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SOME APPLICATIONS OF MINIMAL P_γ-OPEN SETS

Abstract. We characterize minimal P_γ-open sets in topological spaces. We show that any nonempty subset of a minimal P_γ-open set is pre P_γ-open. As an application of a theory of minimal P_γ-open sets, we obtain a sufficient condition for a P_γ-locally finite space to be a pre P_γ-Hausdorff space.

1. Introduction

Mashhour et al [3], introduced and investigated the notions of preopen sets, and Kasahara [4], defined the concept of an operation on topological spaces. Ogata [5], introduced the concept of γ-open sets and investigated the related topological proporties of the associated topology τ_γ and τ, where τ_γ is the collection of all γ-open sets

In this paper, we study fundamental properties of minimal P_γ-open sets and apply them to obtain some results in topological spaces. Also we give some characterizations of minimal P_γ-open sets. Moreover, we define and study P_γ-locally finite spaces and we apply minimal P_γ-open sets to define pre P_γ-open sets.

Finally, we show that any P_γ-locally finite space containing a minimal P_γ-open subset is pre P_γ-Hausdorff.

2. Preliminaries

Definition 2.1. [3] A subset A of a topological space (X, τ) is said to be preopen if $A \subseteq Int(Cl(A))$. The family of all preopen sets is denoted by $PO(X, \tau)$.

Definition 2.2. [4] Let (X, τ) be a topological space. An operation γ on the topology τ is a mapping from τ to power set $P(X)$ of X such that

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$V \subseteq \gamma(V)$ for each $V \in \tau$, where $\gamma(V)$ denotes the value of γ at V. It is denoted by $\gamma : \tau \to P(X)$.

Definition 2.3. [5] A subset A of a topological space (X, τ) is called γ-open set if for each $x \in A$ there exists an open set U such that $x \in U$ and $\gamma(U) \subseteq A$.

Definition 2.4. [2] Let X be a space and $A \subseteq X$ a γ-open set. Then A is called a minimal γ-open set if ϕ and A are the only γ-open subsets of A.

The following definitions and results are obtained from [1]. We defined γ to be a mapping on $PO(X)$ into $P(X)$ and $\gamma : PO(X) \to P(X)$ is called an operation on $PO(X)$, such that $V \subseteq \gamma(V)$ for each $V \in PO(X)$ [1].

Definition 2.5. A subset A of a space X is called P_γ-open if for each $x \in A$, there exists a preopen set U such that $x \in U$ and $\gamma(U) \subseteq A$.

Definition 2.6. Let A be a subset of (X, τ), and $\gamma : PO(X) \to P(X)$ be an operation. Then the P_γ-closure (resp., P_γ-interior) of A is denoted by $p_\gamma Cl(A)$ (resp., $p_\gamma Int(A)$) and defined as follows:

(1) $p_\gamma Cl(A) = \bigcap\{F : F \text{ is } P_\gamma\text{-closed and } A \subseteq F\}$.
(2) $p_\gamma Int(A) = \bigcup\{U : U \text{ is } P_\gamma\text{-open and } U \subseteq A\}$.

Theorem 2.7. *For a point $x \in X$, $x \in p_\gamma Cl(A)$ if and only if for every P_γ-open set V of X containing x, $A \cap V \neq \phi$.*

Definition 2.8. An operation γ on $PO(X)$ is said to be pre regular if for every preopen sets U and V of each $x \in X$, there exists a preopen set W of x such that $\gamma(W) \subseteq \gamma(U) \cap \gamma(V)$.

Proposition 2.9. *Let γ be a pre regular operation on $PO(X)$. If A and B are P_γ-open sets in X, then $A \cap B$ is also a P_γ-open set.*

3. Minimal P_γ-open sets

In view of the definition of minimal γ-open sets [2], we define minimal P_γ-open sets as:

Definition 3.1. Let X be a space and $A \subseteq X$ a P_γ-open set. Then A is called a minimal P_γ-open set if ϕ and A are the only P_γ-open subsets of A.

The following example shows that minimal P_γ-open sets and minimal γ-open sets are independent of each other.

