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COMMON FIXED POINT OF FOUR MAPPINGS SATISFYING IMPLICIT GENERALIZED WEAK CONTRACTIVE TYPE CONDITION

Abstract. In this paper, common fixed point for four maps using an implicit contractive condition in a complete metric space is proved. Some periodic point results for such mappings are also obtained. These results extend and generalize several comparable results in the current literature.

1. Introduction and preliminaries

Alber and Guerre-Delabriere [3] defined weakly contractive maps on a Hilbert space and established a fixed point theorem for such maps. Afterwards, Rhoades [17] using the notion of weakly contractive maps, obtained a fixed point theorem in a complete metric space. Dutta *et al.* [9] generalized the weak contractive condition and proved a fixed point theorem for a self-map, which in turn generalizes Theorem 1 in [17] and the corresponding result in [3]. The study of common fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity. The area of common fixed point theory, involving four single valued maps, began with the assumption that all of the maps commuted. Introducing weakly commuting maps, Sessa [19] generalized the concept of commuting maps. Then Jungck generalized this idea, first to compatible mappings [12] and then to weakly compatible mappings [13]. After then, many fixed point results have been obtained using weakly compatible mappings on ordinary metric spaces, (see [2, 6, 7], [21]). On the other hand, Beg and Abbas [5] obtained a common fixed point theorem extending weak contractive condition for two maps. In this direction, Zhang and Song [22] introduced the concept of a generalized φ-weak contraction condition and obtained a com-

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mon fixed point for two maps. Recently, Doric [8] proved a common fixed point theorem for generalized (ψ, φ)-weak contractions. The purpose of this paper is to obtain a common fixed point theorem for four maps that satisfy contractive condition using implicit condition. Our result extend, unify and generalize the comparable results in ([1, 5, 8, 9] and [22]).

For the sake of convenience, we gather some basic definitions and set out our terminology needed in the sequel.

Definition 1.1. Let $f, g : X \to X$. A point $x \in X$ is called *fixed point of* f if $f(x) = x$; *coincidence point of a pair* (f, g) if $fx = gx$; *common fixed point of a pair* (f, g) if $x = fx = gx$.

$F(f), C(f, g)$ and $F(f, g)$ denote set of all fixed points of f, set of all coincidence points of the pair (f, g) and the set of all common fixed points of the pair (f, g), respectively.

We give here only the definition of a weakly compatible map.

Definition 1.2. Let f and g be mappings from a metric space (X, d) into itself. f and g are said to be weakly compatible if they commute at their coincidence points, that is, $fx = gx$ for some $x \in X$ implies that $fgx = gfx$.

Definition 1.3. Define $\Phi = \{\varphi : [0, \infty) \to [0, \infty) : \varphi$ is lower semi continuous, $\varphi(t) > 0$ for all $t > 0$, $\varphi(0) = 0\}$ and $\Psi = \{\psi : [0, \infty) \to [0, \infty) :$ ψ is continuous and nondecreasing mapping with $\psi(t) = 0$ if and only if $t = 0\}$.

For other related definitions, results and their applications, we refer to [4], [11], [16], [20], and references mentioned therein.

2. Main results

In what follows, $\mathbb{N}$ is the set of all natural numbers and $\mathbb{R}^+$ is the set of all nonnegative real numbers.

A class of implicit relation. Let Θ be the set of all continuous functions $\theta : {\mathbb{R}^+}^5 \to \mathbb{R}^+$, increasing in any coordinate and

$\theta(s, s, s, \alpha s, \beta s) = s$ for every $\alpha, \beta \geq 0$ such that $\alpha + \beta = 2$,

$\theta(x, y, z, u, v) > 0$ if at least one of the x, y, z, u and v is non zero.

Example 2.1. Let $\theta : {\mathbb{R}^+}^5 \to \mathbb{R}^+$ is define by

(i) $\theta(x, y, z, u, v) = \max\{x, y, z, \frac{u + v}{2}\}$.

(ii) $\theta(x, y, z, u, v) = \frac{x + y + z + u + v}{5}$.

