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# ROUGH $\mathcal{I}$-CONVERGENCE

**Abstract.** In this work, using the concept of $\mathcal{I}$-convergence and using the concept of rough convergence, we introduced the notion of rough $\mathcal{I}$-convergence and the set of rough $\mathcal{I}$-limit points of a sequence and obtained two rough $\mathcal{I}$-convergence criteria associated with this set. Later, we proved that this set is closed and convex. Finally, we examined the relations between the set of $\mathcal{I}$-cluster points and the set of rough $\mathcal{I}$-limit points of a sequence.

## 1. Background and introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [4] and Schoenberg [15]. A lot of developments have been made in this area after the works of Aytar [1], Fridy [5], Miller [8] and Šalát [14]. In general, statistically convergent sequences satisfy many of the properties of ordinary convergent sequences in metric spaces.

The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [6] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers. Nuray and Ruckle [9] indepedently introduced the same with another name generalized statistical convergence. Kostyrko et al. [7] studied the idea of $\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points and Demirci [3] studied the concepts of $\mathcal{I}$-limit superior and limit inferior. Šalát, Tripathy and Ziman [13] introduced the notion of $c_A^{\mathcal{I}}$ and $m_A^{\mathcal{I}}$, the $\mathcal{I}$-convergence field and bounded $\mathcal{I}$-convergence field of an infinite matrix $A$.

The idea of rough convergence was first introduced by Phu [10] in finite-dimensional normed spaces. In [10], he showed that the set $LIM^r x$ is bounded, closed, and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other

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convergence types and the dependence of $LIM^r x$ on the roughness degree $r$. In another paper [11] related to this subject, he defined the rough continuity of linear operators and showed that every linear operator $f : X \to Y$ is $r$-continuous at every point $x \in X$ under the assumption $dimY < \infty$ and $r > 0$ where $X$ and $Y$ are normed spaces. In [12], he extended the results given in [10] to infinite-dimensional normed spaces.

In [1], Aytar studied rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also in [2], Aytar studied that the $r$-limit set of the sequence is equal to the intersection of these sets and that $r$-core of the sequence is equal to the union of these sets.

In this paper, using the concept of $\mathcal{I}$-convergence and using the concept of rough convergence, we introduce the notion of rough $\mathcal{I}$-convergence. Defining the set of rough $\mathcal{I}$-limit points of a sequence, we obtain two $\mathcal{I}$-convergence criteria associated with this set. Later, we prove that this set is closed and convex. Finally, we examine the relations between the set of $\mathcal{I}$-cluster points and the set of rough $\mathcal{I}$-limit points of a sequence. We note that our results and proof techniques presented in this paper are $\mathcal{I}$ analogues of those in Phu's [10] paper and Aytar' s [1] paper. The actual origin of most of these results and proof techniques is in those papers. Our theorems and results are the $\mathcal{I}$-extension of theorems and results in [1, 10].

Let $K$ be a subset of the set of positive integers $\mathbb{N}$ and let us denote the set $K_i = \{k \in K : k \leq i\}$. Then the natural density of $K$ is given by

$$\delta(K) = \lim_{i \to \infty} \frac{|K_i|}{i},$$

where $|K_i|$ denotes the number of elements in $K_i$.

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers, $\chi_A$-the characteristic function of $A \subset \mathbb{N}$, $\mathbb{R}$ the set of all real numbers. Recall that a subset $A$ of $\mathbb{N}$ is said to have asymptotic density $d(A)$ if

$$d(A) = \lim_{i \to \infty} \frac{1}{i} \sum_{k=1}^{i} \chi_A(k).$$

**Definition 1.1.** [4] A sequence $x = (x_i)_{i \in \mathbb{N}}$ of real numbers is said to be statistically convergent to $L \in \mathbb{R}$ if for any $\varepsilon > 0$ we have $d(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{i \in \mathbb{N} : |x_i - L| \geq \varepsilon\}$.

Throughout the paper, $\mathbb{R}^n$ denotes the real $n$-dimensional space with the norm $\|.\|$. Consider a sequence $x = (x_i)$ such that $x_i \in \mathbb{R}^n$.

**Definition 1.2.** [1] A sequence $x = (x_i)$ is said to be statistically convergent to $L \in \mathbb{R}^n$, written as st-lim $x = L$, provided that the set

$$\{i \in \mathbb{N} : \|x_i - L\| \geq \varepsilon\}$$

has natural density zero for every $\varepsilon > 0$. In this case, $L$ is called the statistical limit of the sequence $x$.

**Definition 1.3.** Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of $X$ is said to be an ideal in $X$ provided:

i) $\emptyset \in \mathcal{I}$, ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$.

**Definition 1.4.** Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of $X$ is said to be a filter in $X$ provided:

i) $\emptyset \notin \mathcal{F}$, ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

**Lemma 1.5.** [6] *If $\mathcal{I}$ is a nontrivial ideal in $X$, $X \neq \emptyset$, then the class*

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$$

*is a filter on $X$, called the filter associated with $\mathcal{I}$.*

A nontrivial ideal $\mathcal{I}$ in $X$ is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

**Example 1.6.** ([6], Example 3.1.) Denote by $\mathcal{I}_d$ the class of all $A \subset \mathbb{N}$ with $d(A) = 0$. Then $\mathcal{I}_d$ is non-trivial admissible ideal and $\mathcal{I}_d$-convergence coincides with the statistical convergence.

