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WEAK HOMOMORPHISMS OF COALGEBRAS BEYOND SET

Abstract. We study the notion of weak homomorphisms between coalgebras of different types generalizing thereby that of homomorphisms for similarly typed coalgebras. This helps extend some results known so far in the theory of Universal coalgebra over Set. We find conditions under which coalgebras of a set of types and weak homomorphisms between them form a category. Moreover, we establish an Isomorphism Theorem that extends the so-called First Isomorphism Theorem, showing thereby that this category admits a canonical factorization structure for morphisms.

1. Introduction

In Universal (Co)Algebra, the notion of homomorphism is usually considered for similarly typed (co)algebras. A useful way of weakening this notion is to define it for differently typed (co)algebras. Following E. Marczewski who proposed the concept of weak homomorphisms for non-indexed universal algebras in 1961, K. Glazeck in 1980 proposed this concept for indexed ones (see [16] for references). F. M. Schneider generalized the latter to F-algebras for an arbitrary Set-endofunctor F in [16]. Inspired by the preprint of his work and the manuscript of the thesis of K. Saengsura, see [15], K. Denecke and W. Supaporn extended the concept to (F_1, F_2)-systems for some Set-endofunctors F_1 and F_2 in [7]. One of the interesting problems posed by F. M. Schneider in [16] was whether and under what conditions it is possible to generalize the results he obtained by replacing Set with an arbitrary category C with a suitable factorization structure for morphisms and considering those C-endofunctors which are compatible with the chosen factorization structure in some sense. In this paper, we attempt at solving this problem in the dual context. Indeed, considering an arbitrary strongly well-powered category C with epi-strong

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mono factorizations and considering types preserving at least strong monos, we study the concept of weak homomorphisms for differently typed coalgebras.

The main goal of the present paper is to show that for differently typed coalgebras, weak homomorphisms in many respects behave very similarly to proper ones so that many of the results known for coalgebras of a common type actually still hold.

The paper is organized as follows. In Section 2, we give some basics in connection with category theory as well as that of Universal Coalgebra in a more general context. In particular, we focus on the extension of some of the results from [8, 10, 14, 13] where coalgebras taking into consideration are similarly typed with Set as base category. Therefore most of these help in the sequel to extend, in the dual context, results from [16]. In particular, we extend a result from [8, 13] which helps show that a morphism is actually a homomorphism between coalgebras of a given type, a result which is at the origin of the so-called First Diagram Lemma and Second Diagram Lemma in [8]. These lemmas helped inter alia characterize the so-called *closed subsets*, a concept that is also found in [16] for functorial algebras and in [7], where they are rather known as *open subsets* when dealing with (F_1, F_2)-systems. Extending the concept of closed subset, we define that of *closed subobject* and characterize them in a category endowed with a suitable factorization system in Section 3. Closed subobjects constitute one of the main ingredients in the definition of weak homomorphisms. This inter alia helps extend the above diagram lemmas to differently typed coalgebras. Furthermore, we show that, under reasonable circumstances, differently typed coalgebras and weak homomorphisms between them form a category $\mathsf{C}_{\mathcal{F}}$ where $\mathcal{F}$ denote the set of types. We end the section with an Isomorphism Theorem, which enables us to see that this category admits a canonical factorization structure for morphisms. The fourth section is devoted to the conclusion.

2. Preliminaries

Let C be a category and F a C-endofunctor.

2.1. Basics. By *F-coalgebra* (or *F-system*) or simply *coalgebra* (or *system*) is meant a pair $\mathcal{A} = (A, \alpha : A \to F(A))$. F is called the *type* and the morphism α the *transition structure* (or *dynamics* or *costructure*) of the coalgebra $\mathcal{A}$. An *F-homomorphism*, i.e., a morphism from a coalgebra (A, α) to a coalgebra (B, β) is a morphism $\varphi : A \to B$ such that $F(\varphi) \circ \alpha = \beta \circ \varphi$. With morphisms between them, F-coalgebras form a category [1, 3]. Denote it by C_F and by $U : \mathsf{C}_F \to \mathsf{C}$ *the forgetful functor*; i.e., the functor sending (A, α) to A and $\varphi : (A, \alpha) \to (B, \beta)$ to $\varphi : A \to B$. Although U always reflect monos, it doesn't preserve them in general.

The category C_F has all colimits that exist in C as well as all limits that are preserved by F; these (co)limits are created by the forgetful functor [1, 4]. As an immediate consequence thereof, epis (isos) in the former are precisely morphisms which are epis (isos) in the latter, see ibid. Therefore, the forgetful functor reflects epis (isos).

Functors weakly preserving kernels (i.e., transforming a kernel into a weak kernel) and those preserving powers will be of interest in the sequel. In case $\mathsf{C} = \mathsf{Set}$, *polynomial functors* (see [3, 8, 14]) (built from the identity, constant functors, sums, products and composition and are instances of the so-called *partial product functors* which are generalized pullbacks preserving functors [6, 12]), as well as the *finite powerset functor* $\mathcal{P}_\omega$ do weakly preserve kernels and the latter does not preserve binary powers (and thus products) for there exist coalgebras which are nonempty but are isomorphic to their binary power (with nonempty carriers whose products are the empty coalgebra), see [9].

The following definitions of some special morphisms as well as related properties can be found in [2]. A morphism in C is a *regular epi* provided it is the coequalizer of some parallel pair of morphisms. Every regular epi is an epi. By a *strong mono* is meant a mono $m : C \rightarrowtail D$ in C such that for all morphisms $f : A \to C$, $g : B \to D$ and all epi $e : A \twoheadrightarrow B$ such that $m \circ f = g \circ e$, there exists a unique morphism $d : B \to C$ such that $d \circ e = f$ and $m \circ d = g$. The class of epis is as well as that of strong monos, closed under compositions and the latter under pullbacks, too (whenever they exist).

The following, which will be frequently used throughout this paper, is straightforwardly checked.

Lemma 2.1. *Let m and n be morphisms. If $m \circ n$ is a strong monomorphism and m is a monomorphism, then n is a strong monomorphism.*

By an ***M**-subobject* of an object A in C for some class $\mathbf{M} \subseteq \mathsf{Mono}(\mathsf{C})$ is meant one represented by a morphism belonging to $\mathbf{M}$. In particular, a *strong subobject* of A is one represented by a strong mono. A strong subobject S of A will alternatively be designated by the strong mono $S \rightarrowtail A$ by which it is represented. By *subcoalgebra* of an F-coalgebra (A, α) is meant a strong subobject of (A, α).

If C has epi-strong mono factorizations, then every morphism $\varphi : A \to B$ factors in a unique way (up to isomorphisms) in an epi $\psi : A \twoheadrightarrow Q$ followed by a strong mono $\theta : Q \rightarrowtail B$. If $\varphi : A \to B$ is a morphism and $\mu : U \rightarrowtail A$ is a strong mono so that the composite $\varphi \circ \mu$ factors in C in an epi $\psi' : U \twoheadrightarrow Q'$ followed by a strong mono $\mu' : Q' \rightarrowtail B$ as $\varphi \circ \mu = \mu' \circ \psi'$, then Q' is denoted by $\varphi[U]$ and is called the *image of U (or of $\mu : U \rightarrowtail A$) under φ.*

Lemma 2.2. *If a morphism $\varphi : A \to B$ factors in an epi $\psi : A \twoheadrightarrow Q$ followed by a strong mono $\theta : Q \rightarrowtail B$ as $\varphi = \theta \circ \psi$, then $\varphi[A] = Q$. If $\sigma : U \rightarrowtail A$ is a strong subobject of A, then $\varphi[U]$ is a strong subobject of $\varphi[A]$ and $\psi[U] \cong \varphi[U]$.*

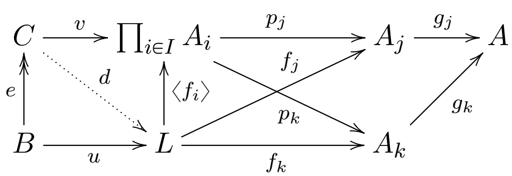
Proof. $Q = \varphi[A]$ is straightforward. Factorizing $\varphi \circ \sigma$ in an epi $\psi' : U \twoheadrightarrow \varphi[U]$ followed by a strong mono $\theta' : \varphi[U] \rightarrowtail B$ as $\varphi \circ \sigma = \theta' \circ \psi'$, we have $\theta \circ \psi \circ \sigma = \theta' \circ \psi'$. But then θ strong mono and ψ' epi imply that there exists a unique morphism $\delta : \varphi[U] \to Q$ such that $\delta \circ \psi' = \psi \circ \sigma$ and $\theta' = \theta \circ \delta$ and since θ' is a strong mono and θ is a (strong) mono, Lemma 2.1 yields that δ is a strong mono, too. Hence $\varphi[U]$ is a strong subobject of $\varphi[A]$. $\psi[U] \cong \varphi[U]$ follows straightforwardly from the fact that epi-strong mono factorizations of the same morphism are isomorphic. ∎

It is well known that strong monos are stable under the formation of pullbacks [2]. If C has pullbacks (along strong monomorphisms) and $\mu : U \rightarrowtail B \leftarrow A : \varphi$ is a cospan where μ is a strong monomorphism, then denote by $\mu' : \varphi^{-}[U] \rightarrowtail A$ the pullback of μ along φ.

Recall that a mono $m : A \rightarrowtail B$ in C is called *extremal* provided whenever $m = f \circ e$ where e is an epi, then e must be an isomorphism; and strong implies extremal, a morphism which is both an epi and an extremal mono is an iso. The following has been discussed for arbitrary limits in [2], where it is shown that the mediating morphism $\langle f_i \rangle$, is always an extremal mono. In particular, it holds:

Lemma 2.3. *Assume that the generalized pullback $(L, (f_i : L \to A_i)_{i\in I})$ of a sink $(g_i : A_i \to A)_{i\in I}$ and the product $\prod_{i\in I} A_i$ of the A_i's exist in C. Then the unique morphism $\langle f_i \rangle : L \to \prod_{i\in I} A_i$, given by the universal property of the product, is a strong monomorphism.*

Proof. Let $e : B \twoheadrightarrow C$ be an epimorphism, and $u : B \to L$ and $v : C \to \prod_{i\in I} A_i$ be morphisms such that $\langle f_i \rangle \circ u = v \circ e$.



We want to show that there exists a unique morphism $d : C \to L$ such that $v = \langle f_i \rangle \circ d$ and $u = d \circ e$. Let $p_i : \prod_{i\in I} A_i \to A_i$ be the i^{th} projection of $\prod_{i\in I} A_i$. $(p_i \circ v : C \to A_i)_{i\in I}$ is a source and by a straightforward diagram chase one can prove: for all $j, k \in I$, $g_j \circ p_j \circ v \circ e = g_k \circ p_k \circ v \circ e$ and, since e is an epimorphism, $g_j \circ p_j \circ v = g_k \circ p_k \circ v$. Therefore, by the universal property of the generalized pullback, there exists a unique morphism $d : C \to L$ such that, for each $j \in I$, $p_j \circ v = f_j \circ d$; that is, $p_j \circ v = p_j \circ \langle f_i \rangle \circ d$. But then, since the $(\prod_{i \in I} A_i, (p_i)_{i \in I})$ is a mono-source, we deduce that $v = \langle f_i \rangle \circ d$. On the other hand,

$$\begin{aligned} p_j \circ v = p_j \circ \langle f_i \rangle \circ d &\Rightarrow p_j \circ v \circ e = p_j \circ \langle f_i \rangle \circ d \circ e \\ &\Rightarrow p_j \circ \langle f_i \rangle \circ u = p_j \circ \langle f_i \rangle \circ d \circ e \\ &\Rightarrow u = d \circ e, \end{aligned}$$

since the p_i's are jointly monic and $\langle f_i \rangle$ is a monomorphism. ∎

2.2. Factorization of homomorphisms and coalgebras of covarietors. It is not hard to check the following. Assertion 2. has been proved by Hughes in [11] for $\mathbf{M}$ taken to be the class of all regular monomorphisms in C whereas, in the setting of category Set, Rutten in [13] and Gumm in [8] have proven the two assertions. It helps show that some morphisms in the base category are in fact homomorphisms.