Theorem 2.2. *Let (X, d) be a complete metric space. If f, g, S and T are self maps of X satisfying:*

(i) *$f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$ and one of the ranges $f(X)$, $g(X)$, $T(X)$ and $S(X)$ is closed,*

(ii) *The pairs (f, S) and (g, T) are weakly compatible,*

(iii)

$$\psi(d(fx, gy)) \leq \psi(M(x, y)) - \varphi(M(x, y)), \tag{2.1}$$

for each $x, y \in X$, $\varphi \in \Phi$, $\psi \in \Psi$ holds, where

$$M(x, y) = \theta \begin{pmatrix} d(Sx, Ty), \ d(fx, Sx), \\ d(gy, Ty), \ d(Sx, gy), d(fx, Ty) \end{pmatrix},$$

for every $\theta \in \Theta$. Then, f, g, S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Choose a point $x_1 \in X$ such that $y_0 = fx_0 = Tx_1$. This can be done, since the range of T contains the range of f. Similarly, a point $x_2 \in X$ can be chosen such that $y_1 = gx_1 = Sx_2$ as $g(X) \subseteq S(X)$. Continuing this process, we obtain a sequence $\{y_n\}$ in X such that $y_{2n} = fx_{2n} = Tx_{2n+1}$ and $y_{2n+1} = gx_{2n+1} = Sx_{2n+2}$.

First, we show that $\{y_n\}$ is a Cauchy sequence in X. Consider two cases.

1. If for some n, $y_n = y_{n+1}$, then $y_{n+1} = y_{n+2}$. If not, then for $n = 2m$, where $m \in \mathbb{N}$, we have

$$\begin{aligned} M(x_{2m+2}, x_{2m+1}) & \\ &= \theta \begin{pmatrix} d(Sx_{2m+2}, Tx_{2m+1}), \ d(fx_{2m+2}, Sx_{2m+2}), \\ d(gx_{2m+1}, Tx_{2m+1}), \ d(Sx_{2m+2}, gx_{2m+1}), d(fx_{2m+2}, Tx_{2m+1}) \end{pmatrix} \\ &= \theta \begin{pmatrix} d(y_{2m+1}, y_{2m}), \ d(y_{2m+2}, y_{2m+1}), \\ d(y_{2m+1}, y_{2m}), \ d(y_{2m+1}, y_{2m+1}), d(y_{2m+2}, y_{2m}) \end{pmatrix} \\ &= \theta \begin{pmatrix} 0, \ d(y_{2m+2}, y_{2m+1}), \\ 0, \ 0, d(y_{2m+2}, y_{2m}) \end{pmatrix}. \end{aligned}$$

Since $d(y_{2m}, y_{2m+2}) \leq d(y_{2m}, y_{2m+1}) + d(y_{2m+1}, y_{2m+2}) = d(y_{2m+1}, y_{2m+2})$. By property of θ and using the above inequality we obtain that

$$\begin{aligned} M(x_{2m+2}, x_{2m+1}) &\leq \theta \begin{pmatrix} d(y_{2m+2}, y_{2m+1}), \ d(y_{2m+2}, y_{2m+1}), \\ d(y_{2m+2}, y_{2m+1}), \ d(y_{2m+2}, y_{2m+1}), d(y_{2m+2}, y_{2m+1}) \end{pmatrix} \\ &= d(y_{2m+2}, y_{2m+1}). \end{aligned}$$

Now (2.1) gives

$$\begin{aligned}\psi(d(y_{2m+2}, y_{2m+1})) &= \psi(d(fx_{2m+2}, gx_{2m+1}))\\ &\le \psi(M(x_{2m+2}, x_{2m+1})) - \varphi(M(x_{2m+2}, x_{2m+1}))\\ &\le \psi(d(y_{2m+2}, y_{2m+1})) - \varphi(M(x_{2m+2}, x_{2m+1}))\\ &< \psi(d(y_{2m+2}, y_{2m+1})),\end{aligned}$$

which is a contradiction. Hence we must have $y_{n+1} = y_{n+2}$, when n is even. Following the similar arguments to those given above, it is noted that this equality holds in case n is odd. Therefore, in any case for all those n for which $y_n = y_{n+1}$ holds, we always obtain $y_{n+1} = y_{n+2}$. Repeating above process inductively, one obtains $y_n = y_{n+k}$, for all $k \ge 1$. Therefore, in this case $\{y_n\}$ turns out to be eventually a constant sequence and hence a Cauchy one.