Throughout the paper, we take $\mathcal{I}$ as a nontrivial admissible ideal in $\mathbb{N}$.

**Definition 1.7.** [6] Let $(X, \rho)$ be a linear metric space and $\mathcal{I} \subset 2^{\mathbb{N}}$ be a non-trivial ideal. A sequence $(x_i)_{i \in \mathbb{N}}$ of elements of $X$ is said to be $\mathcal{I}$-convergent to $\xi \in X$ ($\mathcal{I} - \lim_{i \to \infty} x_i = \xi$) if and only if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{i \in \mathbb{N} : \rho(x_i, \xi) \geq \varepsilon\}$ belongs to $\mathcal{I}$. The element $\xi$ is called the $\mathcal{I}$-limit of the sequence $x = (x_i)_{i \in \mathbb{N}}$.

Note that if $\mathcal{I}$ is an admissible ideal, then usual convergence in $X$ implies $\mathcal{I}$-convergence in $X$.

**Definition 1.8.** [3] For a sequence $x = (x_i)$ of real numbers, the notions of ideal limit superior and ideal limit inferior are defined as follows:

$$\mathcal{I} - \limsup x = \begin{cases} \sup B_x, & \text{if } B_x \neq \emptyset, \\ -\infty, & \text{if } B_x = \emptyset, \end{cases}$$

and

$$\mathcal{I} - \liminf x = \begin{cases} \inf A_x, & \text{if } A_x \neq \emptyset, \\ +\infty, & \text{if } A_x = \emptyset, \end{cases}$$

where $A_x = \{a \in \mathbb{R} : \{i \in \mathbb{N} : x_i < a\} \notin \mathcal{I}\}$ and $B_x = \{b \in \mathbb{R} : \{i \in \mathbb{N} : x_i > b\} \notin \mathcal{I}\}$.

Throughout the paper, let $r$ be a nonnegative real number. The sequence $x = (x_i)$ is said to be $r$-convergent to $x_*$, denoted by $x_i \to^r x_*$ provided that

$$\forall \varepsilon > 0\ \exists i_\varepsilon \in \mathbb{N} : i \geq i_\varepsilon \Rightarrow \|x_i - x_*\| < r + \varepsilon.$$

The set

$$\mathrm{LIM}^r x := \{x_* \in \mathbb{R}^n : x_i \to^r x_*\}$$

is called the $r$-limit set of the sequence $x = (x_i)$. A sequence $x = (x_i)$ is said to be $r$-convergent if $LIM^r x \neq \emptyset$. In this case, $r$ is called the convergence degree of the sequence $x = (x_i)$. For $r = 0$, we get the ordinary convergence. There are several reasons for this interest (see [10]).

A sequence $x = (x_i)$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}^n$, written as $\mathcal{I}\text{-}\lim x = L$, provided that the set

$$\{i \in \mathbb{N} : \|x_i - L\| \geq \varepsilon\}$$

belongs to $\mathcal{I}$ for every $\varepsilon > 0$. In this case, $L$ is called the $\mathcal{I}$-limit of the sequence $x$.

$c \in \mathbb{R}^n$ is called a $\mathcal{I}$-cluster point of a sequence $x = (x_i)$ provided that

$$\{i \in \mathbb{N} : \|x_i - c\| < \varepsilon\} \notin \mathcal{I},$$

for every $\varepsilon > 0$. We denote the set of all $\mathcal{I}$-cluster points of the sequence $x$ by $\mathcal{I}(\Gamma_x)$.

A sequence $x = (x_i)$ is said to be $\mathcal{I}$-bounded if there exists a positive real number $M$ such that

$$\{i \in \mathbb{N} : \|x_i\| \geq M\} \in \mathcal{I}.$$

## 2. Main results

**Definition 2.1.** A sequence $x = (x_i)$ is said to be rough $\mathcal{I}$-convergent to $x_*$, denoted by $x_i \xrightarrow{r-\mathcal{I}} x_*$ provided that

$$\{i \in \mathbb{N} : \|x_i - x_*\| \geq r + \varepsilon\}$$

belongs to $\mathcal{I}$ for every $\varepsilon > 0$; or equivalently, if the condition

$$\mathcal{I} - \limsup \|x_i - x_*\| \leq r \tag{2.1}$$

is satisfied. In addition, we can write $x_i \xrightarrow{r-\mathcal{I}} x_*$ iff the inequality $\|x_i - x_*\| < r + \varepsilon$ holds for every $\varepsilon > 0$ and almost all $i$.

**Remark 2.2.** If $\mathcal{I}$ is an admissible ideal, then usual rough convergence implies rough $\mathcal{I}$-convergence.

Here $r$ is called the roughness degree. If we take $r = 0$, then we obtain the ordinary ideal convergence. In a similar fashion to the idea of classic rough convergence, the idea of rough $\mathcal{I}$-convergence of a sequence can be interpreted as follows.