Example 3.2. Consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, X\}$. Define an operation $\gamma : PO(X) \to P(X)$ by $\gamma(A) = A$. The P_γ-open sets are ϕ, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, {b, c} and X. Here $\{a\}$ is minimal P_γ-open but not minimal γ-open. Also we consider $X = \{a, b, c\}$ with the topology

$\tau = \{\phi, \{a,b\}, X\}$. Define $\gamma : PO(X) \to P(X)$ as $\gamma(A) = A$, the set $\{a,b\}$ is minimal γ-open but not minimal P_γ-open.

Proposition 3.3. *Let X be a space. Then:*

(1) *Let A be a minimal P_γ-open set and B a P_γ-open set. Then $A \cap B = \phi$ or $A \subseteq B$, where γ is pre regular.*
(2) *Let B and C be minimal P_γ-open sets. Then $B \cap C = \phi$ or $B = C$, where γ is pre regular.*

Proof. (1) Let B be a P_γ-open set such that $A \cap B \neq \phi$. Since A is a minimal P_γ-open set and $A \cap B \subseteq A$, we have $A \cap B = A$. Therefore $A \subseteq B$.

(2) If $B \cap C \neq \phi$, then we see that $B \subseteq C$ and $C \subseteq B$ by (1). Therefore $B = C$. ∎

Proposition 3.4. *Let A be a minimal P_γ-open set. If x is an element of A, then $A \subseteq B$ for any P_γ-open neighborhood B of x, where γ is pre regular.*

Proof. Let B be a P_γ-open neighborhood of x such that $A \nsubseteq B$. Since γ is pre regular operation, then $A \cap B$ is P_γ-open set such that $A \cap B \subseteq A$ and $A \cap B \neq \phi$. This contradicts our assumption that A is a minimal P_γ-open set. ∎

The following example shows that the condition that γ is pre regular is necessary for the above proposition.

Example 3.5. Consider $X = \{a,b,c\}$ with the topology $\tau = \{\phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}, X\}$. Define an operation γ on $PO(X)$ by

$$\gamma(A) = \begin{cases} A & \text{if } b \in A, \\ Cl(A) & \text{if } b \notin A. \end{cases}$$

Then calculations show that the operation γ is not pre regular. Clearly $A = \{a,c\}$ is a minimal P_γ-open set. Thus for $a \in A$, there is no P_γ-open set B containing a such that $A \subseteq B$.

Proposition 3.6. *Let A be a minimal P_γ-open set. Then for any element x of A, $A = \cap\{B : B$ is P_γ-open neighborhood of $x\}$, where γ is pre regular.*

Proof. By Proposition 3.4 and the fact that A is P_γ-open neighborhood of x, we have $A \subseteq \cap\{B : B \text{ is } P_\gamma\text{-open neighborhood of } x\} \subseteq A$. Therefore we have the result. ∎

Proposition 3.7. *Let A be a minimal P_γ-open set in X and $x \in X$ such that $x \notin A$. Then for any P_γ-open neighborhood C of x, $C \cap A = \phi$ or $A \subseteq C$, where γ is pre regular.*

Proof. Since C is a P_γ-open set, we have the result by Proposition 3.3. ∎

Corollary 3.8. *Let A be a minimal P_γ-open set in X and $x \in X$ such that $x \notin A$. Define $A_x = \cap\{B$: B is P_γ-open neighborhood of $x\}$. Then $A_x \cap A = \phi$ or $A \subseteq A_x$, where γ is pre regular.*

Proof. If $A \subseteq B$ for any P_γ-open neighborhood B of x, then $A \subseteq \cap\{B$: B is P_γ-open neighborhood of $x\}$. Therefore $A \subseteq A_x$. Otherwise there exists a P_γ-open neighborhood B of x such that $B \cap A = \phi$. Then we have $A_x \cap A = \phi$. ∎

Corollary 3.9. *If A is a nonempty minimal P_γ-open set of X, then for a nonempty subset C of A, $A \subseteq p_\gamma Cl(C)$, where γ is pre regular.*

