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SINGLE-VALLEY-EXTENDED CONTINUOUS SOLUTIONS FOR THE FEIGENBAUM'S FUNCTIONAL EQUATION $f(\varphi(x)) = \varphi^2(f(x))$

Abstract. This work deals with the Feigenbaum's functional equation in the broad sense

$$\begin{cases} f(\varphi(x)) = \varphi^2(f(x)), \\ \varphi(0) = 1, \quad 0 \le \varphi(x) \le 1, \quad x \in [0,1], \end{cases}$$

where φ^2 is the 2-fold iteration of φ, $f(x)$ is a strictly increasing continuous function on $[0,1]$ and satisfies $f(0) = 0, f(x) < x, (x \in (0,1])$. Using constructive method, we discuss the existence of single-valley-extended continuous solutions of the above equation.

1. Introduction

In the early 1978s, Feigenbaum [6], [7] and independently Coullet and Tresser [2] introduced the notion of renormalization for real dynamical systems. In 1992, Sullivan [10] proved the uniqueness of the fixed point for the period doubling renormalization operator. This fixed point of renormalization satisfies a functional equation known as the Cvitanović–Feigenbaum equation:

$$(1.1) \qquad \begin{cases} g(x) = -\frac{1}{\lambda} g(g(-\lambda x)), & 0 < \lambda < 1, \\ g(0) = 1, \quad -1 \le g(x) \le 1, & x \in [-1,1]. \end{cases}$$

As mentioned above, this equation and its solution play an important role in the theory initiated by Feigenbaum [6], [7]. However, finding an exact solution of the above equation, in general, is not an easy task. Such a problem can be studied in either classes of smooth functions or classes of continuous functions context. In classes of smooth functions, for equation (1.1), an existence theory of smooth solutions has been established in [1], [3], [4], [5],

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[10] and references therein. As far as we know, seeking a solution of (1.1) in classes of continuous functions has been relatively little researched. In this area, we refer the reader to [9] and [11]. In particular, Yang and Zhang [11] replaced (1.1) by the following equation

$$(1.2)\qquad \begin{cases} \varphi(x) = \frac{1}{\lambda}\varphi(\varphi(\lambda x)), & 0 < \lambda < 1, \\ \varphi(0) = 1, \quad 0 \le \varphi(x) \le 1, & x \in [0,1], \end{cases}$$

which is called the second type of Feigenbaum's functional equations, to study the existence of a single-valley continuous solution of (1.2) by constructive methods. Liao [8] proved that (1.2) has non-single-valley continuous solutions.

In the present paper, we will consider the Feigenbaum's functional equations in the broad sense

$$(1.3)\qquad \begin{cases} f(\varphi(x)) = \varphi^2(f(x)), \\ \varphi(0) = 1, \quad 0 \le \varphi(x) \le 1, \quad x \in [0,1], \end{cases}$$

where $f(x)$ is a strictly increasing continuous function on $[0,1]$ and satisfies $f(0) = 0, f(x) < x, (x \in (0,1])$. We will prove that the existence of a single-valley-extended continuous solution of (1.3) by constructive method.

2. Basic definitions and lemmas

In this section, we will give some characterizations of single-valley-extended continuous solution of (1.3) and will prove them in Appendix.

Definition 2.1. We call φ a single-valley-extended continuous solution of (1.3), if (1) φ is a continuous solution of (1.3); (2) there exists an $\alpha \in (f(1), 1)$ and φ is strictly decreasing in $[f(1), \alpha]$ and strictly increasing in $[\alpha, 1]$.

Definition 2.2. We call φ a single-valley continuous solution of (1.3), if (1) φ is a single-valley-extended continuous solution of (1.3); (2) there exists an $\alpha \in (f(1), 1)$ and φ is strictly decreasing in $[0, \alpha]$ and strictly increasing in $[\alpha, 1]$.

Definition 2.3. We call φ a non-single-valley continuous solution of (1.3), if (1) φ is a single-valley-extended continuous solution of (1.3); (2) φ has at least an extreme point in $(0, f(1))$.

Obviously, single-valley-extended continuous solution only contains single-valley continuous solution and non-single-valley continuous solution.

Lemma 2.1. *Suppose that $\varphi(x)$ is a single-valley-extended continuous solution of (1.3), and α is the extreme point of φ in $(f(1), 1)$. Then the following conclusions hold:*

(i) *$\varphi(x)$ has a unique minimum point α with $\varphi(\alpha) = 0$;*
(ii) *0 is a recurrent but not periodic point of φ;*

(iii) *For* $x \in [0, f(1)]$, $\varphi(x) = \alpha$ *if and only if* $x = f(\alpha)$*;*
(iv) $\varphi(x)$ *has a unique fixed point* $\beta = \varphi(\beta)$ *in* $[0, 1]$*, and*

$$\varphi(1) = f(1) = \lambda < \beta < \alpha \tag{2.1}$$

and $\varphi(\lambda) > \lambda$*;*
(v) *The equation* $\varphi(x) = f(x)$ *has only one solution* $x = 1$ *in* $(\alpha, 1]$.