Assume that a sequence $y = (y_i)$ is $\mathcal{I}$-convergent and cannot be measured or calculated exactly; one has to do with an approximated (or $\mathcal{I}$ approximated) sequence $x = (x_i)$ satisfying $\|x_i - y_i\| \leq r$ for all $i$ (i.e., $\{i \in \mathbb{N} : \|x_i - y_i\| > r\} \in \mathcal{I}$). Then the sequence $x$ is not $\mathcal{I}$-convergent any more, but as the inclusion

$$\{i \in \mathbb{N} : \|y_i - y_*\| \geq \varepsilon\} \supseteq \{i \in \mathbb{N} : \|x_i - y_*\| \geq r + \varepsilon\} \tag{2.2}$$

holds and we have $\{i \in \mathbb{N} : \|y_i - y_*\| \geq \varepsilon\} \in \mathcal{I}$, we get $\{i \in \mathbb{N} : \|x_i - y_*\| \geq r + \varepsilon\} \in \mathcal{I}$, i.e., the sequence $x$ is rough $\mathcal{I}$-convergent in the sense of Definition 2.1.

In general, the rough $\mathcal{I}$-limit of a sequence may not be unique for the roughness degree $r > 0$. So we have to consider the so-called rough $\mathcal{I}$-limit set of a sequence $x = (x_i)$, which is defined by

$$\mathcal{I} - \mathrm{LIM}^r x := \{x_* \in \mathbb{R}^n : x_i \xrightarrow{r-\mathcal{I}} x_*\}.$$

A sequence $x = (x_i)$ is said to be rough $\mathcal{I}$-convergent if $\mathcal{I} - \mathrm{LIM}^r x \neq \emptyset$. It is clear that if $\mathcal{I} - \mathrm{LIM}^r x \neq \emptyset$ for a sequence $x = (x_i)$ of real numbers, then we have

$$\mathcal{I} - \mathrm{LIM}^r x = [\mathcal{I} - \limsup x - r, \mathcal{I} - \liminf x + r]. \tag{2.3}$$

We know that $\mathrm{LIM}^r x = \emptyset$ for an unbounded sequence $x = (x_i)$. But such a sequence might be rough $\mathcal{I}$-convergent. For instance, let $\mathcal{I}$ be the $\mathcal{I}_d$ of $\mathbb{N}$ and define

$$x_i = \begin{cases} \cos i\pi, & \text{if } i \neq k^2 (k \in \mathbb{N}), \\ i, & \text{otherwise} \end{cases} \tag{2.4}$$

in $\mathbb{R}^1$. Because the set $\{1, 4, 9, 16, \dots\}$ belongs to $\mathcal{I}$, we have

$$\mathcal{I} - \mathrm{LIM}^r x = \begin{cases} \emptyset, & \text{if } r < 1, \\ [1 - r, r - 1], & \text{otherwise} \end{cases}$$

and $\mathrm{LIM}^r x = \emptyset$, for all $r \geq 0$.

As can be seen by the example above, the fact that $\mathcal{I} - \mathrm{LIM}^r x \neq \emptyset$ does not imply $\mathrm{LIM}^r x \neq \emptyset$. Because $\mathcal{I}$ is a admissible ideal, $\mathrm{LIM}^r x \neq \emptyset$ implies $\mathcal{I} - \mathrm{LIM}^r x \neq \emptyset$, i.e., if $x = (x_i) \in \mathrm{LIM}^r x$ then, by Remark 2.2 $x = (x_i) \in \mathcal{I} - \mathrm{LIM}^r x$, for each sequence $x = (x_i)$. Also, if we define all the rough convergence sequences by $\mathrm{LIM}^r$ and if we define all the rough $\mathcal{I}$-convergence sequences by $\mathcal{I} - \mathrm{LIM}^r$, then we get $\mathrm{LIM}^r \subseteq \mathcal{I} - \mathrm{LIM}^r$. This obvious fact means

$$\{r \geq 0 : \mathrm{LIM}^r x \neq \emptyset\} \subseteq \{r \geq 0 : \mathcal{I} - \mathrm{LIM}^r x \neq \emptyset\}$$

in the language of sets and yields immediately

$$\inf\{r \geq 0 : \mathrm{LIM}^r x \neq \emptyset\} \geq \inf\{r \geq 0 : \mathcal{I} - \mathrm{LIM}^r x \neq \emptyset\},$$

for each $x = (x_i)$ sequence. Moreover, it also yields directly

$$diam(\mathrm{LIM}^r x) \leq diam(\mathcal{I} - \mathrm{LIM}^r x).$$

As noted above, we cannot say that the rough $\mathcal{I}$-limit of a sequence is unique for the roughness degree $r > 0$. The following result is related to the this fact.