Theorem 2.4. *Let (A, α), (B, β) and (C, γ) be coalgebras in C_F, $\varphi : (A, \alpha) \to (C, \gamma)$ a homomorphism, $f : A \to B$ and $g : B \to C$ morphisms in C with $\varphi = g \circ f$.*

1. *If f is an epimorphism in C_F, then g is a homomorphism.*
2. *If F preserves or takes $\mathbf{M}$-morphisms to monomorphisms for some $\mathbf{M} \subseteq \mathcal{M}ono(\mathsf{C})$ and g is a homomorphism which is an $\mathbf{M}$-morphism in C, then f is a homomorphism.*

Using the fact that epis in C_F are precisely homomorphisms carried by epis in C, it is not hard to check that every homomorphism carried by a strong mono in C is a strong mono in C_F. Summarizing some results from [1], we obtain the following which will play a key role throughout our work.

Lemma 2.5. *Assume that C has epi-strong mono factorizations.*

1. *If F preserves strong monomorphisms, then strong monomorphisms in C_F are precisely the morphisms which are strong monomorphisms in C.*
   In case $\mathsf{C} = \mathsf{Set}$, strong monomorphisms are precisely injective homomorphisms.
2. *If $\mathsf{C} = \mathsf{Set}$ or F preserves strong monomorphisms, then C_F has also epi-strong mono factorizations. They are, in fact, created by the forgetful functor U.*

Recall that every *strongly complete category*; that is, a category which has all small limits and all intersections of subobjects (possibly large), has epi-strong mono factorizations [2], and a type F is called a *covarietor* provided the forgetful functor has a right adjoint [1, 3]. In Set, examples of covarietors are partial product functors (and hence polynomial ones, see also [3], for instance) [12], the so-called *bounded functors* (e.g., $\mathcal{P}_\omega$) and the functors which *preserve ω-mono-sources*, see [9].

For later use, we also give the following summary of some essential results from [1]:

Lemma 2.6.

1. Limit Theorem: *If C is a strongly complete category and F is a covarietor preserving strong monomorphisms, then C_F is complete.*
2. *For every covarietor over Set, the category of coalgebras is complete.*

2.3. Congruences and bisimulations. Congruences/bisimulations are special relations on/between carriers of coalgebras and will be of interest in the sequel. We don't give a great treatment of them but just definitions and some slight insights thereof needed for our purpose.

Definition 2.7. Let $(A_k)_{k\in\kappa}$ be a family of objects in a category C with κ-products, for some ordinal κ. Then a $\kappa - \mathbf{M}$*-relation* between the A_k's for some $\mathbf{M} \subseteq \mathcal{M}ono(\mathsf{C})$ is an $\mathbf{M}$-subobject of the product $\prod_{k\in\kappa} A_k$. This is represented by an $\mathbf{M}$-morphism $R \rightarrowtail \prod_{k\in\kappa} A_k$, or equivalently by a family of morphisms $(r_k : R \to A_k)_{k\in\kappa}$ with the property that the morphism $\langle r_k\rangle : R \rightarrowtail \prod_{k\in\kappa} A_k$ obtained by the universal property of the product is an $\mathbf{M}$-morphism. In case $\kappa = 2$, we talk of *binary* $\mathbf{M}$*-relation*. In case $\mathbf{M} = \mathcal{M}ono(\mathsf{C})$, we simply talk of *relation*.

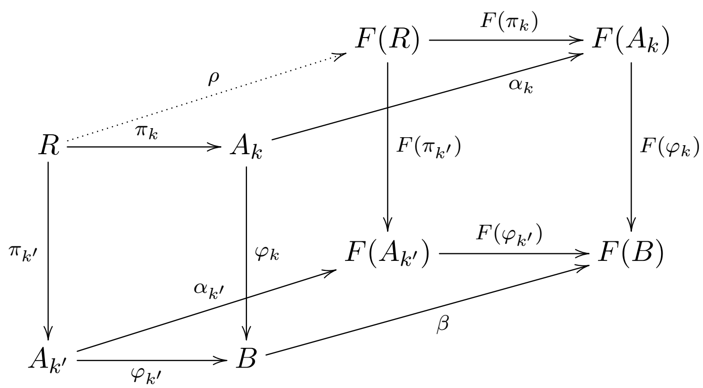
Let $(r_k : R \to A_k)_{k\in\kappa}$ and $(r'_k : R' \to A_k)_{k\in\kappa}$ be $\kappa - \mathbf{M}$-relations. We say that R *is contained in* R', written as $R \sqsubseteq R'$, provided there exists a strong monomorphism $u : R \rightarrowtail R'$ with $r_k = r'_k \circ u$ for every $k \in \kappa$; that is, $\langle r_k\rangle = \langle r'_k\rangle \circ u$.

Definition 2.8. Let κ be an ordinal and $(A_k, \alpha_{A_k})_{k\in\kappa}$ a family of coalgebras in C_F such that the product $\prod_{k\in\kappa} A_k$ exists. An $\mathbf{M} - \kappa$*-simulation* between coalgebras $\mathcal{A}_k$'s in C_F for some class $\mathbf{M} \subseteq \mathcal{M}ono(\mathsf{C})$ is an $\mathbf{M}$-relation $(r_k : R \to A_k)_{k\in\kappa}$ such that there exists a dynamics $\rho : R \to F(R)$ on R turning the projections r_k's into homomorphisms. For $\kappa = 2$, we speak of $\mathbf{M}$*-bisimulations*. In case $\mathbf{M} = \mathcal{M}ono(\mathsf{C})$, we just speak of κ*-simulation* and *bisimulation* in case $\kappa = 2$. If for every $k \in \kappa$ we have $\mathcal{A}_k = \mathcal{A}$, then we speak of an $\mathbf{M} - \kappa$-simulation *on* $\mathcal{A}$.

Thus, $\mathbf{M}-\kappa$-simulations are special $\mathbf{M}$-subobjects and are $\mathbf{M}-\kappa$-relations between the carriers of coalgebras which respect their coalgebraic structure. In case $\mathbf{M}$ is the class of strong monomorphisms and $\mathsf{C} = \mathsf{Set}$, we retrieve the definition given in [13, 8, 14].

Theorem 2.9. *Assume that* C*, with* κ*-products for some ordinal* $\kappa \geq 2$*, has generalized* κ*-pullbacks and* F *weakly preserves* κ*-pullbacks. For every sink* $(\varphi_k : \mathcal{A}_k \to \mathcal{B})_{k\in\kappa}$ *in* C_F *with* $\mathcal{A}_k = (A_k, \alpha_k)$ *and* $\mathcal{B} = (B, \beta)$*, the* κ*-pullback* $(\pi_k : R \to A_k)_{k\in\kappa}$ *of the* φ_k*'s in* C *is a* κ*-strong simulation between the* $\mathcal{A}_k$*'s. If in addition* C *has epi-strong mono factorizations,* F *preserves strong monomorphisms and there exists* $k_0 \in \kappa$ *such that* π_{k_0} *is a strong monomorphism, then it is also the* κ*-pullback of the* φ_k*'s in* C_F*.*

Proof. For all $k \neq k'$ in κ, a straightforward diagram chase yields $F(\varphi_k) \circ \alpha_k \circ \pi_k = F(\varphi_{k'}) \circ \alpha_{k'} \circ \pi_{k'}$. Thus, since $(F(\pi_k) : F(R) \to F(A_k))_{k\in\kappa}$ is the weak κ-pullback of the $F(\varphi_k)$'s, there exists a morphism $\rho : R \to F(R)$ such that for every $k \in \kappa$, $F(\pi_k) \circ \rho = \alpha_k \circ \pi_k$. On the other hand, by Lemma 2.3, R is a strong κ-relation between the A_k's.



Assume in addition that there exists $k_0 \in \kappa$ such that π_{k_0} is a strong monomorphism and let $(\mu_k : R' \to \mathcal{A}_k)_{k\in\kappa}$ be a κ-source in C_F such that $\varphi_k \circ \mu_k = \varphi_{k'} \circ \mu_{k'}$ for all $k \neq k'$. Considering these equalities in C, the universal property of the κ-pullback yields a unique morphism $u : R' \to R$ such that for every k, $\pi_k \circ u = \mu_k$. To end the proof, we need to show that u is a homomorphism. Now, in particular, we have $\pi_{k_0} \circ u = \mu_{k_0}$ where π_{k_0} is a homomorphism which is a strong monomorphism in C and F preserves strong monomorphisms. Thus item 2. of Theorem 2.4 applies. ∎

Definition 2.10. Assume that C is a category with kernel pairs. A *congruence* on a coalgebra $\mathcal{A}$ over C is the kernel in C of a homomorphism φ with domain $\mathcal{A}$.

Although the following does not give a characterization of regular epimorphisms in a category of coalgebras as in [10], it can help show that a homomorphism is a regular epimorphism in some cases.

Lemma 2.11. *If C has kernel pairs and binary powers and F weakly preserves kernels, then:*

1. *every congruence is a strong bisimulation;*
2. *the forgetful functor $U : \mathsf{C}_F \to \mathsf{C}$ reflects regular epimorphisms.*

Proof. 1. Let (θ, π_1, π_2) be a congruence on a coalgebra (A, α). Then there exists a homomorphism $\varphi : (A, \alpha) \to (B, \beta)$ such that (θ, π_1, π_2) is the kernel pair in C of φ. By Lemma 2.3, the unique morphism $\langle \pi_1, \pi_2 \rangle : \theta \to A \times A$, given by the universal property of the product, is a strong monomorphism. Thus, by Theorem 2.9, (θ, π_1, π_2) is a strong bisimulation on $\mathcal{A}$.

2. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism which is a regular epimorphism in C and let $\pi_1, \pi_2 : ker(\varphi) \to A$ be its kernel in C. We want to show that φ is the coequalizer of π_1 with π_2 in C_F. Since φ is a regular epimorphism in C, then it is the coequalizer of π_1 with π_2 in C [2]. Therefore, from 1. above, we deduce that $ker(\varphi)$ is a strong bisimulation on $\mathcal{A}$. Now let $\psi : \mathcal{A} \to \mathcal{C}$ be a homomorphism such that $\psi \circ \pi_1 = \psi \circ \pi_2$. By the universal property of the coequalizer, there exists a unique morphism $\varepsilon : B \to C$ in C such that $\psi = \varepsilon \circ \varphi$. But then, since φ is an epimorphism in C_F, it follows from Theorem 2.4 that ε is a homomorphism. ∎

In some categories, every epimorphism is regular. E.g., the categories Grp of groups, Ab of abelian groups and homomorphisms of groups, Lat of lattices and homomorphisms of lattices, HComp of compact Hausdorff spaces and continuous functions [2], and every topos [5]. The following result is an extension of the work in [10] and helps create such categories in the field of the Universal coalgebra.