Lemma 2.2. *Suppose that* $\varphi(x)$ *is a single-valley-extended continuous solution of* (1.3)*. Then the following conclusions hold:*

(i) *If* $\varphi(\lambda) > \alpha$ *then* $\varphi(x) \geq \alpha$ *for all* $x \in [0, \lambda]$ *and* φ *is respectively strictly increasing in* $[f^n(\alpha), f^n(1)]$ *and strictly decreasing in* $[f^{n+1}(1), f^n(\alpha)]$ *for all* $n \geq 0$*, thus* φ *has infinite many extreme points* $f^n(1)$ *and* $f^n(\alpha)$*;*
(ii) *If* $\varphi(\lambda) < \alpha$ *then* φ *is strictly decreasing in* $[0, \lambda]$*, thus* φ *is a single-valley continuous solution.*

Lemma 2.3. *Let* φ_1*,* φ_2 *be two single-valley-extended continuous solutions of* (1.3)*. If*

$$\varphi_1(x) = \varphi_2(x), \quad x \in [\lambda, 1]$$

then $\varphi_1(x) = \varphi_2(x)$ *on* $[0, 1]$.

3. Constructive method of solutions

In this section, we prove the existence of single-valley-extended continuous solutions of (1.3) by the constructive method.

Theorem 3.1. *Let* $f(x)$ *be an arbitrary fixed strictly increasing continuous function on* $[0, 1]$ *with* $f(0) = 0$*,* $f(x) < x$ $(x \in (0, 1])$*. Denote* $f(1) = \lambda$*. If* $\varphi_0(x)$ *is a continuous function on* $[\lambda, 1]$ *and satisfies the following conditions:*

(i) *there exists an* $\alpha \in (\lambda, 1)$ *such that* $\varphi_0(\alpha) = 0$ *and* φ_0 *is strictly decreasing in* $[\lambda, \alpha]$ *and strictly increasing in* $[\alpha, 1]$*;*
(ii) $\varphi_0(1) = f(1) = \lambda$*,* $\varphi_0(\lambda) > \lambda$*,* $\varphi_0(\lambda) \neq \alpha$*,* $\varphi_0^2(\lambda) = f(\varphi_0(1)) = f(\lambda)$*;*
(iii) *the equation* $\varphi_0(x) = f(x)$ *has only one solution* $x = 1$ *in* $[\alpha, 1]$*;*
then there exists a uniquely single-valley-extended continuous function $\varphi(x)$ *satisfying the equation*

$$\begin{cases} f(\varphi(x)) = \varphi^2(f(x)), & x \in [0, 1], \\ \varphi(x) = \varphi_0(x), & x \in [\lambda, 1]. \end{cases} \tag{3.1}$$

In particular, φ *is a single-valley continuous solution when* $\varphi_0(\lambda) < \alpha$ *and* φ *has infinitely many extreme points when* $\varphi_0(\lambda) > \alpha$*. Conversely, if* φ_0 *is the restriction on* $[\lambda, 1]$ *of a single-valley-extended continuous solution to* (1.3)*, then above conditions* (i)–(iii) *must hold.*

Proof. Suppose that φ_0 satisfies the conditions (i)–(iii). Define

$$\psi_+ = \varphi_0|_{[\alpha,1]}, \quad \psi_- = \varphi_0|_{[\lambda,\alpha]}.$$

By condition (i), we know that ψ_+ is strictly increasing and ψ_- is strictly decreasing. It is trivial that $\{f^n(1)\}$ is decreasing and $\lim_{n\to\infty} f^n(1) = 0$. Let

$$\Delta_n = [f^{n+1}(1), f^n(1)], \quad (n = 0, 1, 2, \cdots), \tag{3.2}$$

then $[0,1] = \bigcup_{n=0}^{\infty} \Delta_n$.

We define φ on Δ_n for all $n \geq 0$ by induction as follows. From $\lambda < \varphi_0(\lambda) < 1$ and $\varphi_0(\lambda) \neq \alpha$, we consider the following two cases.

Case 1. Assume $\varphi_0(\lambda) > \alpha$. Firstly, we prove that $\varphi(x)$ is well defined as a continuous function $\varphi_n(x)$ on Δ_n for all $n \geq 0$. Obviously, $\varphi = \varphi_0$ is well defined on Δ_0. Suppose that $\varphi(x)$ is well defined as $\varphi_n(x)$ on Δ_n for all $n \leq k$, where $k \geq 0$ is a certain integer. Let

$$\varphi_{k+1}(x) = \psi_+^{-1}(f(\varphi_k(f^{-1}(x)))), \quad (x \in \Delta_{k+1}), \tag{3.3}$$

then $\varphi(x)$ is well defined as a continuous function $\varphi_n(x)$ on Δ_n for all $n \geq 0$.