**Theorem 2.3.** *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. For a sequence $x = (x_i)$, we have $diam(\mathcal{I} - \mathrm{LIM}^r x) \leq 2r$. In general, $diam(\mathcal{I} - \mathrm{LIM}^r x)$ has no smaller bound.*

**Proof.** Assume that $diam(\mathcal{I} - \mathrm{LIM}^r x) > 2r$. Then there exist $y, z \in \mathcal{I} - \mathrm{LIM}^r x$ such that $\|y - z\| > 2r$. Take $\varepsilon \in (0, \frac{\|y-z\|}{2} - r)$. Because $y, z \in \mathcal{I} - \mathrm{LIM}^r x$, we have $A_1(\varepsilon) \in \mathcal{I}$ and $A_2(\varepsilon) \in \mathcal{I}$ for every $\varepsilon > 0$, where

$$A_1(\varepsilon) = \{i \in \mathbb{N} : \|x_i - y\| \geq r + \varepsilon\} \text{ and } A_2(\varepsilon) = \{i \in \mathbb{N} : \|x_i - z\| \geq r + \varepsilon\}.$$

Using the properties of $\mathcal{F}(\mathcal{I})$, we get

$$(A_1(\varepsilon)^c \cap A_2(\varepsilon)^c) \in \mathcal{F}(\mathcal{I}).$$

Thus, we can write

$$\|y - z\| \leq \|x_i - y\| + \|x_i - z\| < 2(r + \varepsilon) < 2\left(r + \frac{\|y - z\|}{2} - r\right) = \|y - z\|,$$

for all $i \in A_1(\varepsilon)^c \cap A_2(\varepsilon)^c$, which is a contradiction.

Now let us prove the second part of the theorem. Consider a sequence $x = (x_i)$ such that $\mathcal{I}\text{-}\lim x_i = x_*$. Let $\varepsilon > 0$. Then, we can write

$$\{i \in \mathbb{N} : \|x_i - x_*\| \geq \varepsilon\} \in \mathcal{I}.$$

Thus, we have

$$\|x_i - y\| \leq \|x_i - x_*\| + \|x_* - y\| \leq \|x_i - x_*\| + r,$$

for each $y \in \overline{B_r}(x_*) := \{y \in \mathbb{R}^n : \|y - x_*\| \leq r\}$. Then, we get

$$\|x_i - y\| < r + \varepsilon,$$

for each $i \in \{i \in \mathbb{N} : \|x_i - x_*\| < \varepsilon\}$. Because the sequence $x$ is $\mathcal{I}$-convergent to $x_*$, we have

$$\{i \in \mathbb{N} : \|x_i - x_*\| < \varepsilon\} \in \mathcal{F}(\mathcal{I}).$$

Therefore, we get $y \in \mathcal{I} - \mathrm{LIM}^r x$. Consequently, we can write

$$\mathcal{I} - \mathrm{LIM}^r x = \overline{B_r}(x_*). \tag{2.5}$$

Because $diam(\overline{B_r}(x_*)) = 2r$, this shows that in general, the upper bound $2r$ of the diameter of the set $\mathcal{I} - \mathrm{LIM}^r x$ cannot be decreased anymore. ■

By [10, Proposition 2.2], there exists a nonnegative real number $r$ such that $\mathrm{LIM}^r x \neq \emptyset$ for a bounded sequence. Because the fact $\mathrm{LIM}^r x \neq \emptyset$ implies $\mathcal{I} - \mathrm{LIM}^r x \neq \emptyset$, we have the following result.

**Result 2.1.** *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. If a sequence $x = (x_i)$ is bounded, then there exists a nonnegative real number $r$ such that $\mathcal{I} - \mathrm{LIM}^r x \neq \emptyset$.*

The converse implication of the above result is not valid. If we take the sequence as $\mathcal{I}$-bounded, then the converse of Result 2.1 holds. Thus we have the following theorem.

**Theorem 2.4.** *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence $x = (x_i)$ is $\mathcal{I}$-bounded if and only if there exists a nonnegative real number $r$ such that $\mathcal{I} - \mathrm{LIM}^r x \neq \emptyset$. And also, for all $r > 0$, an $\mathcal{I}$-bounded sequence $x = (x_i)$ always contains a subsequence $(x_{i_j})$ with $\mathcal{I} - \mathrm{LIM}^{(x_{i_j}),r} x_{i_j} \neq \emptyset$.*

**Proof.** Because the sequence $x$ is $\mathcal{I}$-bounded, there exists a positive real number $M$ such that $\{i \in \mathbb{N} : \|x_i\| \geq M\} \in \mathcal{I}$. Define $r' := sup\{\|x_i\| : i \in K^c\}$, where $K = \{i \in \mathbb{N} : \|x_i\| \geq M\}$. Then the set $\mathcal{I} - \mathrm{LIM}^{r'} x$ contains the origin of $\mathbb{R}^n$. So we have $\mathcal{I} - \mathrm{LIM}^{r'} x \neq \emptyset$.

If $\mathcal{I} - \mathrm{LIM}^r x \neq \emptyset$ for some $r \geq 0$, then there exists $x_*$ such that $x_* \in \mathcal{I} - \mathrm{LIM}^r x$, i.e.,

$$\{i \in \mathbb{N} : \|x_i - x_*\| \geq r + \varepsilon\} \in \mathcal{I},$$

for each $\varepsilon > 0$. Then we say that almost all $x_i$' s are contained in some ball with any radius greater than $r$. So the sequence $x$ is $\mathcal{I}$-bounded.