Proof. Let C be any nonempty subset of A. Let $y \in A$ and B be any P_γ-open neighborhood of y. By Proposition 3.4, we have $A \subseteq B$ and $C = A \cap C \subseteq B \cap C$. Thus we have $B \cap C \neq \phi$ and hence $y \in p_\gamma Cl(C)$. This implies that $A \subseteq p_\gamma Cl(C)$. This copmletes the proof. ∎

Proposition 3.10. *Let A be a nonempty P_γ-open subset of a space X. If $A \subseteq p_\gamma Cl(C)$ then $p_\gamma Cl(A) = p_\gamma Cl(C)$, for any nonempty subset C of A.*

Proof. For any nonempty subset C of A, we have $p_\gamma Cl(C) \subseteq p_\gamma Cl(A)$. On the other hand, by supposition we see $p_\gamma Cl(A) \subseteq p_\gamma Cl(p_\gamma Cl(C)) = p_\gamma Cl(C)$ implies $p_\gamma Cl(A) \subseteq p_\gamma Cl(C)$. Therefore, we have $p_\gamma Cl(A) = p_\gamma Cl(C)$ for any nonempty subset C of A. ∎

Proposition 3.11. *Let A be a nonempty P_γ-open subset of a space X. If $p_\gamma Cl(A) = p_\gamma Cl(C)$, for any nonempty subset C of A, then A is a minimal P_γ-open set.*

Proof. Suppose that A is not a minimal P_γ-open set. Then there exists a nonempty P_γ-open set B such that $B \subsetneq A$ and hence there exists an element $x \in A$ such that $x \notin B$. Then we have $p_\gamma Cl(\{x\}) \subseteq (X \backslash B)$ implies that $p_\gamma Cl(\{x\}) \neq p_\gamma Cl(A)$. This contradiction proves the proposition. ∎

Combining Corollary 3.9 and Propositions 3.10 and 3.11, we have:

Theorem 3.12. *Let A be a nonempty P_γ-open subset of space X. Then the following are equivalent:*

(1) *A is minimal P_γ-open set, where γ is pre regular.*
(2) *For any nonempty subset C of A, $A \subseteq p_\gamma Cl(C)$.*
(3) *For any nonempty subset C of A, $p_\gamma Cl(A) = p_\gamma Cl(C)$.*

Definition 3.13. A subset A of a space X is called a pre P_γ-open set if $A \subseteq p_\gamma Int(p_\gamma Cl(A))$. The family of all pre P_γ-open sets of X will be denoted by $PO_\gamma(X)$.

Definition 3.14. A space X is called pre P_γ-Hausdorff if for each $x, y \in X$, $x \neq y$ there exist subsets U and V of $PO_\gamma(X)$ such that $x \in U$, $y \in V$, and $U \cap V = \phi$.

Theorem 3.15. *Let A be a minimal P_γ-open set. Then any nonempty subset C of A is a pre P_γ-open set, where γ is pre regular.*

Proof. By Corollary 3.9, we have $A \subseteq p_\gamma Cl(C)$ implies $p_\gamma Int(A) \subseteq p_\gamma Int(p_\gamma Cl(C))$. Since A is a P_γ-open set, we have $C \subseteq A = p_\gamma Int(A) \subseteq p_\gamma Int(p_\gamma Cl(C))$ or $C \subseteq p_\gamma Int(p_\gamma Cl(C))$, that is C pre P_γ-open. Hence the proof. ∎

Theorem 3.16. *Let A be a minimal P_γ-open set and let B be a nonempty subset of X. If there exists a P_γ-open set C containing B such that $C \subseteq p_\gamma Cl(B \cup A)$, then $B \cup D$ is a pre P_γ-open set for any nonempty subset D of A, where γ is pre regular.*