2. If $y_n \ne y_{n+1}$, for every positive integer n, then for $n = 2m + 1$, for some $m \in \mathbb{N}$,

$$\begin{aligned}&M(x_{2m+2}, x_{2m+1})\\ &\quad= \theta\begin{pmatrix} d(Sx_{2m+2}, Tx_{2m+1}),\ d(fx_{2m+2}, Sx_{2m+2}),\\ d(gx_{2m+1}, Tx_{2m+1}),\ d(Sx_{2m+2}, gx_{2m+1}), d(fx_{2m+2}, Tx_{2m+1})\end{pmatrix}\\ &\quad= \theta\begin{pmatrix} d(y_{2m+1}, y_{2m}),\ d(y_{2m+2}, y_{2m+1}),\\ d(y_{2m+1}, y_{2m}),\ 0, d(y_{2m+2}, y_{2m})\end{pmatrix}.\end{aligned}$$

If $d(y_{2m}, y_{2m+1}) \le d(y_{2m+1}, y_{2m+2})$, by property θ and using the above inequality we get

$$\begin{aligned}&M(x_{2m+2}, x_{2m+1})\\ &\qquad\le \theta\begin{pmatrix} d(y_{2m+1}, y_{2m+2}),\ d(y_{2m+1}, y_{2m+2}),\\ d(y_{2m+1}, y_{2m+2}),\ 0, d(y_{2m}, y_{2m+1}) + d(y_{2m+1}, y_{2m+2})\end{pmatrix}\\ &\qquad\le \theta\begin{pmatrix} d(y_{2m+1}, y_{2m+2}),\ d(y_{2m+1}, y_{2m+2}),\\ d(y_{2m+1}, y_{2m+2}),\ 0, 2d(y_{2m+1}, y_{2m+2})\end{pmatrix}\\ &\qquad= d(y_{2m+1}, y_{2m+2}).\end{aligned}$$

From (2.1), we obtain

$$\begin{aligned}\psi(d(y_{2m+2}, y_{2m+1})) &= \psi(d(fx_{2m+2}, gx_{2m+1}))\\ &\le \psi(M(x_{2m+2}, x_{2m+1})) - \varphi(M(x_{2m+2}, x_{2m+1}))\\ &\le \psi(d(y_{2m+2}, y_{2m+1})) - \varphi(M(x_{2m+2}, x_{2m+1}))\\ &< \psi(d(y_{2m+2}, y_{2m+1})),\end{aligned}$$

which gives a contradiction. Thus $d(y_{2m}, y_{2m+1}) > d(y_{2m+1}, y_{2m+2})$, by property θ we get

$$M(x_{2m+2}, x_{2m+1}) \leq \theta \begin{pmatrix} d(y_{2m+1}, y_{2m+2}),\ d(y_{2m+1}, y_{2m+2}), \\ d(y_{2m+1}, y_{2m+2}),\ 0, d(y_{2m}, y_{2m+1}) + d(y_{2m+1}, y_{2m+2}) \end{pmatrix} \leq \theta \begin{pmatrix} d(y_{2m}, y_{2m+1}),\ d(y_{2m}, y_{2m+1}), \\ d(y_{2m}, y_{2m+1}),\ 0, 2d(y_{2m}, y_{2m+1}) \end{pmatrix} = d(y_{2m}, y_{2m+1}). \tag{2.2}$$

Therefore

$$M(x_{2m+2}, x_{2m+1}) \leq d(y_{2m+1}, y_{2m}).$$

Hence from (2.1), we get

$$\begin{aligned} \psi(d(y_{n+1}, y_n)) &= \psi(d(fx_{n+1}, gx_n)) \\ &\leq \psi(M(x_{n+1}, x_n)) - \varphi(M(x_{n+1}, x_n)) \\ &\leq \psi(M(x_{2m+2}, x_{2m+1})) - \varphi(M(x_{2m+2}, x_{2m})) \\ &\leq \psi(d(y_{2m+1}, y_{2m})) - \varphi(M(x_{2m+2}, x_{2m+1})) \\ &< \psi(d(y_n, y_{n-1})). \end{aligned}$$