As $(x_i)$ is a $\mathcal{I}$-bounded sequence in a finite-dimensional normed space, it certainly contains a $\mathcal{I}$-convergent subsequence $(x_{i_j})$. Let $x_*$ be its $\mathcal{I}$-limit point, then $\mathcal{I} - \mathrm{LIM}^r x_{i_j} = \overline{B}_r(x_*)$ and, for $r > 0$,

$$\mathcal{I} - \mathrm{LIM}^{(x_{i_j}),r} x_{i_j} \neq \emptyset. \ ■$$

Also, we have the following theorem.

**Theorem 2.5.** *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. If $(x_{i_j})$ is a subsequence of $(x_i)$, then*

$$\mathcal{I} - \mathrm{LIM}^r x_i \subseteq \mathcal{I} - \mathrm{LIM}^r x_{i_j}.$$

**Proof.** The proof is trivial (see [10], Proposition 2.3). ■

Now we give the topological and geometrical properties of the rough $\mathcal{I}$-limit set of a sequence.

**Theorem 2.6.** *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. The rough $\mathcal{I}$-limit set of a sequence $x = (x_i)$ is closed.*

**Proof.** If $\mathcal{I} - \mathrm{LIM}^r x = \emptyset$, then there is nothing to prove. Assume that $\mathcal{I} - \mathrm{LIM}^r x \neq \emptyset$. Then we can choose a sequence $(y_i) \subseteq \mathcal{I} - \mathrm{LIM}^r x$ such that $y_i \to y_*$ for $i \to \infty$. If we show that $y_* \in \mathcal{I} - \mathrm{LIM}^r x$, then the proof will be complete.

Let $\varepsilon > 0$ be given. Because $y_i \to y_*$, there exists $i_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that

$$\|y_i - y_*\| < \frac{\varepsilon}{2}, \quad \text{for all } i > i_{\frac{\varepsilon}{2}}.$$

Now choose an $i_0 \in \mathbb{N}$ such that $i_0 > i_{\frac{\varepsilon}{2}}$. Then we can write

$$\|y_{i_0} - y_*\| < \frac{\varepsilon}{2}.$$

On the other hand, because $(y_i) \subseteq \mathcal{I} - \mathrm{LIM}^r x$, we have $y_{i_0} \in \mathcal{I} - \mathrm{LIM}^r x$, namely,

$$A(\frac{\varepsilon}{2}) = \left\{ i \in \mathbb{N} : \|x_i - y_{i_0}\| \geq r + \frac{\varepsilon}{2} \right\} \in \mathcal{I}. \tag{2.6}$$

Now let us show that the inclusion

$$A^c(\frac{\varepsilon}{2}) \subseteq A^c(\varepsilon) \tag{2.7}$$

holds, where $A(\varepsilon) = \{i \in \mathbb{N} : \|x_i - y_*\| \geq r + \varepsilon\}$. Take $j \in A^c(\frac{\varepsilon}{2})$. Then we have

$$\|x_j - y_{i_0}\| < r + \frac{\varepsilon}{2}$$

and hence

$$\|x_j - y_*\| \leq \|x_j - y_{i_0}\| + \|y_{i_0} - y_*\| < r + \varepsilon,$$

that is, $j \in A^c(\varepsilon)$, which proves (2.7). So, we have

$$A(\varepsilon) \subseteq A\left(\frac{\varepsilon}{2}\right).$$

Because $A(\frac{\varepsilon}{2}) \in \mathcal{I}$ by (2.6), we get $A(\varepsilon) \in \mathcal{I}$ (i.e., $y_* \in \mathcal{I} - \mathrm{LIM}^r x$), which completes the proof. ∎

**Theorem 2.7.** *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. The rough $\mathcal{I}$-limit set of a sequence $x = (x_i)$ is convex.*

**Proof.** Assume that $y_0, y_1 \in \mathcal{I} - \mathrm{LIM}^r x$ for the sequence $x = (x_i)$ and let $\varepsilon > 0$ be given. Define

$$A_1(\varepsilon) = \{i \in \mathbb{N} : \|x_i - y_0\| \geq r + \varepsilon\} \text{ and } A_2(\varepsilon) = \{i \in \mathbb{N} : \|x_i - y_1\| \geq r + \varepsilon\}.$$

Because $y_0, y_1 \in \mathcal{I} - \mathrm{LIM}^r x$, we have $A_1(\varepsilon) \in \mathcal{I}$ and $A_2(\varepsilon) \in \mathcal{I}$. Thus we have

$$\|x_i - [(1 - \lambda)y_0 + \lambda y_1]\| = \|(1 - \lambda)(x_i - y_0) + \lambda(x_i - y_1)\| < r + \varepsilon,$$

for each $i \in A_1^c(\varepsilon) \cap A_2^c(\varepsilon)$ and each $\lambda \in [0,1]$. Because $(A_1^c(\varepsilon) \cap A_2^c(\varepsilon)) \in \mathcal{F}(\mathcal{I})$ by definition $\mathcal{F}(\mathcal{I})$, we get

$$\{i \in \mathbb{N} : \|x_i - [(1-\lambda)y_0 + \lambda y_1]\| \geq r + \varepsilon\} \in \mathcal{I},$$

that is,

$$[(1-\lambda)y_0 + \lambda y_1] \in \mathcal{I} - \mathrm{LIM}^r x,$$

which proves the convexity of the set $\mathcal{I} - \mathrm{LIM}^r x$. ∎

**Theorem 2.8.** *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. Suppose $r > 0$. Then a sequence $x = (x_i)$ is rough $\mathcal{I}$-convergent to $x_*$ if and only if there exists a sequence $y = (y_i)$ such that*

$$\mathcal{I} - \lim y = x_* \text{ and } \|x_i - y_i\| \leq r, \quad \text{for each } i \in \mathbb{N}. \tag{2.8}$$