Corollary 2.12. *If in addition to the assumptions of Lemma 2.11, every epimorphism in C is regular, then the following hold:*

1. *every epimorphism in C_F is regular.*
2. *if in addition C has epi-strong mono factorizations and F preserves strong monomorphisms, then every monomorphism in C_F is strong.*

Proof. 1. It follows directly from 2. of Lemma 2.11.

2. By Lemma 2.5, every monomorphism φ in C_F factors in an epimorphism ψ followed by a strong monomorphism μ as $\varphi = \mu \circ \psi$. But then, since φ is a monomorphism, this equality implies that so is ψ. Now, by 1., ψ is also a regular epimorphism. Thus, it is an isomorphism [2]. Hence the desired result. ∎

Lemma 2.13. *Assume that C, with epi-strong mono factorizations, has and F preserves products as well as strong monos, $(\mathcal{A}_i)_{i\in I}$ a family of F-coalgebras where $\mathcal{A}_i = (A_i, \alpha_i)$ and $\pi_i : \prod_{i\in I} A_i \to A_i$ denotes the canonical projection* *for each i. Moreover, let $\mu : S \rightarrowtail \prod_{i\in I} A_i$ be a strong mono and $\sigma : S \to F(S)$ a morphism. Then the following are equivalent:*

(i) *$\mu : (S,\sigma) \to \prod_{i\in I} \mathcal{A}_i$ is a strong mono in C_F.*
(ii) *For each $i \in I$, the composite $\pi_i \circ \mu : S \to A_i$ is a homomorphism from (S,σ) to (A_i,α_i).*

Proof. Since C has and F preserves products, then it follows that C_F has products and they are created by the forgetful functor (see Subsection 2.1). On the other hand, by Lemma 2.5, strong monos in C_F are carried by strong monos in C. Taking all this into account, the desired result is straightforward. ∎

The next result, which has been proven in [8] in case $\mathsf{C} = \mathsf{Set}$ without assuming that F preserves strong monos (which are just monos in this case), gives a characterization of congruences and shows that every congruence gives rise to a unique factor in some cases.

Lemma 2.14. *Assume that C has kernel pairs, binary powers, coequalizers of equivalence relations and epi-strong mono factorizations, every epimorphism is regular and F preserves strong monomorphisms. Let $\mathcal{A} = (A,\alpha)$ be a coalgebra, $\pi_1,\pi_2 : \theta \to A$ a strong equivalence relation on A and $\pi_\theta : A \twoheadrightarrow A/\theta$ the coequalizer of π_1 and π_2. Then, the following are equivalent:*

1. *$\pi_1,\pi_2 : \theta \to A$ is a congruence on $\mathcal{A}$.*
2. *There exists a unique dynamics $\alpha_\theta : A/\theta \to F(A/\theta)$ turning π_θ into a homomorphism.*
3. *$\theta \sqsubseteq ker(F(\pi_\theta) \circ \alpha)$.*

Proof. 1. ⇒ 2. Let $\varphi : \mathcal{A} \to \mathcal{B}$ with $\mathcal{B} = (B,\beta)$ be a homomorphism whose kernel pair (in C) is θ. Let $\varphi = \mu \circ \varphi'$ be the epi-strong mono factorization of φ in C_F with $\varphi' : \mathcal{A} \twoheadrightarrow \varphi[\mathcal{A}]$ and $\mu : \varphi[\mathcal{A}] \rightarrowtail \mathcal{B}$ and denote by δ the dynamics on $\varphi[A]$. Clearly (θ,π_1,π_2) is also the kernel of φ' in C. Thus, since by assumption φ' is a regular epimorphism, it follows that φ' is also the coequalizer of π_1 and π_2 in C [2]. Thus, there exists an isomorphism $\psi : A/\theta \to \varphi[A]$ such that $\varphi' = \psi \circ \pi_\theta$. We have:

$$\begin{aligned} F(\psi)\circ F(\pi_\theta)\circ\alpha\circ\pi_1 &= F(\varphi')\circ\alpha\circ\pi_1 \\ &= \delta\circ\varphi'\circ\pi_1 = \delta\circ\varphi'\circ\pi_2 \\ &= F(\varphi')\circ\alpha\circ\pi_2 = F(\psi)\circ F(\pi_\theta)\circ\alpha\circ\pi_2. \end{aligned}$$

Thus, canceling $F(\psi)$ at the left, the universal property of the coequalizer yields that there exists a unique morphism $\alpha_\theta : A/\theta \to F(A/\theta$ such that $\alpha_\theta \circ \pi_\theta = F(\pi_\theta) \circ \alpha$.

2. ⇒ 3. It follows directly from the equality $F(\pi_\theta)\circ\alpha\circ\pi_1 = F(\pi_\theta)\circ\alpha\circ\pi_2$, the universal property of the pullbacks, Lemmas 2.1 and 2.3.

3. ⇒ 2. and 2. ⇒ 1. are straightforward. ∎

Definition 2.15. The factor coalgebra $(A/\theta, \alpha_\theta)$ given by Lemma 2.14 is denoted by $\mathcal{A}/\theta$ and called *the factor of the coalgebra $\mathcal{A}$ w.r.t. the congruence θ*. The epimorphism $\pi_\theta : \mathcal{A} \twoheadrightarrow \mathcal{A}/\theta$ is called the *canonical projection from $\mathcal{A}$ onto $\mathcal{A}/\theta$* .

3. The category $\mathsf{C}_{\mathcal{F}}$

C is a strongly well-powered category with epi-strong mono factorizations.

3.1. Closed subobjects.

Definition 3.1. Let $\mathcal{A} = (A, \alpha)$ be an F-coalgebra for some type F. A strong subobject $\mu : S \rightarrowtail A$ is called *closed* in (A, α) if there exists a morphism $\sigma : S \to F(S)$ making μ an F-morphism from (S, σ) to (A, α).

It is not hard to see that if F preserves strong monomorphisms, then for each strong subobject S of A, the dynamics σ is unique and it follows from Lemma 2.5 that (S, σ) is precisely a subcoalgebra of (A, α). Since C is strongly well-powered, closed strong subobjects of $\mathcal{A}$ form a set. Denote it by $Sub_F(\mathcal{A})$.

Lemma 3.2. *Assume that F preserves strong monos. Let (A, α) be an F-coalgebra and S an object in C. Then $S \in Sub_F((A, \alpha))$ if and only if there exists a coalgebra (P, π) and a homomorphism $\varphi : (P, \pi) \to (A, \alpha)$ with $S \cong \varphi[P]$.*

Proof. If $\mu : S \rightarrowtail A$ is a strong subobject closed in (A, α), then there exists a dynamics $\sigma : S \to F(S)$ such that $\mu : (S, \sigma) \to (A, \alpha)$ is a homomorphism. Set $(P, \pi) := (S, \sigma)$ and $\varphi := \mu$ and consider the epi-strong mono factorization $S \overset{\psi}{\twoheadrightarrow} \mu[S] \overset{\theta}{\rightarrowtail} A$ of μ in C_F given by Lemma 2.5. $\mu = \theta \circ \psi$ is a strong mono and θ is a strong mono and therefore a mono. Therefore, by Lemma 2.1 ψ is a strong mono, too. Now it is also an epi; thus it is an iso. Hence $S \cong \mu[S]$ for as it has been mentioned above, isos in C_F are carried by isos in C.

Conversely, assume that there exist a coalgebra (P, π) and a homomorphism $\varphi : (P, \pi) \to (A, \alpha)$ with $S \cong \varphi[P]$ and let $\varphi = \varepsilon \circ \eta$ be the epi-strong mono factorization of φ in C_F. Then $\varphi[(P, \pi)]$ is carried by $\varphi[P]$ and $\varepsilon : \varphi[P] \rightarrowtail A$ is a strong subobject of A such that $\varepsilon : \varphi[(P, \pi)] \rightarrowtail (A, \alpha)$ is a homomorphism again by Lemma 2.5. Hence the desired result. ■

The following result will be very useful in the sequel.

Theorem 3.3. *Assume that C has binary powers, and kernel pairs and F preserves binary powers as well as strong monos and let $\mathcal{A} = (A, \alpha)$ be an F-coalgebra and $\varphi : A \to B$ a morphism. Then the following statements hold:*

(i) *If F weakly preserves kernels and there exists a dynamics $\beta : B \to F(B)$ such that φ is a homomorphism from $\mathcal{A}$ to $\mathcal{B} = (B, \beta)$, then $ker(\varphi) \in Sub_F(\mathcal{A} \times \mathcal{A})$.*

(ii) *If φ is an epimorphism in C which is the coequalizer of its kernel pair and $ker(\varphi) \in Sub_F(\mathcal{A} \times \mathcal{A})$, then there exists a dynamics $\beta : B \to F(B)$ such that φ is a homomorphism from $\mathcal{A}$ to $\mathcal{B} = (B, \beta)$.*

Proof. Since C has and F preserves binary powers, then C_F has binary powers and they are carried by binary powers in C, see Subsection 2.1.

(i) It follows directly from item 1. of Lemma 2.11.

(ii) Let $\pi_1, \pi_2 : ker(\varphi) \to A$ denote the canonical projections of $ker(\varphi)$ and let $\gamma : ker(\varphi) \to F(ker(\varphi)$ be a dynamics such that the strong mono $\langle \pi_1, \pi_2 \rangle : ker(\varphi) \rightarrowtail A \times A$ given by the universal property of the product according to Lemma 2.3 is a homomorphism from $(ker(\varphi), \gamma)$ to $\mathcal{A} \times \mathcal{A}$. Let $p_1, p_2 : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ denote the canonical projections and $\pi : A \times A \to F(A \times A)$ the dynamics of $\mathcal{A} \times \mathcal{A}$. $\langle \pi_1, \pi_2 \rangle : (ker(\varphi), \gamma) \to (A \times A, \pi)$ is a homomorphism. On the other hand, for all i, $\alpha \circ p_i = F(p_i) \circ \pi$ and post-composing by $\langle \pi_1, \pi_2 \rangle$ yields $\alpha \circ \pi_i = F(\pi_i) \circ \gamma$. Now it is not hard to check that $F(\varphi) \circ \alpha \circ \pi_1 = F(\varphi) \circ \alpha \circ \pi_2$. Thus, by the universal property of the coequalizer, there exists a unique morphism $\beta : B \to F(B)$ such that $\beta \circ \varphi = F(\varphi) \circ \alpha$. ∎

It is not hard to see that a combination of Lemmas 2.5 and 2.6 yields:

Remark 3.4. Items (i) and (ii) from Theorem 3.3 hold if C is strongly complete and F is a covarietor preserving strong monomorphisms.

An analog of the following for differently typed algebras of functors over Set can be found in [16] and, also in [8] for coalgebras over Set without assumption of preservation of strong monos.