Secondly, we prove that φ_n and φ_{n+1} have the same value on the common endpoint $f^{n+1}(1)$ of Δ_n and $\Delta_{n+1}(n = 1, 2, \cdots)$ for all $n \geq 0$. For $n = 0$, from (3.3) and condition (ii), we have

$$\begin{aligned}\varphi_1(f(1)) &= \psi_+^{-1}(f(\varphi_0(1))) = \psi_+^{-1}(\varphi_0^2(f(1))) \\ &= \psi_+^{-1}(\psi_+(\varphi_0(f(1)))) = \varphi_0(f(1)),\end{aligned} \tag{3.4}$$

i.e. φ_0 and φ_1 have the same value on the common endpoint $f(1)$ of Δ_0 and Δ_1. Suppose that

$$\varphi_k(f^k(1)) = \varphi_{k-1}(f^k(1)), \tag{3.5}$$

where $k \geq 1$ is a certain integer. Let $x = f^{k+1}(1)$ in (3.3) then we have

$$\begin{aligned}\varphi_{k+1}(f^{k+1}(1)) &= \psi_+^{-1}(f(\varphi_k(f^k(1)))) \\ &= \psi_+^{-1}(f(\varphi_{k-1}(f^k(1)))) = \varphi_k(f^{k+1}(1)),\end{aligned} \tag{3.6}$$

i.e. φ_n and φ_{n+1} have the same value on the common endpoint $f^{n+1}(1)$ of Δ_n and Δ_{n+1} for all $n \geq 0$, by induction. Therefore, we can put

$$\varphi(x) = \begin{cases} 1, & (x = 0), \\ \varphi_n(x), & (x \in \Delta_n). \end{cases} \tag{3.7}$$

Since φ_n is continuous on $\Delta_n(n \geq 0)$ and (3.4), (3.6), we have that φ is a continuous function in $(0, 1]$.

Thirdly, we prove that φ is continuous at $x = 0$. It is trivial that $\{f^n(\alpha)\}$ is strictly decreasing and $\lim_{n\to\infty} f^n(\alpha) = 0$. We prove that $\{\varphi_n(f^n(\alpha))\}|_{n=1}^{\infty}$ is strictly increasing in $[\alpha, 1]$ by induction. Since $f(\varphi_1(f(\alpha))) > 0$ and (3.3), we get

$$\varphi_2(f^2(\alpha)) = \psi_+^{-1}(f(\varphi_1(f(\alpha)))) > \psi_+^{-1}(0) = \psi_+^{-1}(f(\varphi_0(\alpha))) = \varphi_1(f(\alpha)).$$

Suppose that $\varphi_k(f^k(\alpha)) > \varphi_{k-1}(f^{k-1}(\alpha))$ holds for $n = k$, where $k \geq 2$ is a certain integer. Therefore, by (3.3) and since $\psi_+^{-1} \circ f$ is strictly increasing, we have that

$$\begin{aligned}\varphi_{k+1}(f^{k+1}(\alpha)) &= \psi_+^{-1}(f(\varphi_k(f^k(\alpha)))) > \psi_+^{-1}(f(\varphi_{k-1}(f^{k-1}(\alpha)))) \\ &= \varphi_k(f^k(\alpha)).\end{aligned}$$

Hence, $\{\varphi_n(f^n(\alpha))\}|_{n=1}^{\infty}$ is strictly increasing in $[\alpha, 1]$ by induction. Let

$$\lim_{n\to\infty} \varphi_n(f^n(\alpha)) = \gamma,$$

then $\gamma \in [\alpha, 1]$. From (3.3), we have

$$\varphi_0(\varphi_{n+1}(f^{n+1}(\alpha))) = \psi_+(\varphi_{n+1}(f^{n+1}(\alpha))) = f(\varphi_n(f^n(\alpha))).$$

If $n \to \infty$, we get $\varphi_0(\gamma) = f(\gamma)$. By condition (iii), we know $\gamma = 1 = \varphi(0)$. Similarly, $\{\varphi_n(f^n(1))\}|_{n=1}^{\infty}$ is strictly increasing in $[\alpha, 1]$ and

$$\lim_{n\to\infty} \varphi_n(f^n(1)) = \varphi(0),$$

we omit the proof here. By the condition (i) and (3.3), we have that φ_n has at $f_n(\alpha)$ the minimum and at $f_{n+1}(1)$ the maximum on Δ_n. This proves that φ is continuous at $x = 0$. Thereby, φ is a continuous function in $[0, 1]$. We have that $\varphi(x)$ satisfies (3.1) by (3.3) and $\varphi(x)$ is unique from Lemma 2.3. And $\varphi(x)$ has infinitely many extreme points from Lemma 2.2 (i).