**Proof.** Assume that $x = (x_i)$ is rough $\mathcal{I}$-convergent to $x_*$. Then, by (2.1) we have

$$\mathcal{I} - \limsup \|x_i - x_*\| \leq r. \tag{2.9}$$

Now, define

$$y_i = \begin{cases} x_*, & \text{if } \|x_i - x_*\| \leq r, \\ x_i + r\frac{x_* - x_i}{\|x_i - x_*\|}, & \text{otherwise.} \end{cases}$$

Then, we have

$$\|y_i - x_*\| = \begin{cases} 0, & \text{if } \|x_i - x_*\| \leq r, \\ \|x_i - x_*\| - r, & \text{otherwise,} \end{cases}$$

and by definition of $y_i$,

$$\|x_i - y_i\| \leq r, \tag{2.10}$$

for all $i \in \mathbb{N}$. By (2.9) and the definition of $y_i$, we get

$$\mathcal{I} - \limsup \|y_i - x_*\| = 0,$$

which implies that $\mathcal{I} - \lim y_i = x_*$.

Assume that (2.8) holds. Because $\mathcal{I} - \lim y = x_*$, we have

$$A(\varepsilon) = \{i \in \mathbb{N} : \|y_i - x_*\| \geq +\varepsilon\} \in \mathcal{I},$$

for each $\varepsilon > 0$. Now, define the set

$$B(\varepsilon) = \{i \in \mathbb{N} : \|x_i - x_*\| \geq r + \varepsilon\}.$$

It is easy to see that the inclusion

$$B(\varepsilon) \subseteq A(\varepsilon)$$

holds. Since $A(\varepsilon) \in \mathcal{I}$, we get $B(\varepsilon) \in \mathcal{I}$. Hence, $x = (x_i)$ is rough $\mathcal{I}$-convergent to $x_*$. ∎

If we replace the condition "$\|x_i - y_i\| \leq r$ for all $i \in \mathbb{N}$" in the hypothesis of the above theorem with the condition "$\{i \in \mathbb{N} : \|x_i - y_i\| > r\} \in \mathcal{I}$" then the theorem will also be valid.

Now we give an important property of the set of rough $\mathcal{I}$-limit points of a sequence.

**Lemma 2.9.** *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. For an arbitrary $c \in \mathcal{I}(\Gamma_x)$ of a sequence $x = (x_i)$, we have*

$$\|x_* - c\| \leq r \text{ for all } x_* \in \mathcal{I} - \mathrm{LIM}^r x.$$

**Proof.** Assume on the contrary that there exist a point $c \in \mathcal{I}(\Gamma_x)$ and $x_* \in \mathcal{I} - \mathrm{LIM}^r x$ such that $\|x_* - c\| > r$. Define $\varepsilon := \frac{\|x_* - c\| - r}{3}$. Then we can write

$$\{i \in \mathbb{N} : \|x_i - c\| < \varepsilon\} \subseteq \{i \in \mathbb{N} : \|x_i - x_*\| \geq r + \varepsilon\}. \tag{2.11}$$

Since $c \in \mathcal{I}(\Gamma_x)$, we have

$$\{i \in \mathbb{N} : \|x_i - c\| < \varepsilon\} \notin \mathcal{I}.$$

But from definition of $\mathcal{I}$-convergence, since

$$\{i \in \mathbb{N} : \|x_i - x_*\| \geq r + \varepsilon\} \in \mathcal{I},$$

so by (2.11) we have

$$\{i \in \mathbb{N} : \|x_i - c\| < \varepsilon\} \in \mathcal{I},$$

which contradicts the fact $c \in \mathcal{I}(\Gamma_x)$. On the other hand, if $c \in \mathcal{I}(\Gamma_x)$ (i.e., $\{i \in \mathbb{N} : \|x_i - c\| < \varepsilon\} \notin \mathcal{I}$) then

$$\{i \in \mathbb{N} : \|x_i - x_*\| \geq r + \varepsilon\}$$

must not belong to $\mathcal{I}$, which contradicts the fact $x_* \in \mathcal{I} - \mathrm{LIM}^r x$. This completed the proof of theorem. ∎

Now we give two $\mathcal{I}$-convergence criteria associated with the rough $\mathcal{I}$-limit set.