Lemma 3.5. *If F preserves strong monos, then the following hold for any homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ and any object U:*

(i) *If $U \in Sub_F(\mathcal{A})$, then $\varphi[U] \in Sub_F(\mathcal{B})$.*

(ii) *If C has and F preserves pullbacks along strong monos and $U \in Sub_F(\mathcal{B})$, then $\varphi^{-}[U] \in Sub_F(\mathcal{A})$.*

Proof. (i) Let $\mathcal{A} = (A, \alpha)$, $\mathcal{B} = (B, \beta)$, $\sigma : U \rightarrowtail A$ be a strong subobject of A such that $U \in Sub_F(\mathcal{A})$ with the dynamics $\mu : U \to F(U)$ and factorize $\varphi \circ \sigma$ in C as an epi $\psi' : U \twoheadrightarrow \varphi[U]$ followed by a strong mono $\theta' : \varphi[U] \rightarrowtail B$. A straightforward diagram chase yields $\beta \circ \theta' \circ \psi' = F(\theta') \circ F(\psi') \circ \mu$. On the other hand F preserves strong monos, thus $F(\theta')$ is a strong mono and since ψ' is an epi, it holds there exists a unique morphism $\phi : \varphi[U] \to F(\varphi[U])$ such that $\phi \circ \psi' = F(\psi') \circ \mu$ and $F(\theta') \circ \phi = \beta \circ \theta'$. Hence θ' is a strong mono which is a homomorphism from $(\varphi[U], \phi)$ to (B, β).

(ii) Since C has and F preserves pullbacks along strong monos, then C_F has pullbacks along strong monos too (see Subsection 2.1); and if F preserves strong monos, then monos therein are carried by strong monos in C, by Lemma 2.5. This immediately yields the desired result. ■

3.2. Some facts about coalgebraically equivalent coalgebras. Borrowing notations from [16], we have the following, where for each coalgebra $\mathcal{A}$ and each set I, $\mathcal{A}^I := \Pi_{i \in I}\mathcal{A}_i$ with $\mathcal{A}_i = \mathcal{A}$ is the I-power of $\mathcal{A}$, whenever it exists:

Definition 3.6. An F-coalgebra $\mathcal{A}$ and a G-coalgebra $\mathcal{A}'$ on a common carrier A, for some C-endofunctors F and G, are said to be *coalgebraically equivalent*, written as $\mathcal{A}_F \equiv_G \mathcal{A}'$ or simply $\mathcal{A} \equiv \mathcal{A}'$ if it is clear from the context, provided for each set I such that $\mathcal{A}^I$ and $\mathcal{A}'^I$ exist, $Sub_F(\mathcal{A}^I) = Sub_G(\mathcal{A}'^I)$.

Recall (see Subsection 2.1) that if C has and F preserves powers or if C is strongly complete and F is a covarietor preserving strong monomorphisms (see Lemma 2.6), arbitrary powers of coalgebras exist in C_F; i.e., for every coalgebra $\mathcal{A}$ and each set I, $\mathcal{A}^I$ exists.

[16] contains an analog of the following, for differently typed algebras over Set.

Lemma 3.7. *Assume that C has arbitrary powers and let F and G be C-endofunctors preserving arbitrary powers and strong monos, (A, α) and (B, β) F-coalgebras and (A, α') and (B, β') G-coalgebras. If there exists a strong mono $\varphi : A \rightarrowtail B$ which is a homomorphism from (A, α) to (B, β) and from (A, α') to (B, β'), then $(B, \beta)_F \equiv_G (B, \beta') \Rightarrow (A, \alpha)_F \equiv_G (A, \alpha')$.*

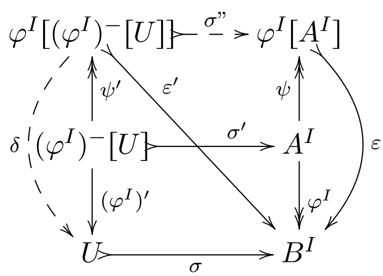
Proof. Since F and G preserve arbitrary powers, then C_F and C_G have powers and they are carried by powers in C, see Subsection 2.1. Assume that $(B, \beta)_F \equiv_G (B, \beta')$ and let $U \in Sub_F((A, \alpha)^I)$ for some set I such that $(A^I, \alpha^I) := (A, \alpha)^I$ and $(A'^I, \alpha'^I) := (A', \alpha')^I$. Then there exists a strong mono $\psi : U \rightarrowtail A^I$ and a dynamics $\mu_F : U \to F(U)$ such that ψ is a homomorphism from (U, μ_F) to (A^I, α^I). But then, the composite $\varphi^I \circ \psi : U \rightarrowtail B^I$ is a strong mono because the class of strong monos is closed under composition and under products [2], and an F-homomorphism, too. Hence $U \in Sub_F((B, \beta)^I)$. Therefore by assumption, $U \in Sub_G((B, \beta')^I)$, too. Thus, there exists a dynamics $\mu_G : U \to G(U)$ such that $\varphi^I \circ \psi : (U, \mu_G) \to (B^I, \beta'^I)$ is a homomorphism. We have $\beta'^I \circ \varphi^I \circ \psi = G(\varphi^I) \circ G(\psi) \circ \mu_G$; i.e., $G(\varphi^I) \circ \alpha'^I \circ \psi = G(\varphi^I) \circ G(\psi) \circ \mu_G$, because φ^I is a homomorphism from $(A, \alpha')^I$ to $(B, \beta')^I$. Now in addition, G preserves strong monomorphisms; thus $G(\varphi^I)$ is a mono so that canceling it from the left yields $\alpha'^I \circ \psi = G(\psi) \circ \mu_G$. Hence

$\psi : (U, \mu_G) \to (A, \alpha')^I$ is a homomorphism; i.e., $U \in Sub_G((A, \alpha')^I)$. By symmetry, we have $Sub_G((A, \alpha')^I) \subseteq Sub_F((A, \alpha)^I)$. ∎

A category *is said to satisfy the Axiom of Choice* (AC, for short) in the sense of [11] provided every epimorphism splits, i.e., every epimorphism is right-invertible. E.g., Set and the category Vec of real vector spaces and linear transformations between them see, for instance, [2] where it is shown that in a category satisfying the AC, epimorphisms are closed under products and pullbacks but this is not true in an arbitrary one. As far as epimorphisms are concerned, a category satisfying the AC satisfies the conditions in the following lemma, whose analog can be found in [16] for differently typed algebras over Set.

Lemma 3.8. *Assume that C has and F and G preserve pullbacks along strong monos and arbitrary powers as well as strong monos and epis are closed under pullbacks along strong monos and arbitrary powers and let $\mathcal{A} = (A, \alpha)$ and $\mathcal{B} = (B, \beta)$ be F-coalgebras, $\mathcal{A}' = (A, \alpha')$ and $\mathcal{B}' = (B, \beta')$ be G-coalgebras. If there exists an epi $\varphi : A \twoheadrightarrow B$ which is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$ as well as from $\mathcal{A}'$ to $\mathcal{B}'$, then $\mathcal{A}_F \equiv_G \mathcal{A}' \Rightarrow \mathcal{B}_F \equiv_G \mathcal{B}'$.*

Proof. Assume that $\mathcal{A}_F \equiv_G \mathcal{A}'$ and let $U \in Sub_F(\mathcal{B}^I)$, for some set I, with $\sigma : U \rightarrowtail B^I$ as corresponding strong mono. Then there exists a dynamics $\mu_F : U \to F(U)$ such that $\sigma : (U, \mu_F) \to \mathcal{B}^I$ is a homomorphism. Then by Lemma 3.5 $(\varphi^I)^-[U] \in Sub_F(\mathcal{A}^I)$ so that by assumption $(\varphi^I)^-[U] \in Sub_G(\mathcal{A}'^I)$ too. Thus, there exists a strong mono $\nu : (\varphi^I)^-[U] \rightarrowtail A^I$ and a dynamics $\nu_G : (\varphi^I)^-[U] \to G((\varphi^I)^-[U])$ such that $\nu : ((\varphi^I)^-[U], \nu_G) \to \mathcal{A}'^I$ is a homomorphism. But then, again by Lemma 3.5 $\varphi^I[(\varphi^I)^-[U]] \in Sub_G(\mathcal{B}'^I)$. Now φ^I factors in an epi followed by a strong mono in C_F and therefore in C according to Lemma 2.5 as $\varphi^I = \varepsilon \circ \psi$ where $\psi : A^I \twoheadrightarrow \varphi^I[A^I]$ and by definition $\varphi^I[(\varphi^I)^-[U]]$ is the image of the strong subobject $\sigma' : (\varphi^I)^-[U] \rightarrowtail A^I$, where σ' is the pullback of σ along φ^I. If $\varphi^I \circ \sigma'$ factors in an epi $\psi' : (\varphi^I)^-[U] \twoheadrightarrow \varphi^I[(\varphi^I)^-[U]]$ followed by a strong mono $\varepsilon' : \varphi^I[(\varphi^I)^-[U]] \rightarrowtail B^I$ as $\varphi^I \circ \sigma' = \varepsilon' \circ \psi'$, i.e., $\varepsilon \circ \psi \circ \sigma' = \varepsilon' \circ \psi'$, then ψ' epi and ε strong mono imply that there exists a unique morphism $\sigma" : \varphi^I[(\varphi^I)^-[U]] \rightarrowtail \varphi^I[A^I]$ such that $\sigma" \circ \psi' = \psi \circ \sigma'$ and $\varepsilon \circ \sigma" = \varepsilon'$; and, $\sigma"$ is a strong mono by Lemma 2.1.



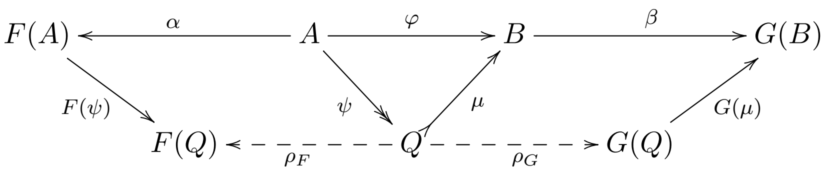
On the other hand, we have $\varepsilon \circ \sigma" \circ \psi' = \varepsilon \circ \psi \circ \sigma' = \varphi^I \circ \sigma' = \sigma \circ (\varphi^I)'$ with ψ' an epi and σ a strong mono. Thus there exists a unique morphism $\delta : \varphi^I[(\varphi^I)^-[U]] \to U$ such that $\delta \circ \psi' = (\varphi^I)'$ and $\sigma \circ \delta = \varepsilon \circ \sigma"$. Now as a composite of strong monos, $\varepsilon \circ \sigma"$ is a strong mono and σ is a (strong) mono. Thus by Lemma 2.1, δ is a strong mono, too. Moreover, since epis are closed under pullbacks along strong monos and under powers and φ is an epi, so is $(\varphi^I)'$. Therefore, the last but one equality implies that δ is an epi as well. Hence, it is an iso so that $U \in Sub_G(\mathcal{B}'^I)$. By symmetry, we obtain that $Sub_G(\mathcal{B}'^I) \subseteq Sub_F(\mathcal{B}^I)$ and this ends the proof. ∎

Taking into account Lemma 2.5, it is not hard to check the following:

Remark 3.9. In case $\mathsf{C} = \mathsf{Set}$ and without the assumption of preservation of strong monos by the types, Lemmas 3.2, 3.5 (see [8, 14]), 3.7 and 3.8 are valid.