Case 2. Assume $\varphi_0(\lambda) < \alpha$. Firstly, for $x \in \Delta_1 = [f^2(1), f(1)]$, we set

$$\varphi_1(x) = \begin{cases} \psi_+^{-1}(f(\varphi_0(f^{-1}(x)))), & (x \in [f^2(1), f(\alpha)]), \\ \psi_-^{-1}(f(\varphi_0(f^{-1}(x)))), & (x \in [f(\alpha), f(1)]). \end{cases}$$

For $n > 1$, $\varphi_n(x)$ is defined as (3.3). Let $\varphi(x)$ be defined as in (3.7). The proof is similar to Case 1, and we omit it here. Hence, $\varphi(x)$ is a single-valley continuous solution of (1.3) from Lemma 2.2(ii).

Obviously, if φ_0 is the restriction on $[\lambda, 1]$ of a single-valley-extended continuous solution to (1.3), then conditions (i)–(iii) must hold by the lemmas in Section 2. ∎

Example 3.1. Let $\varphi_0(x) : [1/4, 1] \mapsto [0, 1]$ be defined as

$$\varphi_0(x) = \begin{cases} -\frac{13}{8}x + \frac{39}{32}, & (\frac{1}{4} \leq x \leq \frac{3}{4}), \\ x - \frac{3}{4}, & (\frac{3}{4} \leq x \leq 1). \end{cases}$$

It is trivial that φ_0 satisfies the conditions of Theorem 3.1, where $f(x) = x/4$ and $\lambda = f(1) = 1/4, \alpha = 3/4$. Thereby, it is the restriction on $[1/4,\ 1]$ of a single-valley-extended continuous solution φ to (1.3). Since φ_0 has the minimum point $\alpha = 3/4$ and $\varphi_0(1/4) = 13/16 > 3/4$, the solution φ has infinitely many extreme points. See Figure 1.

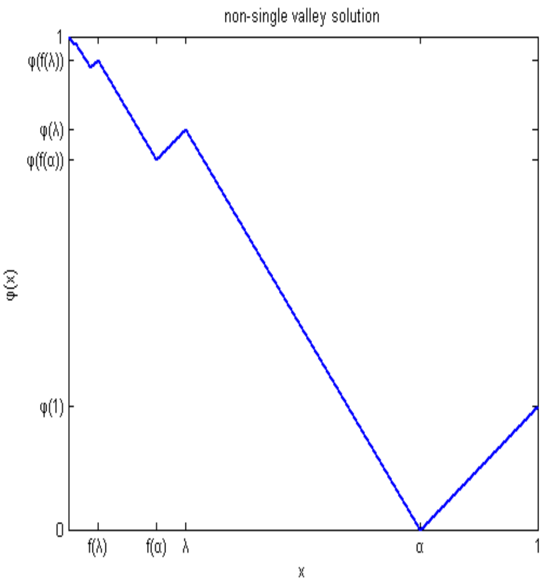

Fig. 1. The graph of non-single-valley solution

Example 3.2. Let $\varphi_0(x) : [1/4, 1] \mapsto [0, 1]$ be defined as

$$\varphi_0(x) = \begin{cases} -\frac{49}{32}x + \frac{147}{128}, & (\frac{1}{4} \le x \le \frac{3}{4}), \\ x - \frac{3}{4}, & (\frac{3}{4} \le x \le 1). \end{cases}$$

It is trivial that φ_0 satisfies the conditions of Theorem 3.1, where $f(x) = x^2/4$ and $\lambda = f(1) = 1/4, \alpha = 3/4$. Hence, it is the restriction on $[1/4,\ 1]$ of a single-valley-extended continuous solution φ to (1.3). Since φ_0 has the minimum point $\alpha = 3/4$ and $\varphi_0(1/4) = 49/64 > 3/4$, the solution φ has infinitely many extreme points. The graph is similar to Figure 1.

Example 3.3. Let $\varphi_0(x) : [1/4, 1] \mapsto [0, 1]$ be defined as

$$\varphi_0(x) = \begin{cases} -\frac{3+\sqrt{7}}{4}x + \frac{9+3\sqrt{7}}{16}, & (\frac{1}{4} \le x \le \frac{3}{4}), \\ x - \frac{3}{4}, & (\frac{3}{4} \le x \le 1), \end{cases}$$

Trivially, φ_0 satisfies the conditions of Theorem 3.1, where $f(x) = x/4$ and $\lambda = f(1) = 1/4, \alpha = 3/4$. Hence, it is the restriction on $[1/4,\ 1]$ of a single-valley-extended continuous solution φ to (1.3). Since φ_0 has the minimum point $\alpha = 3/4$ and $1/4 < \varphi_0(1/4) = (3+\sqrt{7})/8 < 3/4$, the solution φ is a single-valley continuous solution. See Figure 2.