**Theorem 2.10.** *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence $x = (x_i)$ $\mathcal{I}$-converges to $x_*$ if and only if*

$$\mathcal{I} - \mathrm{LIM}^r x = \overline{B}_r(x_*).$$

**Proof.** Since $x = (x_i)$ $\mathcal{I}$-converges to $x_*$, we have $\mathcal{I} - \mathrm{LIM}^r x = \overline{B}_r(x_*)$ by the proof of the Theorem 2.3.

Let $\mathcal{I} - \mathrm{LIM}^r x = \overline{B}_r(x_*) \neq \emptyset$. Then from Theorem 2.4, we have that the sequence $x = (x_i)$ is $\mathcal{I}$-bounded. Assume on the contrary that the sequence $x$ has another $\mathcal{I}$-cluster point $x'_*$ different from $x_*$. Then the point

$$\overline{x}_* := x_* + \frac{r}{\|x_* - x'_*\|}(x_* - x'_*)$$

satisfies

$$\|\overline{x}_* - x'_*\| = \left(\frac{r}{\|x_* - x'_*\|} + 1\right)\|x_* - x'_*\| = r + \|x_* - x'_*\| > r.$$

Since $x'_*$ is an $\mathcal{I}$-cluster point of the sequence $x$, by Lemma 2.9 this inequality implies that

$$\overline{x}_* \notin \mathcal{I} - \mathrm{LIM}^r x.$$

This contradicts with the fact that $\|\overline{x}_* - x_*\| = r$ and $\mathcal{I} - \mathrm{LIM}^r x = \overline{B}_r(x_*)$. Hence, $x_*$ is the unique $\mathcal{I}$-cluster point of the sequence $x$ as a bounded sequence (by Theorem 2.4) in some finite-dimensional normed space. Consequently, we can say that

$$x_i \rightarrow_{\mathcal{I}} x_*. \blacksquare$$

It is easy to seen that $\mathcal{I} - \lim x = x_*$ yields the existence of $y_1, y_2 \in \mathcal{I} - \mathrm{LIM}^r x$ satisfying $\|y_1 - y_2\| = 2r$. Because $\mathrm{LIM}^r x \subseteq \mathcal{I} - \mathrm{LIM}^r x$, using Phu's example [10, Example 3.2], it can be easily shown that the existence of $y_1, y_2 \in \mathcal{I} - \mathrm{LIM}^r x$ such that $\|y_1 - y_2\| = 2r$ does not imply the $\mathcal{I}$-convergence of the sequence $x = (x_i)$. The following result is related to the this converse implication.

**Theorem 2.11.** *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal, $(\mathbb{R}^n, \|.\|)$ be a strictly convex space and $x = (x_i)$ be a sequence in this space. If there exist $y_1, y_2 \in \mathcal{I} - \mathrm{LIM}^r x$ such that $\|y_1 - y_2\| = 2r$, then this sequence is $\mathcal{I}$-convergent to $\frac{1}{2}(y_1 + y_2)$.*

**Proof.** Let $c \in \mathcal{I}(\Gamma_x)$. Then since $y_1, y_2 \in \mathcal{I} - \mathrm{LIM}^r x$, we have

$$\|y_1 - c\| \leq r \text{ and } \|y_2 - c\| \leq r, \tag{2.12}$$

by Lemma 2.9. On the other hand, we have

$$2r = \|y_1 - y_2\| \leq \|y_1 - c\| + \|y_2 - c\|. \tag{2.13}$$

Therefore, we get $\|y_1 - c\| = \|y_2 - c\| = r$ by inequalities (2.12) and (2.13). Since

$$\frac{1}{2}(y_2 - y_1) = \frac{1}{2}[(c - y_1) + (y_2 - c)] \quad \text{and} \quad \|y_1 - y_2\| = 2r, \tag{2.14}$$

we get $\|\frac{1}{2}(y_2 - y_1)\| = r$. By the strict convexity of the space and from the equality (2.14), we get

$$\frac{1}{2}(y_2 - y_1) = c - y_1 = y_2 - c,$$

which implies that $c = \frac{1}{2}(y_1 + y_2)$. Hence $c$ is the unique $\mathcal{I}$-cluster point of the sequence $x = (x_i)$. On the other hand, the assumption $y_1, y_2 \in \mathcal{I} - \mathrm{LIM}^r x$ implies that $\mathcal{I} - \mathrm{LIM}^r x \neq \emptyset$. By Theorem 2.4, the sequence $x$ is $\mathcal{I}$-bounded.

Consequently, the sequence $x = (x_i)$ must $\mathcal{I}$-convergent to $\frac{1}{2}(y_1 + y_2)$, i.e.,

$$\mathcal{I} - \lim x = \frac{1}{2}(y_1 + y_2). \ \blacksquare$$

**Theorem 2.12.** *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal.*

(i) *If $c \in \mathcal{I}(\Gamma_x)$ then*

$$\mathcal{I} - \mathrm{LIM}^r x \subseteq \overline{B}_r(c). \tag{2.15}$$

(ii)

$$\mathcal{I} - \mathrm{LIM}^r x = \bigcap_{c \in \mathcal{I}(\Gamma_x)} \overline{B}_r(c) = \{x_* \in \mathbb{R}^n : \mathcal{I}(\Gamma_x) \subseteq \overline{B}_r(x_*)\}. \tag{2.16}$$

**Proof.** (i) If $c \in \mathcal{I}(\Gamma_x)$ then by Lemma 2.9, we have

$$\|x_* - c\| \leq r, \text{ for all } x_* \in \mathcal{I} - \mathrm{LIM}^r x,$$

otherwise we get

$$\{i \in \mathbb{N} : \|x_i - x_*\| \geq r + \varepsilon\} \notin \mathcal{I}, \quad \text{for } \varepsilon := \frac{\|x_* - c\| - r}{3}.$$

Because $c$ is an $\mathcal{I}$-cluster point of $(x_i)$, this contradicts with the fact that $x_* \in \mathcal{I} - \mathrm{LIM}^r x$.