3.3. Weak homomorphisms of coalgebras.

Definition 3.10. Let F and G be C-endofunctors, (A, α) an F-coalgebra and (B, β) a G-coalgebra. A morphism $\varphi : A \to B$ is called a *weak homomorphism* from (A, α) to (B, β) if factoring φ in an epi ψ followed by a strong mono $\mu : Q \to B$ as $\varphi = \mu \circ \psi$, there exist dynamics $\rho_F : Q \to F(Q)$ and $\rho_G : Q \to G(Q)$ such that the following conditions are satisfied:
(i) $\mathcal{Q}_F \equiv_G \mathcal{Q}'$, with $\mathcal{Q} := (Q, \rho_F)$ and $\mathcal{Q}' := (Q, \rho_G)$;
(ii) $\psi : (A, \alpha) \to (Q, \rho_F)$ and $\mu : (Q, \rho_G) \to (B, \beta)$ are homomorphisms; i.e., the following diagram commutes.



The concept of weak homomorphism generalizes that of homomorphism for strong mono-preserving endofunctors. Indeed, in case F preserves strong monomorphisms, the category C_F has epi-strong mono factorizations according to Lemma 2.5 and it holds in C that the epi-strong mono factorizations of the same morphism are isomorphic. Hence:

Remark 3.11. If F preserves strong monomorphisms, then every F-homomorphism from $\mathcal{A}$ to $\mathcal{B}$ is a weak homomorphism from $\mathcal{A}$ to $\mathcal{B}$. In particular, $id_{\mathcal{A}}$ is a weak homomorphism from $\mathcal{A}$ to itself.

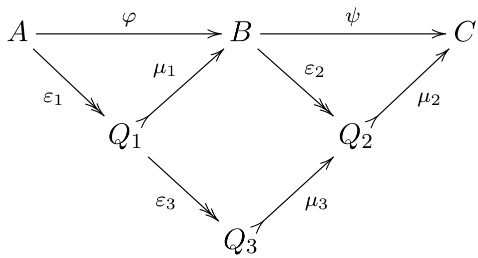
As far as epimorphisms are concerned, the following holds for instance when the category C satisfies the AC.

Theorem 3.12. *Assume that* C *has and* F, G *and* H *preserve arbitrary powers and weakly preserves kernel pairs,* C *has pullbacks along strong monos, epis are closed under pullbacks along strong monos and arbitrary powers and every epimorphism is the coequalizer of its kernel pair,* F *and* G *preserve pullbacks along strong monos,* F*,* G *and* H *preserve strong monos and let* (A, α) *be an* F*-coalgebra,* (B, β) *a* G*-coalgebra and* (C, γ) *a* H*-coalgebra. Whenever* $\varphi : (A, \alpha) \to (B, \beta)$ *and* $\psi : (B, \beta) \to (C, \gamma)$ *are weak homomorphisms,* $\psi \circ \varphi : (A, \alpha) \to (C, \gamma)$ *is a weak homomorphism, too.*

Proof. By assumption φ and ψ factor into an epi followed by a strong mono as $\varphi = \mu_1 \circ \varepsilon_1$ and $\psi = \mu_2 \circ \varepsilon_2$ with $\varepsilon_1 : A \twoheadrightarrow Q_1$ and $\varepsilon_2 : B \twoheadrightarrow Q_2$ and there are F-dynamics $\rho^1_F : Q_1 \to F(Q_1)$, G-dynamics $\rho^1_G : Q_1 \to G(Q_1)$, $\rho^2_G : Q_2 \to G(Q_2)$ and H-dynamics $\rho^2_H : Q_2 \to H(Q_2)$ such that:

(i) $(Q_1, \rho^1_F) \equiv (Q_1, \rho^1_G)$ and $(Q_2, \rho^2_G) \equiv (Q_2, \rho^2_H)$;

(ii) $\varepsilon_1 : (A, \alpha) \to (Q, \rho^1_F)$, $\mu_1 : (Q_1, \rho^1_G) \to (B, \beta)$, $\varepsilon_2 : (B, \beta) \to (Q_2, \rho^2_G)$ and $\mu_2 : (Q_2, \rho^2_H) \to (C, \gamma)$ are homomorphisms. As a composite of G-homomorphisms, $\varepsilon_2 \circ \mu_1 : (Q_1, \rho^1_G) \to (Q_2, \rho^2_G)$ is a G-homomorphism, too. Let $\varepsilon_2 \circ \mu_1 = \mu_3 \circ \varepsilon_3$ be its factorization in C_G into an epimorphism followed by a strong monomorphism with $\varepsilon_3 : (Q_1, \rho^1_G) \twoheadrightarrow (Q_3, \rho^3_G)$. Then $\psi \circ \varphi = (\mu_2 \circ \mu_3) \circ (\varepsilon_3 \circ \varepsilon_1)$ is an epi-strong mono factorization of $\psi \circ \varphi$ in C.



Since G weakly preserves kernels, then by (i) of Theorem 3.3 we obtain $ker(\varepsilon_3) \in Sub_G((Q_1, \rho^1_G) \times (Q_1, \rho^1_G))$ and, because $(Q_1, \rho^1_F) \equiv (Q_1, \rho^1_G)$, it immediately follows that $ker(\varepsilon_3) \in Sub_F((Q_1, \rho^1_F) \times (Q_1, \rho^1_F))$, too. Moreover, F weakly preserves kernels and ε_3 is an epimorphism in C. Thus, by (ii) of Theorem 3.3, there is an F-dynamics $\rho^3_F : Q_3 \to F(Q_3)$ such that $\varepsilon_3 : (Q_1, \rho^1_F) \twoheadrightarrow (Q_3, \rho^3_F)$ is an F-homomorphism. As a consequence of Lemma 3.8, $(Q_3, \rho^3_F) \equiv (Q_3, \rho^3_G)$. On the other hand, by (i) of Lemma 3.5, $\mu_3[Q_3] \in Sub_G((Q_2, \rho^2_G))$ and, taking into account $(Q_2, \rho^2_G) \equiv (Q_2, \rho^2_H)$, $\mu_3[Q_3] \in Sub_H((Q_2, \rho^2_H))$, too. Thus $\mu_3 : (Q_3, \rho^3_H) \rightarrowtail (Q_2, \rho^2_H)$ is an H-homomorphism. Moreover, $\mu_3 : (Q_3, \rho^3_G) \rightarrowtail (Q_2, \rho^2_G)$ is a G-homomorphism, too. Thus, since $\mu_3 : Q_3 \rightarrowtail Q_2$ is a strong mono in C, it follows from Lemma 3.7 that $(Q_3, \rho^3_G) \equiv (Q_3, \rho^3_H)$. Hence $(Q_3, \rho^3_F) \equiv (Q_3, \rho^3_H)$ and this ends the proof. ∎

From Theorem 3.12, the following is straightforward.

Corollary 3.13. *Assume that* C *has arbitrary powers, kernel pairs, pullbacks along strong monos, epis are closed under arbitrary powers and pullbacks along strong monos, every epimorphism is the coequalizer of its kernel pair and let* $\mathcal{F}$ *be a set of* C*-endofunctors that preserve arbitrary powers, pullbacks along strong monos, strong monos and weakly preserve kernels. Then the class* $\mathsf{C}_{\mathcal{F}}$ *of* F*-coalgebras for some* $F \in \mathcal{F}$ *and weak homomorphisms between them forms a category.*

Below is another situation in which the composition of weak homomorphisms is well-behaved:

Theorem 3.14. *Assume that* C *is strongly complete, epis are closed under pullbacks along strong monos and arbitrary powers and every epimorphism is the coequalizer of its kernel pair,* F *and* G *are covarietors and together with* H *preserve strong monos, arbitrary powers and weakly preserve kernels and let* (A, α) *be an* F*-coalgebra,* (B, β) *a* G*-coalgebra and* (C, γ) *a* H*-coalgebra. Whenever* $\varphi : (A, \alpha) \to (B, \beta)$ *and* $\psi : (B, \beta) \to (C, \gamma)$ *are weak homomorphisms,* $\psi \circ \varphi : (A, \alpha) \to (C, \gamma)$ *is a weak homomorphism, too.*

Proof. As a consequence of Lemmas 2.6 and 2.5, the categories C_F and C_G are complete and strong monos therein are carried by strong monos in C. Thus in particular binary powers and pullbacks along strong monos exist in each of them. On the other hand, strong monomorphisms are closed under the formation of pullbacks [2]. The rest of the proof is as that of Theorem 3.12. ∎

Theorem 3.14 immediately yields:

Corollary 3.15. *If* C *is strongly complete, epis are closed under pullbacks along strong monos and arbitrary powers and every epimorphism is the coequalizer of its kernel pair and* $\mathcal{F}$ *is a set of covarietors preserving strong monos, arbitrary powers and weakly preserving kernels, then the class* $\mathsf{C}_{\mathcal{F}}$ *of* F*-coalgebras for some* $F \in \mathcal{F}$ *and weak homomorphisms between them forms a category.*

3.4. Generation of weak homomorphisms. It is well known that if an epimorphism is regular and has a kernel pair, then it is the coequalizer of its kernel pair, see [2, 5], for instance. Therefore, in any category with kernel pairs an epimorphism is regular iff it is the coequalizer of its kernel pair. This is the case, for instance, in every topos, in the category Grp and, by Corollary 2.12, in any category of coalgebras of a weakly preserving kernels endofunctor of a category with kernel pairs, coequalizers and binary powers in which every epimorphism is regular. On the other hand, in some categories, regular epimorphisms are stable under pullbacks. Typical examples are those which are regular in the sense of [5]. Therefore, from the above, a regular category in which every epimorphism is regular (e.g., any topos) satisfies the assumptions in item 1. of the following which is somewhat an analog to Theorem 2.4. It is similar to some results about coalgebras of Set-endofunctors from [8, 13].