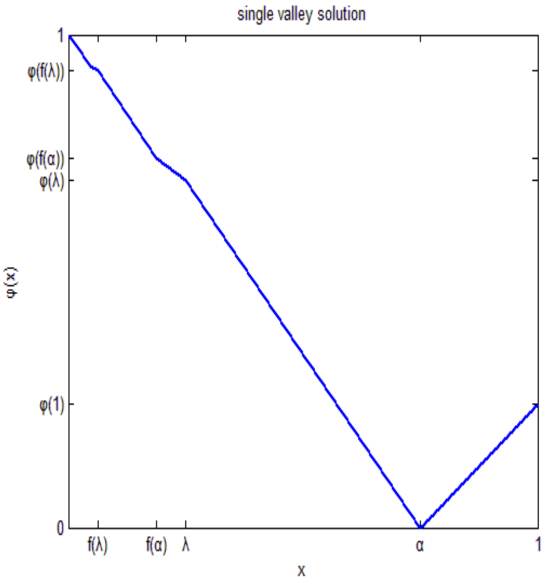

Fig. 2. The graph of single-valley solution

Example 3.4. Let $\varphi_0(x) : [1/4,\ 1] \mapsto [0,\ 1]$ be defined as

$$\varphi_0(x) = \begin{cases} -\frac{6+\sqrt{34}}{8}x + \frac{18+3\sqrt{34}}{32}, & (\frac{1}{4} \le x \le \frac{3}{4}), \\ x - \frac{3}{4}, & (\frac{3}{4} \le x \le 1). \end{cases}$$

Trivially, φ_0 satisfies the conditions of Theorem 3.1, where $f(x) = x^2/4$ and $\lambda = f(1) = 1/4, \alpha = 3/4$. Hence, it is the restriction on $[1/4,\ 1]$ of a single-valley-extended continuous solution φ to (1.3). Since φ_0 has the minimum point $\alpha = 3/4$ and $1/4 < \varphi_0(1/4) = (6+\sqrt{34})/16 < 3/4$, the solution φ is a single-valley continuous solution. The graph is similar to Figure 2.

Appendix A

Proof of Lemma 2.1. (i) Suppose that γ is a minimum point of φ. By (1.3) we have

$$f(\varphi(\gamma)) = \varphi^2(f(\gamma)) \ge \varphi(\gamma).$$

From $f(0) = 0$ and $f(x) < x$ $(x \in (0,1])$, we know that $\varphi(\gamma) = 0$. If $\gamma < f(1)$, then there exists a $\delta \in [0,1]$ such that $f(\delta) = \gamma$ and by (1.3), we have

$$f(\varphi(\delta)) = \varphi^2(f(\delta)) = \varphi^2(\gamma) = \varphi(0) = 1.$$

This contradicts $f(0) = 0, f(x) < x(x \in (0,1])$. Thus $\gamma > f(1)$ and from the definition of α, we know $\gamma = \alpha$ and $\varphi(\alpha) = \varphi(\gamma) = 0$.

(ii) We now prove that for all $n \geq 0$ and each $x \in [0,1]$, we have

$$f^n(\varphi(x)) = \varphi^{2^n}(f^n(x)). \tag{A.1}$$

Obviously, (A.1) holds for $n = 1$ by (1.3). We suppose that (A.1) holds for $n \leq k$, where $k \geq 1$ is an integer. Therefore, by (1.3) we have that

$$\begin{aligned}\varphi^{2^{k+1}}(f^{k+1}(x)) &= (\varphi^{2^k})^2(f^{k+1}(x)) = \varphi^{2^k} \circ \varphi^{2^k}(f^{k+1}(x)) \\ &= \varphi^{2^k}(f^k(\varphi(f(x)))) = f^k(\varphi^2(f(x))) \\ &= f^k(f(\varphi(x))) = f^{k+1}(\varphi(x)),\end{aligned}$$

i.e., (A.1) holds for $n = k+1$. Thereby, (A.1) is proved by induction. Let $x = 0$ in (A.1). We have

$$f^n(1) = f^n(\varphi(0)) = \varphi^{2^n}(f^n(0)) = \varphi^{2^n}(0). \tag{A.2}$$

It is trivial that $\{f^n(1)\}$ is strictly decreasing and $\lim_{n\to\infty} f^n(1) = 0$. Thereby, we have

$$\lim_{n\to\infty} \varphi^{2^n}(0) = \lim_{n\to\infty} f^n(1) = 0, \tag{A.3}$$

i.e., we proved that 0 is a recurrent but not periodic point of φ.

