty}\left(\frac{1}{h_{r,s}}\sum_{n,k\in I_{r,s}}\bar{d}\{(X_{nk}+Y_{nk}),\bar{0}\}\right)^p$$
$$\le \sum_{r,s=1,1}^{\infty,\infty}\left(\frac{1}{h_{r,s}}\sum_{n,k\in I_{r,s}}\bar{d}(X_{nk},\bar{0})\right)^p + \sum_{r,s=1,1}^{\infty,\infty}\left(\frac{1}{h_{r,s}}\sum_{n,k\in I_{r,s}}\bar{d}(Y_{nk},\bar{0})\right)^p < \infty.$$

Thus $(X_{nk}+Y_{nk}) \in ({}_2\ell_p)_\theta^F$. This completes the proof. ∎

Theorem 2. *Let $\theta_{r,s} = \{k_r, \ell_s\}$ be a double lacunary sequence and lim inf $q_r > 1$, and lim inf $\acute{q}_s > 1$, then for $0 < p < 1$, $Ces_2(p) \subset ({}_2\ell_p)_\theta^F$.*

Proof. Suppose *lim inf* $q_r > 1$, and *lim inf* $\acute{q}_s > 1$ then there exists $\delta > 0$ such that $q_r > 1+\delta$ and $\acute{q}_s > 1+\delta$. This implies

$$\frac{k_r}{h_r} = \frac{k_r}{k_r - k_{r-1}} = \frac{q_r}{q_r - 1} \le \frac{1+\delta}{\delta} \text{ and } \frac{\ell_s}{\acute{h}_s} = \frac{\ell_s}{\ell_s - \ell_{s-1}} = \frac{\acute{q}_s}{\acute{q}_s - 1} \le \frac{1+\delta}{\delta}.$$

Let $(X_{nk}) \in Ces_2(p)$, then we can write

$$\text{(i)} \qquad \sum_{r,s=1,1}^{\infty,\infty}\left\{\frac{1}{k_r\ell_s}\sum_{n=1,k=1}^{k_r,\ell_s}\bar{d}(X_{nk},\bar{0})\right\}^p < \infty.$$

We have

$$\sum_{r,s=1,1}^{\infty,\infty}\left[\frac{1}{h_{r,s}}\sum_{n,k\in I_{r,s}}\bar{d}(X_{nk},\bar{0})\right]^p$$
$$= \sum_{r,s=1,1}^{\infty,\infty}\left[\frac{1}{h_{r,s}}\sum_{n=1,k=1}^{k_r,\ell_s}\bar{d}(X_{nk},\bar{0}) - \frac{1}{h_{r,s}}\sum_{n=1,k=1}^{k_{r-1},\ell_{s-1}}\bar{d}(X_{nk},\bar{0})\right]^p$$
$$\le \sum_{r,s=1,1}^{\infty,\infty}\left[\frac{1}{h_{r,s}}\sum_{n=1,k=1}^{k_r,\ell_s}\bar{d}(X_{nk},\bar{0})\right]^p - \sum_{r,s=1,1}^{\infty,\infty}\left[\frac{1}{h_{r,s}}\sum_{n=1,k=1}^{k_{r-1},\ell_{s-1}}\bar{d}(X_{nk},\bar{0})\right]^p$$

$$= \sum_{r,s=1,1}^{\infty,\infty} \left[\frac{1}{h_r \acute{h}_s} \sum_{n=1,k=1}^{k_r,\ell_s} \bar{d}(X_{nk},\bar{0})\right]^p - \sum_{r,s=1,1}^{\infty,\infty} \left[\frac{1}{h_r \acute{h}_s} \sum_{n=1,k=1}^{k_{r-1},\ell_{s-1}} \bar{d}(X_{nk},\bar{0})\right]^p$$

$$= \sum_{r,s=1,1}^{\infty,\infty} \frac{1}{k_r \ell_s} \left[\frac{k_r}{h_r}\frac{\ell_s}{\acute{h}_s} \sum_{n=1,k=1}^{k_r,\ell_s} \bar{d}(X_{nk},\bar{0})\right]^p$$