Following the similar arguments to those given above, we conclude the same inequality when n is taken as even integer. Consequently, we have

$$\psi(d(y_{n+1}, y_n)) < \psi(d(y_n, y_{n-1})), \quad \text{for all } n \geq 1,$$

which further implies that $d(y_{n+1}, y_n) < d(y_n, y_{n-1})$. Therefore, $\{d(y_{n+1}, y_n)\}$ is a strictly decreasing sequence which is bounded below by 0. Therefore there exists $r \geq 0$ such that $d(y_{n+1}, y_n) \to r$ as $n \to \infty$. From (2.2), for $x = x_{n+1}$ and $y = x_n$ for $n = 2m + 1$, we obtain

$$M(x_{n+1}, x_n) = M(x_{2m+2}, x_{2m+1}) \leq \theta \begin{pmatrix} d(y_{2m+1}, y_{2m}),\ d(y_{2m+2}, y_{2m+1}), \\ d(y_{2m+1}, y_{2m}),\ 0, 2d(y_{2m+1}, y_{2m}) \end{pmatrix}.$$

Therefore,

$$\lim_{n\to\infty} M(x_{n+1}, x_n) \leq \theta \begin{pmatrix} r,\ r, \\ r,\ 0, 2r \end{pmatrix} = r.$$

Thus

$$\lim_{n\to\infty} \psi(M(x_{n+1}, x_n)) \leq \psi(r). \tag{2.3}$$

Now (2.1), (2.3) and lower semicontinuity of φ give

$$\limsup_{n\to\infty} \psi(d(y_{n+1}, y_n)) \le \limsup_{n\to\infty} \psi(M(x_{n+1}, x_n)) - \liminf_{n\to\infty} \varphi(M(x_{n+1}, x_n))$$

and $\psi(r) \le \psi(r) - \varphi(r)$. Therefore $r = 0$, and

$$\lim_{n\to\infty} d(y_{n+1}, y_n) = 0. \tag{2.4}$$

Because of (2.4), to show $\{y_n\}_{n\ge1}$ to be a Cauchy sequence in X, it is sufficient to show that $\{y_{2n}\}_{n\ge1}$ is Cauchy in X. If not, there is an $\epsilon > 0$, and there exists even integers $2m(k)$ and $2n(k)$ with $2m(k) > 2n(k) > k$ such that

$$d(y_{2n(k)}, y_{2m(k)}) \ge \epsilon \quad \text{and} \quad d(y_{2n(k)}, y_{2m(k)-2}) < \epsilon.$$

Now (2.4) and inequality

$$\begin{aligned}\epsilon &\le d(y_{2n(k)}, y_{2m(k)}) \\ &\le d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-1}, y_{2m(k)-2}) + d(y_{2m(k)-1}, y_{2m(k)})\end{aligned}$$

implies that

$$\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}) = \epsilon. \tag{2.5}$$

Also, (2.4) and inequality

$$d(y_{2m(k)}, y_{2n(k)}) \le d(y_{2m(k)}, y_{2m(k)+1}) + d(y_{2m(k)+1}, y_{2n(k)})$$

gives that $\epsilon \le \lim_{k\to\infty} d(y_{2m(k)+1}, y_{2n(k)})$, while (2.5) and inequality

$$d(y_{2m(k)+1}, y_{2n(k)}) \le d(y_{2m(k)+1}, y_{2m(k)}) + d(y_{2m(k)}, y_{2n(k)})$$

yields $\lim_{k\to\infty} d(y_{2m(k)+1}, y_{2n(k)}) \le \epsilon$. Hence

$$\lim_{k\to\infty} d(y_{2m(k)+1}, y_{2n(k)}) = \epsilon. \tag{2.6}$$

Similarly, we obtain

$$\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)-1}) = \lim_{k\to\infty} d(y_{2n(k)-1}, y_{2m(k)+1}) = \epsilon. \tag{2.7}$$