(iii) Firstly, we prove the sufficiency. By (1.3), we have $0 = f(\varphi(\alpha)) = \varphi^2(f(\alpha))$. Since α is the unique minimum point of φ, it follows that $\varphi(f(\alpha)) = \alpha$. Thus, the sufficiency is proved.

Secondly, we prove the necessity. Suppose that $\varphi(x) = \alpha$ for some $x \in [0, f(1)]$. By (1.3), we know that

$$f(\varphi(f^{-1}(x))) = \varphi^2(x) = \varphi(\alpha) = 0,$$

and from $f(0) = 0, f(x) < x(x \in (0,1])$, we have $\varphi(f^{-1}(x)) = 0$. Thereby, $f^{-1}(x) = \alpha$, i.e., $x = f(\alpha)$. Thus the necessity is proved.

(iv) Let $x = 0$ in (1.3) then

$$f(1) = f(\varphi(0)) = \varphi^2(f(0)) = \varphi^2(0) = \varphi(1).$$

Let $\varphi(1) = f(1) = \lambda$. Firstly, we prove that $\beta < \alpha$. Suppose that q is a fixed point of $\varphi(x)$. We have $q \neq 1$ by (A.3) and $q \neq \alpha$ by $\varphi(\alpha) = 0$. If $q \in (\alpha, 1)$ then since $\varphi(x)$ is strictly increasing in $[\alpha, 1]$, it follows that $q = \varphi(q) < \varphi(1)$. By induction, for all $m \geq 0$, we have $q = \varphi^m(q) < \varphi^m(1)$, in particular,

$$q = \varphi^{2^n - 1}(q) < \varphi^{2^n - 1}(1) = \varphi^{2^n - 1}(\varphi(0)) = \varphi^{2^n}(0).$$

This contradicts (A.3). Thereby, we proved that $q < \alpha$.

Secondly, by $\varphi(0) = 1, \varphi(\alpha) = 0$, we know that φ has at least one fixed point β. We prove that $\beta > \lambda$ and β is unique, as follows. We claim that

$$\forall x \in [0, f(\alpha)), \quad \varphi(x) > \alpha. \tag{A.4}$$

If there exists $x_1 \in (0, f(\alpha))$ such that $\varphi(x_1) \leq \alpha$, then by $\varphi(0) = 1$ and intermediate value theorem, there exists $x_2 \in (0, x_1)$ such that $\varphi(x_2) = \alpha$. This contradicts conclusion (iii). From $\varphi(f(\alpha)) = \alpha$ and $f(\alpha) < \alpha$, we have $\beta \notin [0, f(\alpha)]$. If $\beta \in (f(\alpha), f(1)]$ then $f^{-1}(\beta) \in (\alpha, 1]$. By (1.3), we get

$$f(\varphi(f^{-1}(\beta))) = \varphi^2(\beta) = \beta.$$

Thus $\varphi(f^{-1}(\beta)) = f^{-1}(\beta)$. Hence, φ has a fixed point in $[\alpha, 1]$. This proves $\beta \notin (f(\alpha), f(1)]$. Thereby, we have $\beta \in (f(1), \alpha)$. Since φ is strictly decreasing in $(f(1), \alpha)$, β is a unique fixed point of φ in $[0, 1]$.

Thirdly, since $\lambda < \beta < \alpha$ and φ is strictly decreasing in (λ, α), we have

$$\varphi(\lambda) > \varphi(\beta) = \beta > \lambda.$$

(v) By (1.3), it is trivial that $x = 1$ is a solution of the equation $\varphi(x) = f(x)$. Suppose that $x_0 \in [\alpha, 1]$ is an arbitrary solution of this equation, i.e., $\varphi(x_0) = f(x_0)$. Since $[\alpha, 1] \subset \varphi([0, f(\alpha)])$, there exists $y \in [0, f(\alpha)]$, such that $\varphi(y) = x_0$. We claim that

$$\forall n \geq 0, \quad \varphi(f^n(y)) = x_0. \tag{A.5}$$

Obviously, (A.5) holds for $n = 0$. Suppose that (A.5) holds for $n = k$, where $k \geq 0$. Therefore, by (1.3) we have

$$\varphi^2(f^{k+1}(y)) = f(\varphi(f^k(y))) = f(x_0) = \varphi(x_0).$$

Since $f^{k+1}(y) < y < f(\alpha)$ and (A.4), we have $\varphi(f^{k+1}(y)) > \alpha$. And, since φ is strictly increasing in $[\alpha, 1]$, we have $\varphi(f^{k+1}(y)) = x_0$, i.e., (A.5) holds for $n = k + 1$. Thereby, (A.5) is proved by induction. Since $\{f^n(y)\}$ is strictly decreasing and $\lim_{n\to\infty} f^n(y) = 0$, we have $x_0 = \lim_{n\to\infty} \varphi(f^n(y)) = \varphi(0) = 1$. ■