$$- \sum_{r,s=1,1}^{\infty,\infty} \frac{1}{k_{r-1}\ell_{s-1}} \left[\frac{k_{r-1}}{h_r}\frac{\ell_{s-1}}{\acute{h}_s} \sum_{n=1,k=1}^{k_{r-1},\ell_{s-1}} \bar{d}(X_{nk},\bar{0})\right]^p$$

$$\leq \left(\frac{1+\delta}{\delta}\right)^2 \sum_{r,s=1,1}^{\infty,\infty} \frac{1}{k_r \ell_s} \Big[\sum_{n=1,k=1}^{k_r,\ell_s} \bar{d}(X_{nk},\bar{0})\Big]^p$$

$$- \left(\frac{1}{\delta}\right)^2 \sum_{r,s=1,1}^{\infty,\infty} \frac{1}{k_{r-1}\ell_{s-1}} \Big[\sum_{n=1,k=1}^{k_{r-1},\ell_{s-1}} \bar{d}(X_{nk},\bar{0})\Big]^p.$$

Using (i) we have

$$\sum_{r,s=1,1}^{\infty,\infty} \left[\frac{1}{h_{r,s}} \sum_{n,k\in I_{r,s}} \bar{d}(X_{nk},\bar{0})\right]^p < \infty.$$

This completes the proof. ∎

Theorem 3. *Let $\theta_{r,s} = \{k_r, \ell_s\}$ be a double lacunary sequence and lim sup $q_r < \infty$ and lim sup $\acute{q}_s < \infty$ then for $0 < p < 1$, $({}_2\ell_p)_\theta^F \subset Ces_2(p)$.*

Proof. Let *lim sup* $q_r < \infty$ and *lim sup* $\acute{q}_s < \infty$. Then there exists $M > 0$ such that $q_r < M$ and $\acute{q}_s < M$ for all r, s. Let $(X_{nk}) \in ({}_2\ell_p)_\theta^F$ and $\varepsilon > 0$ be given, then there exist $r_0 > 0$, $s_0 > 0$ such that

$$A_{ij} = \frac{1}{h_{r,s}} \sum_{n,k\in I_{r,s}} [\bar{d}(X_{nk},\bar{0})]^p < \varepsilon, \text{for every } i > r_0 \text{ and } j > s_0.$$

Let $K = max\{A_{ij} : 1 \leq r \leq r_0; 1 \leq s \leq s_0\}$ and choose m and p such that $k_{r-1} < m \leq k_r$ and $\ell_{s-1} < p \leq \ell_s$. Then we have

$$\frac{1}{mp} \sum_{n=1,k=1}^{m,p} [\bar{d}(X_{nk},\bar{0})]^p \leq \frac{1}{k_{r-1}\ell_{s-1}} \sum_{n=1,k=1}^{k_r,\ell_s} [\bar{d}(X_{nk},\bar{0})]^p$$

$$\leq \frac{1}{k_{r-1}\ell_{s-1}} \sum_{u=1,v=1}^{r,s} \Big\{\sum_{n,k\in I_{u,v}} [\bar{d}(X_{nk},\bar{0})]^p\Big\}$$

$$= \frac{1}{k_{r-1}\ell_{s-1}} \sum_{u=1,v=1}^{r_0,s_0} h_{u,v}A_{u,v} + \frac{1}{k_{r-1}\ell_{s-1}} \sum_{(r_0<u<r)\cup(s_0<v<s)} h_{u,v}A_{u,v}$$

$$\leq \frac{K}{k_{r-1}\ell_{s-1}} \sum_{u=1,v=1}^{r_0,s_0} h_{u,v} + \frac{1}{k_{r-1}\ell_{s-1}} \sum_{(r_0<u<r)\cup(s_0<v<s)} h_{u,v}A_{u,v}$$

$$\leq \frac{Kn_{r_0}k_{s_0}r_0s_0}{k_{r-1}\ell_{s-1}} + \frac{1}{k_{r-1}\ell_{s-1}} \sum_{(r_0<u<r)\cup(s_0<v<s)} h_{u,v}A_{u,v}$$