Proof. By Theorem 3.12 (3), we have $p_\gamma Cl(B \cup D) = p_\gamma Cl(B) \cup p_\gamma Cl(D) = p_\gamma Cl(B) \cup p_\gamma Cl(A) = p_\gamma Cl(B \cup A)$. By supposition $C \subseteq p_\gamma Cl(B \cup A) = p_\gamma Cl(B \cup D)$ implies $p_\gamma Int(C) \subseteq p_\gamma Int(p_\gamma Cl(B \cup D))$. Since C is a P_γ-open neighborhood of B, namely C is a P_γ-open such that $B \subseteq C$, we have $B \subseteq C = p_\gamma Int(C) \subseteq p_\gamma Int(p_\gamma Cl(B \cup D))$. Moreover, we have $p_\gamma Int(A) \subseteq p_\gamma Int(p_\gamma Cl(B \cup A))$, for $p_\gamma Int(A) = A \subseteq p_\gamma Cl(A) \subseteq p_\gamma Cl(B) \cup p_\gamma Cl(A) = p_\gamma Cl(B \cup A)$. Since A is a P_γ-open set, we have $D \subseteq A = p_\gamma Int(A) \subseteq p_\gamma Int(p_\gamma Cl(B \cup A)) = p_\gamma Int(p_\gamma Cl(B \cup D))$. Therefore $B \cup D \subseteq p_\gamma Int(p_\gamma Cl(B \cup D))$ implies $B \cup D$ is a pre P_γ-open set. ∎

Corollary 3.17. *Let A be a minimal P_γ-open set and B be a nonempty subset of X. If there exists a P_γ-open set C containing B such that $C \subseteq p_\gamma Cl(A)$, then $B \cup D$ is a pre P_γ-open set for any nonempty subset D of A, where γ is pre regular.*

Proof. By assumption, we have $C \subseteq p_\gamma Cl(B) \cup p_\gamma Cl(A) = p_\gamma Cl(B \cup A)$. By Theorem 3.16, we see that $B \cup D$ is a pre P_γ-open set. ∎

4. Finite P_γ-open sets

In this section, we study some properties of minimal P_γ-open sets in finite P_γ-open sets and P_γ-locally finite spaces.

Proposition 4.1. *Let X be a space and $\phi \neq B$ be a finite P_γ-open set in X. Then there exists at least one (finite) minimal P_γ-open set A such that $A \subseteq B$.*

Proof. Suppose that B is a finite P_γ-open set in X. Then we have the following two possibilities:

(1) B is a minimal P_γ-open set.
(2) B is not a minimal P_γ-open set.

In case (1), if we choose $B = A$, then the proposition is proved.

If the case (2) is true, then there exists a nonempty (finite) P_γ-open set B_1 which is properly contained in B. If B_1 is minimal P_γ-open, we take $A = B_1$. If B_1 is not a minimal P_γ-open set, then there exists a nonempty (finite) P_γ-open set B_2 such that $B_2 \subseteq B_1 \subseteq B$. We continue this process and have a sequence of P_γ-open sets $\cdots \subseteq B_m \subseteq \cdots \subseteq B_2 \subseteq B_1 \subseteq B$. Since B is finite, this process will end in a finite number of steps. That is, for some natural number k, we have a minimal P_γ-open set B_k such that $B_k = A$. This completes the proof. ∎

Definition 4.2. A space X is said to be a P_γ-locally finite space, if for each $x \in X$ there exists a finite P_γ-open set A in X such that $x \in A$.

Corollary 4.3. *Let X be a P_γ-locally finite space and B be a nonempty P_γ-open set. Then there exists at least one (finite) minimal P_γ-open set A such that $A \subseteq B$, where γ is pre regular.*

Proof. Since B is a nonempty set, there exists an element x of B. Since X is a P_γ-locally finite space, we have a finite P_γ-open set B_x such that $x \in B_x$. Since $B \cap B_x$ is a finite P_γ-open set, we get a minimal P_γ-open set A such that $A \subseteq B \cap B_x \subseteq B$ by Proposition 4.1. ∎

Proposition 4.4. *Let X be a space and for any $\alpha \in I$, B_α be a P_γ-open set and $\phi \neq A$ be a finite P_γ-open set. Then $A \cap (\bigcap_{\alpha\in I} B_\alpha)$ is a finite P_γ-open set, where γ is pre regular.*

Proof. We see that there exists an integer n such that $A \cap (\bigcap_{\alpha\in I} B_\alpha) = A \cap (\bigcap_{i=1}^{n} B_{\alpha_i})$ and hence we have the result. ∎