(ii) From (2.15), we have

$$\mathcal{I} - \mathrm{LIM}^r x \subseteq \bigcap_{c \in \mathcal{I}(\Gamma_x)} \overline{B}_r(c). \tag{2.17}$$

Now, let $y \in \bigcap_{c \in \mathcal{I}(\Gamma_x)} \overline{B}_r(c)$. Then we have

$$\|y - c\| \leq r,$$

for all $c \in \mathcal{I}(\Gamma_x)$, which is equivalent to $\mathcal{I}(\Gamma_x) \subseteq \overline{B}_r(y)$, i.e.,

$$\bigcap_{c \in \mathcal{I}(\Gamma_x)} \overline{B}_r(c) \subseteq \{x_* \in \mathbb{R}^n : \mathcal{I}(\Gamma_x) \subseteq \overline{B}_r(x_*)\}. \tag{2.18}$$

Now, let $y \notin \mathcal{I} - \mathrm{LIM}^r x$. Then, there exists an $\varepsilon > 0$ such that

$$\{i \in \mathbb{N} : \|x_i - y\| \geq r + \varepsilon\} \notin \mathcal{I},$$

which implies the existence of an $\mathcal{I}$-cluster point $c$ of the sequence $x$ with $\|y - c\| \geq r + \varepsilon$, i.e.,

$$\mathcal{I}(\Gamma_x) \nsubseteq \overline{B}_r(y) \quad \text{and} \quad y \notin \{x_* \in \mathbb{R}^n : \mathcal{I}(\Gamma_x) \subseteq \overline{B}_r(x_*)\}.$$

Hence, $y \in \mathcal{I} - \mathrm{LIM}^r x$ follows from $y \in \{x_* \in \mathbb{R}^n : \mathcal{I}(\Gamma_x) \subseteq \overline{B}_r(x_*)\}$, i.e.,

$$\{x_* \in \mathbb{R}^n : \mathcal{I}(\Gamma_x) \subseteq \overline{B}_r(x_*)\} \subseteq \mathcal{I} - \mathrm{LIM}^r x. \tag{2.19}$$

Therefore, the inclusions (2.17)–(2.19) ensure that (2.16) holds i.e.,

$$\mathcal{I}-\mathrm{LIM}^r x = \bigcap_{c\in\mathcal{I}(\Gamma_x)} \overline{B}_r(c) = \{x_* \in \mathbb{R}^n : \mathcal{I}(\Gamma_x) \subseteq \overline{B}_r(x_*)\}. \blacksquare$$

**Example 2.13.** Consider the sequence $x = (x_i)$ defined in (2.4) and let $\mathcal{I}$ be the $\mathcal{I}_d$ of $\mathbb{N}$. Then we have

$$\mathcal{I}(\Gamma_x) = \{-1, 1\}.$$

It follows from (2.16) that

$$\mathcal{I} - LIM^r x = \overline{B}_r(-1) \cap \overline{B}_r(1).$$

We finally complete this work by giving the relation between the set of $\mathcal{I}$-cluster points and the set of rough $\mathcal{I}$-limit points of a sequence.

**Theorem 2.14.** *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal and $x = (x_i)$ be an $\mathcal{I}$-bounded sequence. If $r \geq diam(\mathcal{I}(\Gamma_x))$, then we have $\mathcal{I}(\Gamma_x) \subseteq \mathcal{I} - \mathrm{LIM}^r x$.*

**Proof.** Let $c \notin \mathcal{I} - \mathrm{LIM}^r x$. Then there exists an $\varepsilon > 0$ such that

$$\{i \in \mathbb{N} : \|x_i - c\| \geq r + \varepsilon\} \notin \mathcal{I}. \tag{2.20}$$

Since $x = (x_i)$ is $\mathcal{I}$-bounded and from the inequality (2.20), there exists an $\mathcal{I}$-cluster point $c_1$ such that

$$\|c - c_1\| > r + \varepsilon_1,$$

where $\varepsilon_1 := \frac{\varepsilon}{2}$. So we get

$$diam(\mathcal{I}(\Gamma_x)) > r + \varepsilon_1,$$

which proves the theorem. $\blacksquare$

The converse of this theorem is also true, i.e., if $\mathcal{I}(\Gamma_x) \subseteq \mathcal{I} - \mathrm{LIM}^r x$, then we have $r \geq diam(\mathcal{I}(\Gamma_x))$.

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