Proof of Lemma 2.2. (i) If $\varphi(\lambda) > \alpha$, we claim that

$$\forall x \in [f(\alpha), \lambda], \quad \varphi(x) \geq \alpha. \tag{A.6}$$

If there exists $y \in (f(\alpha), \lambda)$ such that $\varphi(y) < \alpha$, then by $\varphi(0) = 1$ and intermediate value theorem, there exists $z \in (y, \lambda)$ such that $\varphi(z) = \alpha$. This contradicts Lemma 2.1(iii). Thereby we proved (A.6). From (A.4) we have

$$\forall x \in [0, \lambda], \quad \varphi(x) \geq \alpha. \tag{A.7}$$

Define

$$\psi_+ = \varphi|_{[\alpha,1]}, \quad \psi_- = \varphi|_{[\lambda,\alpha]}.$$

Obviously, ψ_+ is strictly increasing and ψ_- is strictly decreasing. It is trivial that $\{f^n(1)\}$ and $\{f^n(\alpha)\}$ are decreasing, $f^{n+1}(1) \leq f^n(\alpha) \leq f^n(1)$ and

$\lim_{n\to\infty} f^n(1) = \lim_{n\to\infty} f^n(\alpha) = 0$. For $n = 0, 1, 2, \ldots$, let

$$\Delta_n = [f^{n+1}(1), f^n(1)], \quad \Delta_n^1 = [f^n(\alpha), f^n(1)], \quad \Delta_n^2 = [f^{n+1}(1), f^n(\alpha)],$$

then

$$[0,1] = \bigcup_{n=0}^{\infty} \Delta_n = \bigcup_{n=0}^{\infty} (\Delta_n^1 \cup \Delta_n^2).$$

We prove by induction, that for all $n \geq 0$, φ is strictly increasing in Δ_n^1 and strictly decreasing in Δ_n^2, respectively.

Obviously, φ is strictly increasing in Δ_0^1 and strictly decreasing in Δ_0^2. Suppose that φ is strictly increasing in Δ_n^1 and strictly decreasing in Δ_n^2 for $n \geq k$, where $k \geq 0$. By (1.3), we have

$$f(\varphi(f^{-1}(x))) = \varphi^2(x), \quad x \in \Delta_{k+1}. \tag{A.8}$$

By (A.7), we have $\varphi(x) \in [\alpha, 1]$. Thus (A.8) is equivalent to the following equation

$$f(\varphi(f^{-1}(x))) = \psi_+(\varphi(x)), \quad x \in \Delta_{k+1}.$$

Thereby,

$$\varphi(x) = \psi_+^{-1}(f(\varphi(f^{-1}(x)))), \quad x \in \Delta_{k+1}. \tag{A.9}$$

Since ψ_+^{-1}, f, f^{-1} are strictly increasing and φ is strictly increasing in Δ_k^1 and strictly decreasing in Δ_k^2, we know φ is strictly increasing in Δ_{k+1}^1 and strictly decreasing in Δ_{k+1}^2. Thereby, φ is strictly increasing in Δ_n^1 and strictly decreasing in Δ_n^2 for all $n \geq 0$. Thus φ has infinite many extreme points $f^n(1)$ and $f^n(\alpha)$.

(ii) If $\varphi(\lambda) < \alpha$, we prove by induction that for all $n \geq 1$, φ is strictly decreasing in Δ_n. By (A.4), we have $\varphi(x) \in [\alpha, 1]$ for $x \in [f^2(1), f(\alpha)]$. By $\lambda < \varphi(\lambda) < \alpha$ and the condition (iii) of Lemma 2.1, we have $\varphi(x) \in [\lambda, \alpha]$ for $x \in [f(\alpha), f(1)]$. Thus for $x \in \Delta_1 = [f^2(1), f(1)]$, from (1.3) we have

$$\varphi(x) = \begin{cases} \psi_+^{-1}(f(\varphi(f^{-1}(x)))), & (x \in [f^2(1), f(\alpha)]), \\ \psi_-^{-1}(f(\varphi(f^{-1}(x)))), & (x \in [f(\alpha), f(1)]). \end{cases} \tag{A.10}$$

Since f, f^{-1}, ψ_+^{-1} are strictly increasing and ψ_-^{-1} is strictly decreasing and φ is strictly increasing in Δ_0^1 and strictly decreasing in Δ_0^2, we know φ is strictly decreasing in Δ_1. For $n > 1$, from (1.3) we have (A.9). The proof is similar to conclusion (i), and we omit it here. ∎