Using Proposition 4.4, we can prove the following:

Theorem 4.5. *Let X be a space and for any $\alpha \in I$, B_α be a P_γ-open set and for any $\beta \in J$, A_β be a nonempty finite P_γ-open set. Then $(\bigcup_{\beta\in J} A_\beta) \cap (\bigcap_{\alpha\in I} B_\alpha)$ is a P_γ-open set, where γ is pre regular.*

5. Applications

Let A be a nonempty finite P_γ-open set. It is clear, by Proposition 3.3 and Proposition 4.1, that if γ is pre regular, then there exists a natural number m such that $\{A_1, A_2, \ldots, A_m\}$ is the class of all minimal P_γ-open sets in A satisfying the following two conditions:

(1) For any l, n with $1 \leq l,\ n \leq m$ and $l \neq n$, $A_l \cap A_n = \phi$.
(2) If C is a minimal P_γ-open set in A, then there exists l with $1 \leq l \leq m$ such that $C = A_l$.

Theorem 5.1. *Let X be a space and $\phi \neq A$ be a finite P_γ-open set such that A is not a minimal P_γ-open set. Let $\{A_1, A_2, \ldots, A_m\}$ be a class of all minimal P_γ-open sets in A and $y \in A \backslash (A_1 \cup A_2 \cup \cdots \cup A_m)$. Define $A_y = \cap\{B : B$ is a P_γ-open neighborhood of $y\}$. Then there exists a natural number $k \in \{1, 2, \ldots, m\}$ such that A_k is contained in A_y, where γ is pre regular.*

Proof. Suppose on the contrary that for any natural number $k \in \{1, 2, \ldots, m\}$, A_k is not contained in A_y. By Corollary 3.8, for any minimal P_γ-open set A_k in A, $A_k \cap A_y = \phi$. By Proposition 4.4, $\phi \neq A_y$ is a finite P_γ-open set. Therefore by Proposition 4.1, there exists a minimal P_γ-open set C such that $C \subseteq A_y$. Since $C \subseteq A_y \subseteq A$, we have C is a minimal P_γ-open set in A. By supposition, for any minimal P_γ-open set A_k, we have $A_k \cap C \subseteq A_k \cap A_y = \phi$. Therefore for any natural number $k \in \{1, 2, \ldots, m\}$, $C \neq A_k$. This contradicts our assumption. Hence the proof. ∎

Proposition 5.2. *Let X be a space and $\phi \neq A$ be a finite P_γ-open set which is not a minimal P_γ-open set. Let $\{A_1, A_2, \ldots, A_m\}$ be a class of all minimal P_γ-open sets in A and $y \in A \backslash (A_1 \cup A_2 \cup \cdots \cup A_m)$. Then there exists a natural number $k \in \{1, 2, \ldots, m\}$ such that for any P_γ-open neighborhood B_y of y, A_k is contained in B_y, where γ is pre regular.*

Proof. This follows from Theorem 5.1, as $\cap\{B :$ B is a P_γ-open of $y\} \subseteq B_y$. Hence the proof. ∎

Theorem 5.3. *Let X be a space and $\phi \neq A$ be a finite P_γ-open set which is not a minimal P_γ-open set. Let $\{A_1, A_2, \ldots, A_m\}$ be the class of all minimal P_γ-open sets in A and $y \in A \backslash (A_1 \cup A_2 \cup \cdots \cup A_m)$. Then there exists a natural number $k \in \{1, 2, \ldots, m\}$ such that $y \in p_\gamma Cl(A_k)$, where γ is pre regular.*

Proof. It follows from Proposition 5.2, that there exists a natural number $k \in \{1, 2, \ldots, m\}$ such that $A_k \subseteq B$ for any P_γ-open neighborhood B of y. Therefore $\phi \neq A_k \cap A_k \subseteq A_k \cap B$ implies $y \in p_\gamma Cl(A_k)$. This completes the proof. ∎