Theorem 3.16. *Let (A,α), (B,β) and (C,γ) be coalgebras in C_F, C_G and C_H, respectively, $\varphi:(A,\alpha)\to(C,\gamma)$ a weak homomorphism, $f:A\to B$ and $g:B\to C$ morphisms in C with $\varphi=g\circ f$.*

1. *Assume that C has pullbacks along strong monos, arbitrary powers and kernel pairs, every epimorphism in C is the coequalizer of its kernel pair, epis are closed under pullbacks along strong monos and arbitrary powers and both F and G preserve arbitrary powers, strong monos as well as pullbacks along strong monos and weakly preserve kernel pairs. If f is an epimorphism in C which is a weak homomorphism from (A,α) to (B,β), then g is a weak homomorphism from (B,β) to (C,γ).*
2. *If C has arbitrary powers, G and H preserve strong monos and arbitrary powers and g is a weak homomorphism which is a strong mono in C, then f is a weak homomorphism.*

Proof. 1. Factorize g in C as an epi followed by a strong mono as $g=\mu\circ\varepsilon$ with $\varepsilon:B\twoheadrightarrow Q$. Then $\varphi=\mu\circ(\varepsilon\circ f)$ is an epi-strong mono factorization of φ in C. Thus, since φ is a weak homomorphism from (A,α) to (C,γ), there exists an F-dynamics $\rho_F:Q\to F(Q)$ and a H-dynamics $\rho_H:Q\to H(Q)$ on Q such that $(Q,\rho_F)\equiv(Q,\rho_H)$, $\varepsilon\circ f:(A,\alpha)\to(Q,\rho_F)$ and $\mu:(Q,\rho_H)\to(C,\gamma)$ are homomorphisms in C_F and C_H, respectively. On the other hand, f is a weak homomorphism from (A,α) to (B,β). Thus considering the trivial epi-strong mono factorization of f in C, there exists an F-dynamics $\beta':B\to F(B)$ such that $f:(A,\alpha)\to(B,\beta')$ is an F-homomorphism and $(B,\beta')_F\equiv_G(B,\beta)$. Moreover, $\varepsilon\circ f:(A,\alpha)\to(Q,\rho_F)$ is a homomorphism in C_F and f is an epimorphism in C_F. Thus by (i) of Theorem 2.4, $\varepsilon:(B,\beta')\to(Q,\rho_F)$ is a homomorphism in C_F, too. Now F weakly preserves kernels; thus it follows from (i) of Theorem 3.3 that $ker(\varepsilon)\in Sub_F((B,\beta')\times(B,\beta'))$ and, because $(B,\beta')_F\equiv_G(B,\beta)$, $ker(\varepsilon)\in Sub_G((B,\beta)\times(B,\beta))$, too. Therefore, applying (ii) of Theorem 3.3 yields a unique G-dynamics $\rho_G:Q\to G(Q)$ such that $\varepsilon:(B,\beta)\to(Q,\rho_G)$ is a G-homomorphism. But then, by Lemma 3.8, $(B,\beta')_F\equiv_G(B,\beta)\Rightarrow(Q,\rho_F)\equiv(Q,\rho_G)$; and since $(Q,\rho_F)\equiv(Q,\rho_H)$, we deduce that $(Q,\rho_G)\equiv(Q,\rho_H)$. Hence the desired result.

2. Factorize f in C as an epi followed by a strong mono as $f=\nu\circ\theta$ with $\nu:K\rightarrowtail B$. Since strong monos are closed under composition [2], $\varphi=(g\circ\nu)\circ\theta$ is an epi-strong mono factorization of φ in C. Thus, φ being a weak homomorphism from (A,α) to (C,γ), there exists an F-dynamics

$\kappa_F : K \to F(K)$ and a H-dynamics $\kappa_H : K \to H(K)$ on K such that $(K, \kappa_F) \equiv (K, \kappa_H)$, $\theta : (A, \alpha) \to (K, \kappa_F)$ and $g \circ \nu : (K, \kappa_H) \to (C, \gamma)$ are homomorphisms in C_F and C_H, respectively. On the other hand, g is a weak homomorphism from (B, β) to (C, γ). Thus taking into account the trivial epi-strong mono factorization of g in C, there exists an H-dynamics $\beta" : B \to H(B)$ such that $g : (B, \beta") \to (C, \gamma)$ is a H-homomorphism and $(B, \beta")_H \equiv_G (B, \beta)$. $g \circ \nu : (K, \kappa_H) \to (C, \gamma)$ is a H-homomorphism, g is a strong monomorphism in C which is an H-homomorphism and in addition H preserves strong monomorphisms. Thus, by 2. of Theorem 2.4, $\nu : (K, \kappa_H) \to (B, \beta")$ is a H-homomorphism. On the other hand, ν is a strong monomorphism in C, G preserves strong monos too, and $(B, \beta")_H \equiv_G (B, \beta)$; thus by Lemma 3.7 $(K, \kappa_H) \equiv (K, \kappa_G)$. Now $(K, \kappa_F) \equiv (K, \kappa_H)$. Hence $(K, \kappa_F) \equiv (K, \kappa_G)$. Hence the desired result. ∎

From Theorem 3.16 one can easily deduce the following corollaries. In case $\mathsf{C} = \mathsf{Set}$, an analog of the first one for similarly typed coalgebras can be found in [8] as the so-called *First Diagram Lemma* and the functorial algebraic version of both of them can be found in [16].

Corollary 3.17. *Assume that the assumptions from* 1. *of Theorem* 3.16 *are satisfied and let* $\varphi : (A, \alpha) \to (B, \beta)$ *and* $\psi : (A, \alpha) \to (C, \gamma)$ *be weak homomorphisms where* (A, α) *is an* F*-coalgebra,* (B, β) *is a* G*-coalgebra,* (C, γ) *is a* H*-coalgebra and* ψ *is an epimorphism in* C*. There is a weak homomorphism* $\pi : (B, \beta) \to (C, \gamma)$ *such that* $\pi \circ \varphi = \psi$ *iff* $ker(\varphi) \sqsubseteq ker(\psi)$*. In this case,* π *is uniquely determined.*

Proof. Assume that $ker(\varphi) \sqsubseteq ker(\psi)$ and denote by $\pi_i : ker(\varphi) \to A$ and $\sigma_i : ker(\psi) \to A$, $i = 1, 2$, the canonical i^{th} projection of $ker(\varphi)$ and $ker(\psi)$, respectively. Then there exists a strong monomorphism $\tau : ker(\varphi) \to ker(\psi)$ such that $\pi_i = \sigma_i \circ \tau$, $i = 1, 2$. But then we have $\psi \circ \pi_1 = \psi \circ \pi_2$. Thus, by the universal property of the coequalizer, there exists a unique morphism $\pi : B \to C$ such that $\psi = \pi \circ \varphi$. Now ψ and φ are weak homomorphisms and φ is an epimorphism in C. Thus, by 1. of Theorem 3.16, $\pi : (B, \beta) \to (C, \gamma)$ is a weak homomorphism, too.

Conversely assume that there exists a (unique) morphism $\pi : B \to C$ such that $\psi = \pi \circ \varphi$. Then we have $\psi \circ \pi_1 = \psi \circ \pi_2$ and, by the universal property of the pullback, there exists a unique morphism $\theta : ker(\varphi) \to ker(\psi)$ such that $\pi_i = \sigma_i \circ \theta$, $i = 1, 2$. From these equalities, it follows that $\langle \pi_1, \pi_2 \rangle = \langle \sigma_1, \sigma_2 \rangle \circ \theta$ with $\langle \pi_1, \pi_2 \rangle$ a strong monomorphism and $\langle \sigma_1, \sigma_2 \rangle$ a (strong) monomorphism. Thus by Lemma 2.1, θ is a strong monomorphism, too. Hence $ker(\varphi) \sqsubseteq ker(\psi)$. π is uniquely determined follows from the fact that φ is an epimorphism in C. ∎

Corollary 3.18. *Assume that the assumptions from* 1. *of Theorem* 3.16 *are satisfied and consider an* F*-coalgebra* (A, α)*, a* G*-coalgebra* (B, β)*, a* H*-coalgebra* (C, γ) *and a* J*-coalgebra* (D, δ) *as well as weak homomorphisms* $\varphi : (A, \alpha) \to (B, \beta)$*,* $\psi : (A, \alpha) \to (C, \gamma)$*,* $\rho : (B, \beta) \to (D, \delta)$ *and* $\mu : (C, \gamma) \to (D, \delta)$*. Assume that* φ *is an epimorphism and* μ *is a strong monomorphism in* C*. If* $\rho \circ \varphi = \mu \circ \psi$*, then there exists a unique weak homomorphism* $\sigma : (B, \beta) \to (C, \gamma)$ *with the property* $\sigma \circ \varphi = \psi$*. In addition,* σ *satisfies* $\mu \circ \sigma = \rho$*.*

Proof. Since φ is an epimorphism, μ is a strong monomorphism in C and $\rho \circ \varphi = \mu \circ \psi$, it follows from the definition of a strong monomorphism that there exists a unique morphism $\sigma : B \to C$ such that $\sigma \circ \varphi = \psi$ and $\mu \circ \sigma = \rho$. But then the last but one equality and item 1. of Theorem 3.16 yield that $\sigma : (B, \beta) \to (C, \gamma)$ is a weak homomorphism. ∎

By 2. of Theorem 3.16, we have the following:

Remark 3.19. If instead of considering the assumptions from 1. of Theorem 3.16 in Corollary 3.18 we assume rather that G and J preserve strong monomorphisms, then the result still holds and looks like the so-called *Second Diagram Lemma* from [8].

Taking into account Lemmas 2.5 and 2.6, the following is straightforward:

Remark 3.20. If C is a strongly complete category in which every epimorphism is the coequalizer of its kernel pair and epis are closed under pullbacks along strong monos and arbitrary powers and F and G are covarietors preserving strong monos and arbitrary powers and weakly preserving kernels, then item 1. of Theorem 3.16 as well as Corollaries 3.17 and 3.18 still hold.

Lemma 3.21. *Assume that* C *has kernel pairs. If* $\mathcal{A}$ *is an* F*-coalgebra,* $\mathcal{B}$ *is a* G*-coalgebra and* $\varphi : \mathcal{A} \to \mathcal{B}$ *is a weak homomorphism, then* $ker(\varphi)$ *is a congruence on* $\mathcal{A}$*. If moreover* C *has binary powers and* F *weakly preserves kernels, then it is a strong bisimulation on* $\mathcal{A}$*.*

Proof. Factorize φ in C in an epimorphism followed by a strong monomorphism as $\varphi = \mu \circ \varepsilon$ where $\varepsilon : A \twoheadrightarrow Q$. Then there exist a F-dynamics $\rho_F : Q \to F(Q)$ and a G-dynamics $\rho_G : Q \to G(Q)$ such that $(Q, \rho_F) \equiv (Q, \rho_G)$, $\varepsilon : \mathcal{A} \twoheadrightarrow (Q, \rho_F)$ and $\mu : (Q, \rho_G) \to \mathcal{B}$ are homomorphisms in C_F and C_G, respectively. Thus $ker(\varepsilon)$ is a congruence on $\mathcal{A}$. Now it is a routine exercise to check that $ker(\varphi) \cong ker(\varepsilon)$. Thus $ker(\varphi)$ is a congruence on $\mathcal{A}$, too. The last assertion follows directly from 1. of Lemma 2.11. ∎

The following, which can be seen as an extension of Lemma 3.5, generalizes some results about coalgebras of a common type over Set from [8]. An analog thereof can be found in [16] for differently typed functorial algebras over Set.

Proposition 3.22. *Let (A, α) be an F-coalgebra, (B, β) a G-coalgebra and (B, β^*) a H-coalgebra and $\varphi : A \to B$ a morphism. Assume that $\varphi : (A, \alpha) \to (B, \beta)$ is a weak homomorphism. Then the following statements are true:*

(i) *If F preserves strong monomorphisms, then for each $U \in Sub_F((A, \alpha))$, $\varphi[U] \in Sub_G((B, \beta))$.*
(ii) *If C has and F and G preserve pullbacks along strong monos as well as strong monos, then for each $U \in Sub_G((B, \beta))$, $\varphi^-[U] \in Sub_F((A, \alpha))$.*
(iii) *If C has and F and G preserve arbitrary powers and strong monos and φ is an isomorphism, then its inverse φ^{-1} is a weak homomorphism from (B, β) to (A, α).*
(iv) *If C has and G and H preserve arbitrary powers and strong monos and $(B, \beta)_G \equiv_H (B, \beta^*)$, then $\varphi : (A, \alpha) \to (B, \beta^*)$ is a weak homomorphism.*

Proof. Let φ factor in an epi followed by a strong mono as $\varphi = \theta \circ \psi$ with $\psi : A \twoheadrightarrow Q$ and denote by $\rho_F : Q \to F(Q)$ and $\rho_G : Q \to G(Q)$ the dynamics such that $\psi : (A, \alpha) \to (Q, \rho_F)$ and $\theta : (Q, \rho_G) \to (B, \beta)$ are homomorphisms in C_F and C_G, respectively and $(Q, \rho_F) \equiv (Q, \rho_G)$.