Now from definition of M and from (2.4), (2.5), (2.6), (2.7), replacing x, y by $x_{2n(k)}, x_{2m(k)+1}$, we have

$$\begin{aligned}&M(x_{2n(k)}, x_{2m(k)+1}) \\ &= \theta\begin{pmatrix} d(Sx_{2n(k)}, Tx_{2m(k)+1}), \quad d(fx_{2n(k)}, Sx_{2n(k)}), \\ d(gx_{2m(k)+1}, Tx_{2m(k)+1}),\ d(Sx_{2n(k)}, gx_{2m(k)+1}), d(fx_{2n(k)}, Tx_{2m(k)+1})\end{pmatrix} \\ &= \theta\begin{pmatrix} d(y_{2n(k)-1}, y_{2m(k)}),\ \ d(y_{2n(k)}, y_{2n(k)-1}), \\ d(y_{2m(k)+1}, y_{2m(k)}),\ d(y_{2n(k)-1}, y_{2m(k)+1}), d(y_{2n(k)}, y_{2m(k)})\end{pmatrix}.\end{aligned}$$

Thus

$$\lim_{k\to\infty} M(x_{2n(k)}, x_{2m(k)+1}) \leq \theta\begin{pmatrix} \epsilon, \ \epsilon, \\ \epsilon, \ \epsilon, \epsilon \end{pmatrix} = \epsilon.$$

Putting $x = x_{2n(k)}$ and $y = x_{2m(k)+1}$ in (2.1), we obtain

$$\begin{aligned}\psi(d(y_{2n(k)}, y_{2m(k)+1})) &= \psi(d(fx_{2n(k)}, gx_{2m(k)+1})) \\ &\leq \psi(M(x_{2n(k)}, x_{2m(k)+1})) - \varphi(M(x_{2n(k)}, x_{2m(k)+1})),\end{aligned}$$

which, on taking limit as $k \to \infty$, implies that

$$\psi(\epsilon) \leq \psi(\epsilon) - \varphi(\epsilon). \tag{2.8}$$

As (2.8) gives a contradiction when $\epsilon > 0$, it follows that $\{y_{2n}\}_{n\geq 1}$ is a Cauchy sequence in X. Since X is complete, there exists a point $z \in X$ such that $\lim_{n\to\infty} y_n = z$. The second step of the proof is to show that z is the fixed point for maps f and S. It is clear that

$$\begin{aligned}\lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} y_{2n+1} &= \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} gx_{2n+1} \\ &= \lim_{n\to\infty} Sx_{2n+2} = \lim_{n\to\infty} Rx_{2n+1} = z.\end{aligned}$$

Assuming $S(X)$ is closed, there exists $u \in X$ such that $z = Su$. We claim that $fu = z$. If not then

$$\begin{aligned}M(u, x_{2n+1}) &= \theta\begin{pmatrix} d(Su, Tx_{2n+1}), & d(fu, Su), \\ d(gx_{2n+1}, Tx_{2n+1}), & d(Su, gx_{2n+1}), d(fu, Tx_{2n+1}) \end{pmatrix} \\ &= \theta\begin{pmatrix} d(z, Tx_{2n+1}), & d(fu, z), \\ d(gx_{2n+1}, Tx_{2n+1}), & d(z, gx_{2n+1}), d(fu, Tx_{2n+1}) \end{pmatrix}.\end{aligned}$$

As $n \to \infty$ we have

$$\begin{aligned}\lim_{n\to\infty} M(u, x_{2n+1}) &= \theta\begin{pmatrix} d(z,z), \ d(fu,z), \\ d(z,z), \ d(z,z), d(fu,z) \end{pmatrix} \\ &\leq \theta\begin{pmatrix} d(fu,z), \ d(fu,z), \\ d(fu,z), \ d(fu,z), d(fu,z) \end{pmatrix} \\ &= d(fu, z).\end{aligned}$$

Since $\lim_{n\to\infty} M(u, x_{2n+1}) \neq 0$, from (2.1) we obtain

$$\psi(d(fu, gx_{2n+1})) \leq \psi(M(u, x_{2n+1})) - \varphi(M(u, x_{2n+1})),$$

which, on taking limit as $n \to \infty$, gives

$$\psi(d(fu, z)) \leq \psi(d(fu, z)) - \varphi(\lim_{n\to\infty} M(u, x_{2n+1})),$$

a contradiction with $d(fu, z) > 0$. Hence $fu = z$. Therefore $fu = Su = z$. Since the maps f and S are weakly compatible, we have $fz = fSu = Sfu = Sz$.