Proposition 5.4. *Let $\phi \neq A$ be a finite P_γ-open set in a space X and suppose that for each $k \in \{1, 2, \ldots, m\}$, A_k is a minimal P_γ-open set in A. If the class $\{A_1, A_2, \ldots, A_m\}$ contains all minimal P_γ-open sets in A, then for any $\phi \neq B_k \subseteq A_k$, $A \subseteq p_\gamma Cl(B_1 \cup B_2 \cup \cdots \cup B_m)$, where γ is pre regular.*

Proof. If A is a minimal P_γ-open set, then this is the result of Theorem 3.12(2). Otherwise A is not a minimal P_γ-open set. If x is any element of $A \backslash (A_1 \cup A_2 \cup \cdots \cup A_m)$, we have $x \in p_\gamma Cl(A_1) \cup p_\gamma Cl(A_2) \cup \cdots \cup p_\gamma Cl(A_m)$

by Theorem 5.3. Therefore,

$$\begin{aligned} A &\subseteq p_\gamma Cl(A_1) \cup p_\gamma Cl(A_2) \cup \cdots \cup p_\gamma Cl(A_m) \\ &= p_\gamma Cl(B_1) \cup p_\gamma Cl(B_2) \cup \cdots \cup p_\gamma Cl(B_m) \\ &= p_\gamma Cl(B_1 \cup B_2 \cup \cdots \cup B_m) \end{aligned}$$

by Theorem 3.12(3). ■

Proposition 5.5. *Let $\phi \neq A$ be a finite P_γ-open set and A_k be a minimal P_γ-open set in A, for each $k \in \{1, 2, \ldots, m\}$. If for any $\phi \neq B_k \subseteq A_k$, $A \subseteq p_\gamma Cl(B_1 \cup B_2 \cup \cdots \cup B_m)$ then $p_\gamma Cl(A) = p_\gamma Cl(B_1 \cup B_2 \cup \cdots \cup B_m)$.*

Proof. For any $\phi \neq B_k \subseteq A_k$ with $k \in \{1, 2, \ldots, m\}$, we have $p_\gamma Cl(B_1 \cup B_2 \cup \cdots \cup B_m) \subseteq p_\gamma Cl(A)$. Also, we have

$$p_\gamma Cl(A) \subseteq p_\gamma Cl(p_\gamma Cl(B_1 \cup B_2 \cup \cdots \cup B_m)) = p_\gamma Cl(B_1 \cup B_2 \cup \cdots \cup B_m).$$

Therefore, we have $p_\gamma Cl(A) = p_\gamma Cl(B_1 \cup B_2 \cup \cdots \cup B_m)$ for any nonempty subset B_k of A_k with $k \in \{1, 2, \ldots, m\}$. ■

Proposition 5.6. *Let $\phi \neq A$ be a finite P_γ-open set and for each $k \in \{1, 2, \ldots, m\}$, A_k is a minimal P_γ-open set in A. If for any $\phi \neq B_k \subseteq A_k$, $p_\gamma Cl(A) = p_\gamma Cl(B_1 \cup B_2 \cup \cdots \cup B_m)$, then the class $\{A_1, A_2, \ldots, A_m\}$ contains all minimal P_γ-open sets in A.*

Proof. Suppose that C is a minimal P_γ-open set in A and $C \neq A_k$ for $k \in \{1, 2, \ldots, m\}$. Then we have $C \cap p_\gamma Cl(A_k) = \phi$ for each $k \in \{1, 2, \ldots, m\}$. It follows that any element of C is not contained in $p_\gamma Cl(A_1 \cup A_2 \cup \cdots \cup A_m)$. This is a contradiction to the fact that

$$C \subseteq A \subseteq p_\gamma Cl(A) = p_\gamma Cl(B_1 \cup B_2 \cup \cdots \cup B_m).$$

This completes the proof. ■

Combining Proposition 5.4, 5.5 and 5.6, we have the following theorem:

Theorem 5.7. *Let A be a nonempty finite P_γ-open set and A_k be a minimal P_γ-open set in A for each $k \in \{1, 2, \ldots, m\}$. Then the following three conditions are equivalent:*