(i) Let $\sigma : U \rightarrowtail A$ be the strong mono and $\mu : U \to F(U)$ the morphism making U a closed strong subobject of (A, α). Item (i) of Lemma 3.5 yields $\psi[U] \in Sub_F((Q, \rho_F))$. Moreover, by Lemma 2.2, $\psi[U] \cong \varphi[U]$. Thus, by invoking 3.10 (i), $\varphi[U] \in Sub_G((Q, \rho_G))$. Now $\theta : Q \rightarrowtail B$ is a strong mono which is a G-morphism and the class of strong monos is closed under composition (see [2], for instance); therefore $\varphi[U] \in Sub_G((B, \beta))$.

(ii) Let $U \in Sub_G((B, \beta))$. Then by item (ii) of Lemma 3.5, $\theta^-[U] \in Sub_G((Q, \rho_G))$. Therefore, again by invoking 3.10 (i), $\theta^-[U] \in Sub_F((Q, \rho_F))$. Thus, applying once more (ii) of Lemma 3.5 we have $\psi^-[\theta^-[U]] \in Sub_F((A, \alpha))$. The desired result follows from the fact that $\varphi^-[U] = \psi^-[\theta^-[U]]$.

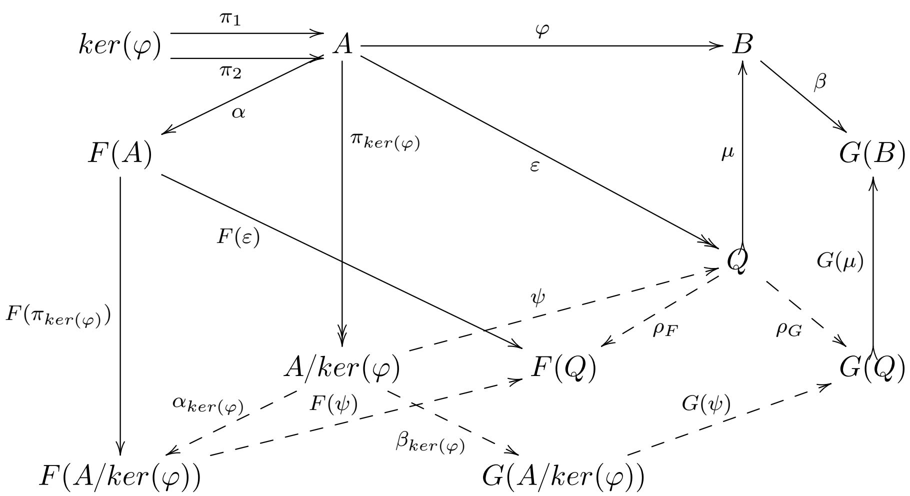
(iii) It follows directly from item 2. of Theorem 3.16 and Remark 3.11.

(iv) Assume that $(B, \beta)_G \equiv_H (B, \beta^*)$. Then since $Q \in Sub_G((B, \beta))$, it follows that $Q \in Sub_H((B, \beta^*))$, too. Thus, there exists a dynamics $\rho_H : Q \to H(Q)$ such that $\theta : (Q, \rho_H) \to (B, \beta^*)$ is a homomorphism. But then by Lemma 3.7, $(B, \beta)_G \equiv_H (B, \beta^*)$ implies $(Q, \rho_G) \equiv (Q, \rho_H)$. Thus we have $(Q, \rho_F) \equiv (Q, \rho_H)$. ■

3.5. An Isomorphism Theorem for weak homomorphisms. In the following we try to establish, for differently typed coalgebras, an analog to a special version of 1. ⇒ 2. from Lemma 2.14.

Theorem 3.23. *Assume that C has kernel pairs, arbitrary powers, coequalizers of kernel pairs and every epimorphism in C is regular and G preserves* *strong monomorphisms and together with F preserve arbitrary powers. If $\mathcal{A}$ is an F-coalgebra, $\mathcal{B}$ is a G-coalgebra, $\varphi : \mathcal{A} \to \mathcal{B}$ is a weak homomorphism and $\pi_{ker(\varphi)} : A \to A/ker(\varphi)$ is the coequalizer of the kernel pair $\pi_1, \pi_2 : ker(\varphi) \to A$ of φ, then there exists a unique morphism $\beta_{ker(\varphi)} : A/ker(\varphi) \to G(A/ker(\varphi))$ such that $\pi_{ker(\varphi)} : \mathcal{A} \to (A/ker(\varphi), \beta_{ker(\varphi)})$ is a weak homomorphism.*

Proof. Let $\varphi = \mu \circ \varepsilon$ be the epi-strong mono factorization of φ in C with $\varepsilon : A \twoheadrightarrow Q$ and $\mu : Q \rightarrowtail B$. Then there exist morphisms $\rho_F : Q \to F(Q)$ and $\rho_G : Q \to G(Q)$ such that $\varepsilon : (A, \alpha) \twoheadrightarrow (Q, \rho_F)$ and $\mu : (Q, \rho_G) \to (B, \beta)$ are homomorphisms in C_F and C_G, respectively and $(Q, \rho_F) \equiv (Q, \rho_G)$. By Lemma 3.21, $\pi_1, \pi_2 : ker(\varphi) \to A$ is a congruence on $\mathcal{A}$. Clearly $(ker(\varphi), \pi_1, \pi_2)$ is also the kernel of ε in C. Thus, since by assumption ε is a regular epimorphism, it is also the coequalizer of π_1 and π_2 in C [2, 5]. Thus, there exists an isomorphism $\psi : A/ker(\varphi) \to Q$ such that $\varepsilon = \psi \circ \pi_{ker(\varphi)}$.



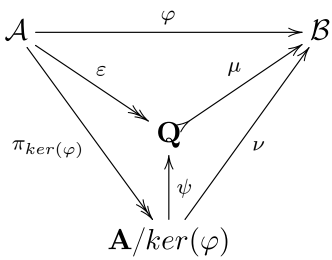
But then, using a diagram chase and arguing as in the proof of 1. ⇒ 2. of Lemma 2.14, we get that there exists a unique morphism $\alpha_{ker(\varphi)} : A/ker(\varphi) \to F(A/ker(\varphi))$ such that $\alpha_{ker(\varphi)} \circ \pi_{ker(\varphi)} = F(\pi_{ker(\varphi)}) \circ \alpha$. Since ε is an F-homomorphism and $\varepsilon = \psi \circ \pi_{ker(\varphi)}$ with $\pi_{ker(\varphi)}$ an epimorphism in C_F, it follows from 1. of Theorem 2.4 that ψ is an F-homomorphism, too. But then, we deduce that $A/ker(\varphi) \in Sub_F(Q, \rho_F)$ and, since $(Q, \rho_F) \equiv (Q, \rho_G)$, it follows that there exists a morphism $\beta_{ker(\varphi)} : A/ker(\varphi) \to G(A/ker(\varphi))$ such that $\psi : (A/Ker(\varphi)) \to (Q, \rho_G)$ a G-homomorphism. ψ is therefore an isomorphism in C_G, too. Let's show that $\pi_{ker(\varphi)} : (A, \alpha) \to (A/ker(\varphi), \beta_{ker(\varphi)})$ is a weak homomorphism. We have $\varphi = \mu \circ \varepsilon = \mu \circ \psi \circ \pi_{ker(\varphi)}$. $\mu \circ \psi$ is a G-homomorphism from $(A/ker(\varphi), \beta_{ker(\varphi)})$ to (B, β).

Thus by Remark 3.11, it is a weak homomorphism. In addition, $\mu \circ \psi$ is a strong monomorphism in C. Thus by 2. of Theorem 3.16, $\pi_{ker(\varphi)}$ is a weak homomorphism from (A, α) to $(A/ker(\varphi), \beta_{ker(\varphi)})$. The uniqueness of $\beta_{ker(\varphi)}$ as G-dynamics making $\pi_{ker(\varphi)} : (A, \alpha) \to (A/ker(\varphi), \beta_{ker(\varphi)})$ a weak homomorphism is straightforward. ∎

Remark 3.24. Instead of assuming in Theorem 3.23 that every epimorphism is regular, one could simply assume that C has regular epi-strong mono factorizations and the result still holds.

If in the above φ is a weak homomorphism which is an epimorphism in C, then μ must be an isomorphism in C. Therefore, from Theorem 3.23 and its proof, we immediately deduce the following, whose analog for similarly typed coalgebras over Set can be found in [8, 14]:

Theorem 3.25. (Isomorphism Theorem) *Assume that C has kernel pairs, arbitrary powers, coequalizers of kernel pairs and every epimorphism in C is regular. If F and G are arbitrary powers preserving C-endofunctors such that G preserves strong monomorphisms, then every weak homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$, where $\mathcal{A}$ is an F-coalgebra and $\mathcal{B}$ a G-coalgebra, can be decomposed as $\varphi = \nu \circ \pi_{ker(\varphi)}$, where $\nu = \mu \circ \psi$ with ψ both an F-isomorphism and a G-isomorphism, $\pi_{ker(\varphi)}$ is a weak homomorphism which is an epimorphism in C and ν is a weak homomorphism which is a strong monomorphism in C. In particular, every codomain of a weak homomorphism which is an epimorphism in C is isomorphic to a factor.*



In the above diagram, the boldface is used when the carrier is endowed with both an F-dynamics and a G-dynamics.

From Corollary 3.13 and Theorem 3.25, the following corollary is straightforward:

Corollary 3.26. *Assume that C has arbitrary powers, pullbacks along strong monomorphisms, kernel pairs, coequalizers of kernel pairs, every epimorphism in C is regular and epis are closed under arbitrary powers and pullbacks along strong monos. If $\mathcal{F}$ denotes a set of C-endofunctors which preserve strong monomorphisms, pullbacks along strong monomorphisms,* *arbitrary powers and weakly preserve kernels, then every weak homomorphism can be decomposed in* $\mathsf{C}_{\mathcal{F}}$ *into a weak homomorphism, which is an epimorphism in* C *followed by a weak homomorphism, which is a strong monomorphism in* C.

Theorem 3.25 and Corollary 3.15 also immediately yield:

Corollary 3.27. *Assume that* C *is a strongly complete category with coequalizers of kernel pairs and in which epimorphisms are regular and closed under pullbacks along strong monos and arbitrary powers. If* $\mathcal{F}$ *denotes a set of covarietors preserving strong monomorphisms, arbitrary powers and weakly preserving kernels, then every weak homomorphism can be decomposed in* $\mathsf{C}_{\mathcal{F}}$ *into a weak homomorphism, which is an epimorphism in* C *followed by a weak homomorphism, which is a strong monomorphism in* C.