Proof of Lemma 2.3. It is trivial that there exist $\alpha \in (\lambda, 1), \beta \in (\lambda, 1)$ such that $\varphi_i(\alpha) = 0, \varphi_i(\beta) = \beta$ $(i = 1, 2)$. Define

$$\psi_+ = \varphi_1|_{[\alpha,1]} = \varphi_2|_{[\alpha,1]}, \quad \psi_- = \varphi_1|_{[\lambda,\alpha]} = \varphi_2|_{[\lambda,\alpha]}. \tag{A.11}$$

Obviously, ψ_+ is strictly increasing and ψ_- is strictly decreasing. It is trivial that $\{f^n(1)\}$ is decreasing and $\lim_{n\to\infty} f^n(1) = 0$. Let $\Delta_n = [f^{n+1}(1), f^n(1)]$, $(n = 0, 1, 2, \dots)$ then $[0, 1] = \bigcup_{n=0}^{\infty} \Delta_n$.

We prove by induction that $\varphi_1(x) = \varphi_2(x)$ on Δ_n for all $n \geq 0$:

Obviously, $\varphi_1(x) = \varphi_2(x)$ on Δ_0. Suppose that $\varphi_1(x) = \varphi_2(x)$ holds on Δ_n for all $n \leq k$, where $k \geq 0$ is an integer. Let

$$\varphi(x) = \varphi_1(x) = \varphi_2(x), \quad x \in [f^{k+1}(1), 1].$$

By (1.3), we have

$$f(\varphi(f^{-1}(x))) = f(\varphi_i(f^{-1}(x))) = \varphi_i(\varphi_i(x)), (i = 1, 2, x \in \Delta_{k+1}). \tag{A.12}$$

Next, we prove the following two cases.

Case 1. $\varphi_1(\lambda) = \varphi_2(\lambda) > \alpha$. Then by (A.7), we have that (A.12) is equivalent to the following equation

$$f(\varphi(f^{-1}(x))) = \psi_+(\varphi_i(x)),$$

thereby

$$\varphi_i(x) = \psi_+^{-1}(f(\varphi(f^{-1}(x)))), \quad (i = 1, 2, x \in \Delta_{k+1}). \tag{A.13}$$

Thus we have $\varphi_1(x) = \varphi_2(x)$ on Δ_{k+1}. By induction, $\varphi_1(x) = \varphi_2(x)$ on Δ_n for all $n \geq 0$.

Case 2. $\varphi_1(\lambda) = \varphi_2(\lambda) < \alpha$. By Lemma 2.1 (iv), we have $\varphi(\lambda) > \lambda$. Thus $\varphi_i(x) > \varphi_i(\lambda) > \lambda$ for $x < \lambda$. If $\varphi_i(x) \in [\alpha, 1]$ then φ_i satisfy (A.13). If $\varphi_i(x) \in [\lambda, \alpha]$ then (A.12) is equivalent to the following equation

$$f(\varphi(f^{-1}(x))) = \psi_-(\varphi_i(x)),$$

thereby

$$\varphi_i(x) = \psi_-^{-1}(f(\varphi(f^{-1}(x)))), \quad (i = 1, 2, x \in \Delta_{k+1}). \tag{A.14}$$

We claim that $\varphi_1(x)$ and $\varphi_2(x)$ either both satisfy (A.13) or both satisfy (A.14), simultaneously. Suppose there exists an $x_0 \in \Delta_{k+1}$ such that

$$\varphi_1(x_0) = \psi_+^{-1}(f(\varphi(f^{-1}(x_0)))), \quad \varphi_2(x_0) = \psi_-^{-1}(f(\varphi(f^{-1}(x_0)))).$$

By $\psi_+^{-1} : [0, \lambda] \mapsto [\alpha, 1]$ and $\psi_-^{-1} : [0, \varphi(\lambda)] \mapsto [\lambda, \alpha]$, there exists x_i such that $x_1 > x_0$, $x_1 \in \Delta_{k+1}$ and

$$\varphi_1(x_0) > \varphi_1(x_1) > \alpha > \varphi_2(x_0) > \varphi_2(x_1).$$

Since $\varphi_i(x)$ are respectively strictly monotone in $[\lambda, \alpha]$ and $[\alpha, 1]$, from (A.12), we have that

$$\varphi_1(\varphi_1(x_1)) < \varphi_1(\varphi_1(x_0)) = \varphi_2(\varphi_2(x_0)) < \varphi_2(\varphi_2(x_1)).$$

This contradicts (A.12) $(x = x_1)$. Thus we have $\varphi_1(x) = \varphi_2(x)$ on Δ_{k+1}. Hence $\varphi_1(x) = \varphi_2(x)$ on Δ_n, for all $n \geq 0$. ∎

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