$$\leq \frac{Kn_{r_0}k_{s_0}r_0s_0}{k_{r-1}\ell_{s-1}} + \sup_{(u>r_0)\cup(v>s_0)} (A_{u,v}).\frac{1}{k_{r-1}\ell_{s-1}} \sum_{(r_0<u<r)\cup(s_0<v<s)} h_{u,v}$$

$$\leq \frac{Kn_{r_0}k_{s_0}r_0s_0}{k_{r-1}\ell_{s-1}} + \frac{1}{k_{r-1}\ell_{s-1}}\ \varepsilon \sum_{(r_0<u<r)\cup(s_0<v<s)} h_{u,v}$$

$$\leq \frac{Kn_{r_0}k_{s_0}r_0s_0}{k_{r-1}\ell_{s-1}} + \varepsilon M^2.$$

We observe that k_r and ℓ_s will tend to infinity as $m, p \to \infty$, thereby

$$\frac{1}{mp} \sum_{n=1,k=1}^{m,p} [\bar{d}(X_{nk}, \bar{0})]^p \to \infty.$$

Thus $(X_{nk}) \in Ces_2(p)$. This completes the proof. ∎

Combining Theorem 2 and Theorem 3, we have the following theorem.

Theorem 4. *Let $\theta_{r,s} = \{k_r, \ell_s\}$ be a double lacunary sequence. If $1 < lim\ inf\ q_r < lim\ sup\ q_r < \infty$ and $1 < lim\ inf\ \acute{q}_s < lim\ sup\ \acute{q}_s < \infty$ then $({}_2\ell_p)_\theta^F = Ces_2(p)$, for $0 < p < 1$.*

Theorem 5. *The class of sequences $({}_2\ell_p)_\theta^F$ is solid.*

Proof. Let $\theta_{r,s} = \{k_r, \ell_s\}$ be a double lacunary sequence, consider the double sequences $(X_{nk}), (Y_{nk}) \in {}_2w_F$ such that $\bar{d}(Y_{nk}, \bar{0}) < \bar{d}(X_{nk}, \bar{0})$ for all $n, k \in N$. Suppose that $(X_{nk}) \in ({}_2\ell_p)_\theta^F$, then

$$\sum_{r=1,s=1}^{\infty,\infty} \left(\frac{1}{h_{r,s}} \sum_{n,k\in I_{r,s}} \bar{d}(X_{nk}, \bar{0}) \right)^p < \infty.$$

We have

$$\sum_{r=1,s=1}^{\infty,\infty} \left(\frac{1}{h_{r,s}} \sum_{n,k\in I_{r,s}} \bar{d}(Y_{nk}, \bar{0}) \right)^p \leq \sum_{r=1,s=1}^{\infty,\infty} \left(\frac{1}{h_{r,s}} \sum_{n,k\in I_{r,s}} \bar{d}(X_{nk}, \bar{0}) \right)^p < \infty.$$

Thus $(Y_{nk}) \in ({}_2\ell_p)_\theta^F$. This completes the proof. ∎

Theorem 6. *The class of sequences $({}_2\ell_p)_\theta^F$ is a sequence algebra.*

Proof. Let $\theta_{r,s} = \{k_r, \ell_s\}$ be a double lacunary sequence and $(X_{nk}), (Y_{nk}) \in ({}_2\ell_p)_\theta^F$.

We observe that

$$\sum_{r=1,s=1}^{\infty,\infty}\left(\frac{1}{h_{r,s}}\sum_{n,k\in I_{r,s}}\bar{d}(X_{nk}\otimes Y_{nk},\bar{0})\right)^p$$
$$\leq \sum_{r=1,s=1}^{\infty,\infty}\left(\frac{1}{h_{r,s}}\sum_{n,k\in I_{r,s}}\bar{d}(X_{nk},\bar{0})\bar{d}(Y_{nk},\bar{0})\right)^p$$
$$\leq \sum_{r=1,s=1}^{\infty,\infty}\left(\frac{1}{h_{r,s}}\sum_{n,k\in I_{r,s}}\bar{d}(X_{nk},\bar{0})\right)^p \sum_{r=1,s=1}^{\infty,\infty}\left(\frac{1}{h_{r,s}}\sum_{n,k\in I_{r,s}}\bar{d}(Y_{nk},\bar{0})\right)^p < \infty.$$

This completes the proof. ∎

Theorem 7. *The class of sequences $({}_2\ell_p)_\theta^F$ is not convergence free in general.*

Proof. We provide the following example in support of the proof.