Next we claim that $fz = z$. If not, then

$$M(z, x_{2n+1}) = \theta\begin{pmatrix} d(Sz, Tx_{2n+1}), & d(fz, Sz), \\ d(gx_{2n+1}, Tx_{2n+1}), & d(Sz, gx_{2n+1}), d(fz, Tx_{2n+1}) \end{pmatrix}$$
$$= \theta\begin{pmatrix} d(fz, Tx_{2n+1}), & d(fz, fz), \\ d(gx_{2n+1}, Tx_{2n+1}), & d(fz, gx_{2n+1}), d(fz, Tx_{2n+1}) \end{pmatrix}.$$

As $n \to \infty$ we have

$$\lim_{n\to\infty} M(z, x_{2n+1}) = \theta\begin{pmatrix} d(fz, z), & 0, \\ 0, & d(fz, z), d(fz, z) \end{pmatrix}$$
$$\leq \theta\begin{pmatrix} d(fz, z), & d(fz, z), \\ d(fz, z), & d(fz, z), d(fz, z) \end{pmatrix}$$
$$= d(fz, z).$$

Since $\lim_{n\to\infty} M(z, x_{2n+1}) \neq 0$, and again by (2.1)

$$\psi(d(fz, gx_{2n+1})) \leq \psi(M(z, x_{2n+1})) - \varphi(M(z, x_{2n+1})),$$

which, on taking limit as $n \to \infty$ gives the contradiction

$$\psi(d(fz, z)) \leq \psi(d(fz, z)) - \varphi(\lim_{n\to\infty} M(z, x_{2n+1})).$$

Therefore $fz = z$.

The next step is to show that z is also fixed point for maps g and T. Since $f(X) \subseteq T(X)$, there is some v in X such that $fz = Tv$. Then $fz = Tv = Sz = z$. We claim that $gv = z$. If $gv \neq z$ then from (2.1), we have

$$d(z, gv) = d(fz, gv) \leq \psi(M(z, v)) - \varphi(M(z, v)),$$

where

$$M(z, v) = \theta\begin{pmatrix} d(Sz, Tv), & d(fz, Sz), \\ d(gv, Tv), & d(Sz, gv), d(fz, Tv) \end{pmatrix}$$
$$\leq \theta\begin{pmatrix} d(gv, z), & d(gv, z), \\ d(gv, z), & d(gv, z), d(gv, z) \end{pmatrix}$$
$$= d(gv, z).$$

Thus

$$\psi(d(z, gv)) \leq \psi(d(z, gv)) - \varphi(M(z, v))$$

gives a contradiction. Therefore $z = gv$. Hence $gv = Tv = z$. By weak compatibility of mappings g and T, we obtain $gz = gTv = TTv = Tz$. Finally, we claim that $gz = z$. If $gz \neq z$ then (2.1) gives

$$\psi(d(z, gz)) = \psi(d(fz, gz)) \leq \psi(M(z, z)) - \varphi(M(z, z)),$$

where

$$
\begin{aligned}
M(z,z) &= \theta\begin{pmatrix} d(Sz,Tz),\ d(fz,Sz), \\ d(gz,Tz),\ d(Sz,gz), d(fz,Tz) \end{pmatrix} \\
&\le \theta\begin{pmatrix} d(z,gz),\ d(z,gz), \\ d(z,gz),\ d(z,gz), d(z,gz) \end{pmatrix} \\
&= d(z,gz).
\end{aligned}
$$

Therefore

$$\psi(d(z,gz)) \le \psi(d(z,gz)) - \varphi(M(z,z))$$

gives a contradiction. Hence, $fz = gz = Sz = Tz = z$. Similar analysis is valid for the case in which $T(X)$ is closed, as well as for the cases in which $f(X)$ or $g(X)$ is closed, since $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$.