(1) *The class $\{A_1, A_2, \ldots, A_m\}$ contains all minimal P_γ-open sets in A.*
(2) *For any $\phi \neq B_k \subseteq A_k$, $A \subseteq p_\gamma Cl(B_1 \cup B_2 \cup \cdots \cup B_m)$.*
(3) *For any $\phi \neq B_k \subseteq A_k$, $p_\gamma Cl(A) = p_\gamma Cl(B_1 \cup B_2 \cup \cdots \cup B_m)$, where γ is pre regular.*

Suppose that $\phi \neq A$ is a finite P_γ-open set and $\{A_1, A_2, \ldots, A_m\}$ is a class of all minimal P_γ-open sets in A such that for each $k \in \{1, 2, \ldots, m\}$, $y_k \in A_k$. Then by Theorem 5.7, it is clear that $\{y_1, y_2, \ldots, y_m\}$ is a pre P_γ-open set.

Theorem 5.8. *Let A be a nonempty finite P_γ-open set and suppose that $\{A_1, A_2, \ldots, A_m\}$ is a class of all minimal P_γ-open sets in A. Let B be any subset of $A \backslash (A_1 \cup A_2 \cup \cdots \cup A_m)$ and B_k be any nonempty subset of A_k for each $k \in \{1, 2, \ldots, m\}$. Then $B \cup B_1 \cup B_2 \cup \cdots \cup B_m$ is a pre P_γ-open set.*

Proof. By Theorem 5.7, we have

$$A \subseteq p_\gamma Cl(B_1 \cup B_2 \cup \cdots \cup B_m) \subseteq p_\gamma Cl(B \cup B_1 \cup B_2 \cup \cdots \cup B_m).$$

Since A is a P_γ-open set, we have

$$\begin{aligned} B \cup B_1 \cup B_2 \cup \cdots \cup B_m &\subseteq A \\ &= p_\gamma Int(A) \subseteq p_\gamma Int(p_\gamma Cl(B \cup B_1 \cup B_2 \cup \cdots \cup B_m)). \end{aligned}$$

Hence, we have the result. ∎

Theorem 5.9. *Let X be a P_γ-locally finite space. If a minimal P_γ-open set $A \subseteq X$ has more than one element, then X is a pre P_γ-Hausdorff space, where γ is pre regular.*

Proof. Let $x, y \in X$ such that $x \neq y$. Since X is a P_γ-locally finite space, there exist finite P_γ-open sets U and V such that $x \in U$ and $y \in V$. By Proposition 4.1, there exists the class $\{U_1, U_2, \ldots, U_n\}$ of all minimal P_γ-open sets in U and the class $\{V_1, V_2, \ldots, V_m\}$ of all minimal P_γ-open sets in V. We consider three possibilities:

(1) If there exists i of $\{1, 2, \ldots, n\}$ and j of $\{1, 2, \ldots, m\}$ such that $x \in U_i$ and $y \in V_j$, then by Theorem 3.15, $\{x\}$ and $\{y\}$ are disjoint pre P_γ-open sets which contains x and y, respectively.

(2) If there exists i of $\{1, 2, \ldots, n\}$ such that $x \in U_i$ and $y \notin V_j$ for any j of $\{1, 2, \ldots, m\}$, then we find an element y_j of V_j for each j such that $\{x\}$ and $\{y, y_1, y_2, \ldots, y_m\}$ are pre P_γ-open sets and $\{x\} \cap \{y, y_1, y_2, \ldots, y_n\} = \phi$ by Theorems 3.15, 5.8 and the assumption.

(3) If $x \notin U_i$ for any i of $\{1, 2, \ldots, n\}$ and $y \notin V_j$ for any j of $\{1, 2, \ldots, m\}$, then we find elements x_i of U_i and y_j of V_j for each i, j such that $\{x, x_1, x_2, \ldots, x_n\}$ and $\{y, y_1, y_2, \ldots, y_m\}$ are pre P_γ-open sets and $\{x, x_1, x_2, \ldots, x_n\} \cap \{y, y_1, y_2, \ldots, y_m\} = \phi$ by Theorem 5.8 and the assumption. Hence X is a pre P_γ-Hausdorff space. ∎

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