Example 1. Let $\theta = (3^r, 3^s)$ be a double lacunary sequence and $p = 1$. Consider the double sequence (X_{nk}) defined by

$$X_{nk}(t) = \begin{cases} \{(n+k)^2 t + 1\}, & \text{for } \frac{-1}{(n+k)^2} \leq t \leq 0; \\ \{1-(n+k)^2 t\}, & \text{for } 0 \leq t \leq \frac{1}{(n+k)^2}; \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\sum_{r=1,s=1}^{\infty,\infty}\left(\frac{1}{h_{r,s}}\sum_{n,k\in I_{r,s}}\bar{d}(X_{nk},\bar{0})\right)^p = \sum_{r=1,s=1}^{\infty,\infty}\left(\frac{1}{4.3^{r+s-2}}\sum_{n,k\in I_{r,s}}(n+k)^{-2}\right) < \infty.$$

Thus $(X_{nk}) \in ({}_2\ell_p)_\theta^F$.

Consider the double sequence (Y_{nk}) defined by

$$Y_{nk}(t) = \begin{cases} \{t\sqrt{n+k}+1\}, & \text{for } \frac{-1}{\sqrt{(n+k)}} \leq t \leq 0; \\ \{1-t\sqrt{n+k}\}, & \text{for } 0 \leq t \leq \frac{1}{\sqrt{(n+k)}}; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\sum_{r=1,s=1}^{\infty,\infty}\left(\frac{1}{h_{r,s}}\sum_{n,k\in I_{r,s}}\bar{d}(Y_{nk},\bar{0})\right)^p = \sum_{r=1,s=1}^{\infty,\infty}\left(\frac{1}{4.3^{r+s-2}}\sum_{n,k\in I_{r,s}}(n+k)^{-\frac{1}{2}}\right) = \infty.$$

This shows that $(Y_{nk}) \notin ({}_2\ell_p)_\theta^F$.

Theorem 8. $({}_2\ell_p)^F \subset ({}_2\ell_p)_\theta^F$, if $\sum_{r=1,s=1}^{\infty,\infty}\frac{1}{h_{r,s}} < \infty$.

Proof. Let $(X_{nk}) \in ({}_2\ell_p)^F$, therefore $\sum_{r=1,s=1}^{\infty,\infty} \left\{\bar{d}(X_{nk}, \bar{0})\right\}^p < \infty$. We can choose a positive integer n_0 such that for all $n, k > n_0$, $\sum_{n,k>n_0} \{\bar{d}(X_{nk}, \bar{0})\}^p < 1$.

Thus we have

$$\sum_{r,s>n_0} \left\{ \frac{1}{h_{r,s}} \sum_{n,k\in I_{r,s}} \bar{d}(X_{nk}, \bar{0}) \right\}^p \leq \sum_{r,s>n_0} \frac{1}{h_{r,s}} < \infty.$$

This implies

$$\sum_{r=1,s=1}^{\infty,\infty} \left\{ \frac{1}{h_{r,s}} \sum_{n,k\in I_{r,s}} \bar{d}(X_{nk}, \bar{0}) \right\}^p < \infty.$$

This completes the proof. ∎

Theorem 9. For $0 < p < q$, $({}_2\ell_p)_\theta^F \subset ({}_2\ell_q)_\theta^F$.

Proof. The proof follows directly from the following inclusion relation:

$$\sum_{r=1,s=1}^{\infty,\infty} \left\{ \frac{1}{h_{r,s}} \sum_{n,k\in I_{r,s}} \bar{d}(X_{nk}, \bar{0}) \right\}^p \subset \sum_{r=1,s=1}^{\infty,\infty} \left\{ \frac{1}{h_{r,s}} \sum_{n,k\in I_{r,s}} \bar{d}(X_{nk}, \bar{0}) \right\}^q. \blacksquare$$

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