As uniqueness of common fixed point z easily follows from the inequality (2.1), the proof is completed. ∎

Corollary 2.3. *Let (X,d) be a complete metric space. If h, f, g, S and T are self maps of X satisfying:*

(i) *h be one to one continuous mapping such that it commutes with f, g, S, T,*
(ii) *$hf(X) \subseteq hT(X)$, $hg(X) \subseteq hS(X)$ and one of the ranges $hf(X)$, $hg(X)$, $hT(X)$ and $hS(X)$ is closed,*
(iii) *the pairs (hf, hS) and (hg, hT) are weakly compatible,*
(iv)

(2.9) $$\psi(d(hfx,hgy)) \le \psi(M(x,y)) - \varphi(M(x,y)),$$

for each $x, y \in X$, $\varphi \in \Phi$, $\psi \in \Psi$ holds, where

$$M(x,y) = \theta\begin{pmatrix} d(hSx,hTy),\ d(hfx,hSx), \\ d(hgy,hTy),\ d(hSx,hgy), d(hfx,hTy) \end{pmatrix},$$

for every $\theta \in \Theta$. Then h, f, g, S and T have a unique common fixed point in X.

Proof. By Theorem 2.2, hf, hg, hS and hT have a unique common fixed point $z \in X$. Since h is one to one, from $hfz = hgz = hSz = hTz = z$, it follows that $fz = gz = Sz = Tz$. We claim that $fz = z$. If $fz \neq z$ then from $hffz = f(hfz) = fz$ and (2.9) gives

$$\psi(d(fz,z)) = \psi(d(hffz,hgz)) \le \psi(M(fz,z)) - \varphi(M(fz,z)),$$

where

$$M(fz,z) = \theta\begin{pmatrix} d(hSfz,hTz), & d(hffz,hSfz), \\ d(hgz,hTz), & d(hSfz,hgz), d(hffz,hTz) \end{pmatrix}$$
$$\le \theta\begin{pmatrix} d(Sz,z), & d(fz,Sz), \\ d(z,z), & d(Sz,z), d(fz,z) \end{pmatrix}$$
$$\le \theta\begin{pmatrix} d(fz,z), & d(fz,z), \\ d(fz,z), & d(fz,z), d(fz,z) \end{pmatrix}$$
$$= d(fz,z).$$

Therefore

$$\psi(d(fz,z)) \le \psi(d(fz,z)) - \varphi(M(fz,z))$$

gives a contradiction. Hence, $fz = gz = Sz = Tz = hz = z$. ∎

As a consequence of Theorem 2.2 we obtain the following result proved in [1].

Corollary 2.4. *Let (X,d) be a complete metric space. If f, g, S and T are self maps of X satisfying:*

(i) *$f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$ and one of the ranges $f(X)$, $g(X)$, $T(X)$ and $S(X)$ is closed,*
(ii) *the pairs (f,S) and (g,T) are weakly compatible,*
(iii)
$$\psi(d(fx,gy)) \le \psi(M(x,y)) - \varphi(M(x,y)),$$
for each $x, y \in X$, $\varphi \in \Phi$, $\psi \in \Psi$ holds where
$$M(x,y) = \max\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{1}{2}(d(Sx,gy) + d(fx,Ty))\}.$$

Then f, g, S and T have a unique common fixed point in X.

Proof. If we define $\theta(x,y,z,u,v) = \max\{x,y,z,\frac{u+v}{2}\}$ then all conditions of Theorem 2.2 hold, thus f, g, S and T have a unique common fixed point in X. ∎

Corollary 2.5. *Let (X,d) be a complete metric space. Let f and g be self maps of X such that one of the ranges $f(X)$, $g(X)$ is closed. If*

$$\psi(d(fx,gy)) \le \psi(M(x,y)) - \varphi(M(x,y)),$$

for each $x, y \in X$, $\varphi \in \Phi$, $\psi \in \Psi$, where

$$M(x,y) = \theta\begin{pmatrix} d(x,y), & d(fx,x), \\ d(gy,y), & d(x,gy), d(fx,y) \end{pmatrix},$$

for every $\theta \in \Theta$, then f and g have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and conversely.