Example 3.28. Set $\mathsf{C} = \mathsf{Grp}$, $F = Id$ and $G = T$ the termination: for all object G in Grp, $T(G) := \{*\}$, the one-element group (recall that it is both initial and terminal object in Grp) and denote by $!_G : G \to \{*\}$ the unique morphism from G to $\{*\}$. Then:

1. For every $n \in \mathbb{Z}$, the F-coalgebra $(n\mathbb{Z}, \zeta)$ where $\zeta(nx) = n^2x$ and the G-coalgebra $(n\mathbb{Z}, !_{n\mathbb{Z}})$ are coalgebraically equivalent.
2. For every integer k, the map $\alpha_k : \mathbb{Z} \to \mathbb{Z}$ defined by $x \mapsto kx$ is a weak homomorphism from $(\mathbb{Z}, \alpha_n)$ to $(\mathbb{Z}, !_{\mathbb{Z}})$, for any integer n.

Proof. 1. Grp has arbitrary powers because it is complete [2]. Therefore, since F and G preserves products, in particular arbitrary powers exist in Grp_F and Grp_G each of which is carried by the corresponding powers of the carrier of the coalgebra considered in Grp, see Subsection 2.1. On the other hand, for every set I, $Sub_F((n\mathbb{Z}, \zeta)^I) = \{(m\mathbb{Z})^I : m \in n\mathbb{Z}\}$ because $\zeta^I_m : (m\mathbb{Z})^I \to (m\mathbb{Z})^I$ defined by $\zeta^I_m(x) = \zeta^I(x)$ is a dynamics on $(m\mathbb{Z})^I$ making the map $(i^n_m)^I : (m\mathbb{Z})^I \to (n\mathbb{Z})^I$, where $i^n_m : m\mathbb{Z} \to n\mathbb{Z}$ is an inclusion, a homomorphism in Grp. In fact, if $p_i : (n\mathbb{Z})^I \to n\mathbb{Z}$ and $q_i : (m\mathbb{Z})^I \to m\mathbb{Z}$ are i^{th} canonical projections, then $(i^n_m \circ q_i : (m\mathbb{Z})^I \to n\mathbb{Z})_{i \in I}$ is the source that gives, $(i^n_m)^I : (m\mathbb{Z})^I \to (n\mathbb{Z})^I$ defined by $p_i \circ (i^n_m)^I = i^n_m \circ q_i$ for each $i \in I$, by the universal property of the product. Moreover, obviously $Sub_G((n\mathbb{Z}, !_{n\mathbb{Z}})^I) = \{(m\mathbb{Z})^I : m \in n\mathbb{Z}\}$ with $(m\mathbb{Z})^I$ endowed with the dynamics defined by $!_{(m\mathbb{Z})^I}$ for each $m \in n\mathbb{Z}$. Hence $(n\mathbb{Z}, \zeta)_{Id} \equiv_T (n\mathbb{Z}, !_{n\mathbb{Z}})$.

2. $\alpha_k = i_k \circ \varepsilon_k$ where $\varepsilon_k : \mathbb{Z} \to k\mathbb{Z}$ is defined by $x \mapsto kx$ and i_k is the inclusion map. By 1. above, $(k\mathbb{Z}, \rho)_{Id} \equiv_T (k\mathbb{Z}, !_{k\mathbb{Z}})$, where $\rho : k\mathbb{Z} \to k\mathbb{Z}$ is defined by $kp \mapsto nkp$, and verification that ε_k and i_k are morphisms in Grp_{Id} and in Grp_T from $(\mathbb{Z}, \alpha_n)$ to $(k\mathbb{Z}, \rho)$ and from $(k\mathbb{Z}, !_{k\mathbb{Z}})$ to $(\mathbb{Z}, !_{\mathbb{Z}})$, respectively, is straightforward. ∎

Remark 3.29. Every topos C is cartesian closed [5]. Thus for every object A in C, the functor $(-)^A$ is a right adjoint to the functor $-\times A$. Therefore, it preserves all limits that exist in C [2, 5].

Since Set is a topos and is complete [2, 5], it follows from Remark 3.29 that for any set Σ, the functor $(-)^\Sigma$ preserves all limits and, in particular, arbitrary powers and pullbacks. Thus invoking subsection 2.1, $Set_{(-)^\Sigma}$ has arbitrary powers and they are created by the forgetful functor. This yields:

Example 3.30. Set $\mathsf{C} = \mathsf{Set}$, $F = (-)^{\Sigma_1}$ and $G = (-)^{\Sigma_0}$ where $\Sigma_1 = \{\emptyset\}$ and $\Sigma_0 = \emptyset$; that is, for every set A, $F(A) = \emptyset$ if $A = \emptyset$, $F(A) = \{f_a : \Sigma_1 \to A : \emptyset \mapsto a, a \in A\}$ if $A \neq \emptyset$ and $G(A) = \{\emptyset \to A\}$. Denote by $!_A : A \to A^{\Sigma_0}$ the unique map from A to $\{\emptyset \to A\}$ and let (A, α_A) be the F-coalgebra with $\alpha_A = id_\emptyset$ if $A = \emptyset$ and $\alpha_A(a) = f_a$, for all $a \in A$, otherwise. Then:

1. $(A, \alpha_A)_F \equiv_G (A, !_A)$.
2. Every map $\varphi : A \to A$ is a weak homomorphism from (A, α_A) to $(A, !_A)$.

Proof. 1. If $A = \emptyset$ then it is obvious. Otherwise, let I be a set. Since powers in $\mathsf{Set}_{(-)^{\Sigma_1}}$ are created by the forgetful functor, we have $(A, \alpha_A)^I = (A^I, \alpha_A^I)$, with $\alpha_A^I : A^I \to (A^I)^{\Sigma_1}$ being the unique map given by the universal property of the product in Set and being such that for all $i \in I$, if $p_i : A^I \to A$ denotes the i^{th} canonical projection of A^I onto A, $\alpha_A \circ p_i = p_i^{\Sigma_1} \circ \alpha_A^I$. Now $p_i^{\Sigma_1} \circ \alpha_A^I = \alpha_A \circ p_i$ means that for all $(a_i)_{i\in I}$, $(p_i^{\Sigma_1} \circ \alpha_A^I)((a_i)_{i\in I}) = (\alpha_A \circ p_i)((a_i)_{i\in I})$; that is, $p_i \circ (\alpha_A^I((a_i)_{i\in I})) = \alpha_A(a_i)$ which amounts to saying that $p_i \circ \alpha_A^I((a_i)_{i\in I}) = f_{a_i}$ for all $i \in I$; that is, $p_i((\alpha_A^I((a_i)_{i\in I}))(\emptyset)) = a_i$ for all $i \in I$. But this means that $(\alpha_A^I((a_i)_{i\in I}))(\emptyset) = (a_i)_{i\in I}$. Thus, $\alpha_A^I((a_i)_{i\in I}) = f_{(a_i)_{i\in I}}$ so that $\alpha_A^I = \alpha_{A^I}$. We want to show that $Sub_{(-)^{\Sigma_1}}((A^I, \alpha_{A^I})) = Sub_{(-)^{\Sigma_0}}((A^I, !_{A^I}))$. Let $S \in Sub_{(-)^{\Sigma_1}}((A^I, \alpha_{A^I}))$. Then there exists an injective map $\mu : S \rightarrowtail A^I$ and a dynamics $\sigma : S \to S^{\Sigma_1}$ such that $\mu^{\Sigma_1} \circ \sigma = \alpha_{A^I} \circ \mu$, where $\mu^{\Sigma_1} : S^{\Sigma_1} \to (A^I)^{\Sigma_1}$ is defined by $\mu^{\Sigma_1} = \emptyset \to (A^I)^{\Sigma_1}$ if $S = \emptyset$ and $\mu^{\Sigma_1}(u) = \mu \circ u$, for all $u \in S^{\Sigma_1}$, otherwise. Obviously it holds $!_{A^I} \circ \mu = \mu^{\Sigma_0} \circ !_S$. Thus $S \in Sub_{(-)^{\Sigma_0}}((A^I, !_{A^I}))$, where μ^{Σ_0} is the trivial bijection $\{\emptyset \to S\} \to \{\emptyset \to A^I\}$. Conversely, assuming rather that $S \in Sub_{(-)^{\Sigma_0}}((A^I, !_{A^I}))$ with $\mu : S \rightarrowtail A^I$ the corresponding injection, α_S is a dynamics on S making μ an F-homomorphism from (S, α_S) to (A^I, α_{A^I}). Indeed, if $S = \emptyset$, it is obvious. Otherwise, let $s \in S$. We have $((\mu^{\Sigma_1} \circ \alpha_S)(s))(\emptyset) = (\mu^{\Sigma_1}(\alpha_S(s)))(\emptyset) = (\mu \circ (\alpha_S(s)))(\emptyset) = (\mu \circ f_s)(\emptyset) = \mu(f_s(\emptyset)) = \mu(s)$ and $((\alpha_{A^I} \circ \mu)(s))(\emptyset) = (\alpha_{A^I}(\mu(s)))(\emptyset) = f_{\mu(s)}(\emptyset) = \mu(s)$. Hence $\mu^{\Sigma_1} \circ \alpha_S = \alpha_{A^I} \circ \mu$.

2. Let $\varphi : A \to A$ be a map. Factorizing φ canonically as an epi followed by a mono as $\varphi = \nu \circ \pi$ where $\pi : A \twoheadrightarrow \varphi(A)$ and $\nu : \varphi(A) \rightarrowtail A$ is the inclusion map, to verify that $\pi : (A, \alpha_A) \to (\varphi(A), \alpha_{\varphi(A)})$ and

$\nu : (\varphi(A), !_{\varphi(A)}) \to (A, !_A)$ are F-homomorphism and G-homomorphism, respectively, is straightforward. In addition by 1. above, $(\varphi(A), \alpha_{\varphi(A)})_F \equiv_G (\varphi(A), !_{\varphi(A)})$. ∎

Moreover, $(-)^\Sigma$ obviously preserves strong monos (resp. Set is strongly complete and $(-)^\Sigma$ is a covarietor). Thus Corollary 3.13 (resp. 3.15) yields:

Example 3.31. If $\mathcal{F} = \{(-)^{\Sigma_k}, k \in K\}$ where K is a nonempty set, Σ_k is a set for every $k \in K$, then $\mathsf{Set}_{\mathcal{F}}$ is a category.

In fact, we have the following example where coalgebras considered are automata which are examples of deterministic systems with input [14]:

Example 3.32. Set is a (strongly) complete topos in which epis are coequalizers of their kernel pairs and are closed under pullbacks. Moreover, elements of the set of types $\mathcal{F}$ from Example 3.31 are simultaneously pullbacks, arbitrary powers, strong monos and weakly kernels preserving functors and are covarietors. Thus as far as coalgebras and (weak) homomorphisms are concerned, Set and $\mathcal{F}$ satisfy assumptions in all the results from this paper.

4. Conclusion

In our paper, we presented an extension of the concept of weak homomorphism between differently typed coalgebras to arbitrary categories endowed with a suitable factorization system. We showed that many of the results known so far for similarly typed coalgebras generalize to differently typed ones. In particular, under some reasonable assumptions on the category and on the set of its endofunctors, we showed that differently typed coalgebras and weak homomorphisms between them form a category which admits a canonical factorization structure for morphisms. Requiring that the base category satisfies the Axiom of Choice would have helped simplify the statement as well as the proof of most of the results we have obtained but this would have considerably restricted their scopes.

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