Proof. Take S and T as identity maps on X. From Theorem 2.2, f and g have a unique common fixed point. Now suppose that z is a fixed point of f and $z \neq gz$. From (2.1), we have

$$\begin{aligned}\psi(d(z, gz)) &= \psi(d(fz, gz))\\ &\leq \psi((M(z, z) - \phi(M(z, z)),\end{aligned}$$

where

$$\begin{aligned}M(z, z) &= \theta\begin{pmatrix} d(z, z), & d(fz, z), \\ d(gz, z), & d(z, gz), d(fz, z)\end{pmatrix}\\ &\leq \theta\begin{pmatrix} d(z, gz), & d(z, gz), \\ d(gz, z), & d(z, gz), d(z, gz)\end{pmatrix}\\ &= d(z, gz).\end{aligned}$$

Therefore

$$\psi(d(z, gz)) \leq \psi(d(z, gz)) - \varphi(M(z, z)),$$

which is a contradiction by virtue of a property of ϕ. Hence $z = gz$. Using a similar argument, we have that any fixed point of g is also a fixed point of f. ∎

Corollary 2.6. *Let (X, d) be a complete metric space. Let f and g be self maps of X such that one of the ranges $f^m(X)$, $g^n(X)$ is closed. Let*

$$\psi(d(f^m x, g^n y)) \leq \psi(M(x, y)) - \varphi(M(x, y)), \tag{2.10}$$

for each $x, y \in X$, $\varphi \in \Phi$, $\psi \in \Psi$, where

$$M(x, y) = \theta\begin{pmatrix} d(x, y), & d(f^m x, x), \\ d(g^n y, y), & d(x, g^n y), d(f^m x, y)\end{pmatrix},$$

for every $\theta \in \Theta$ where m,n are fixed positive integers. Then f and g have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and conversely.

Proof. By Corollary 2.5, f^m and g^n have a unique common fixed point z. On the other hand, $f^m(fz) = f(f^m z) = fz$. By Corollary 2.5, fz is a fixed point of g^n. Hence $fz = z$. It then follows that z is a common fixed point of f and g. Suppose that z is a fixed point of f. Then z is a fixed point of f^m. By Corollary 2.5, z is a fixed point of g^n. From the uniqueness of the common fixed point of f^m and g^n, it follows that z is a fixed point of g. In a similar manner it can be shown that any fixed point of g is also a fixed point of f. Condition (2.10) implies the uniqueness of z. ∎

The following corollary extends the result of [22]

Corollary 2.7. *Let (X,d) be a complete metric space. If f and g are self mappings of X into itself such that one of the ranges $f(X)$, $g(X)$ is closed and*

$$d(fx,gy) \leq M(x,y) - \varphi(M(x,y)),$$

for each $x,y \in X$, $\varphi \in \Phi$, where

$$M(x,y) = \theta \begin{pmatrix} d(x,y), & d(fx,x), \\ d(gy,y), & d(x,gy), d(fx,y) \end{pmatrix},$$

for every $\theta \in \Theta$ then f and g have a unique common fixed point in X.

Proof. Take $\psi = I$, an identity map on $[0,\infty)$, and S and T as identity maps on X in Theorem 2.2. ∎

Corollary 2.8. *Let (X,d) be a complete metric space. If f is a self mapping of X into itself such that the range $f(X)$ is closed. If*

$$\psi(d(fx,fy)) \leq \psi(M(x,y)) - \varphi(M(x,y)),$$

for each $x,y \in X$, $\varphi \in \Phi$, $\psi \in \Psi$ and where

$$M(x,y) = \theta \begin{pmatrix} d(x,y), & d(fx,x), \\ d(fy,y), & d(x,fy), d(fx,y) \end{pmatrix},$$

for every $\theta \in \Theta$. Then, f has a unique fixed point in X.

Corollary 2.9. *Let (X,d) be a complete metric space, and $f : X \to X$ be a mapping such that the range $f(X)$ is closed. If for all $x,y \in X$,*

$$d(fx,fy) \leq M(x,y) - \varphi(M(x,y)),$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous function with $\varphi(t) > 0$ for $t \in (0,\infty)$, $\varphi(0) = 0$, and

$$M(x,y) = \max\{d(x,y), d(fx,x), d(fy,y), \frac{1}{2}(d(x,fy) + d(fx,y))\}.$$

Then f has a unique fixed point in X.

3. Mappings with properties P

It is an obvious fact that if T is a map which has a fixed point p, then p is also a fixed point of T^n for every natural number n. However the converse is false. For example, consider, $X = [0,1]$, and T defined by $Tx = 1 - x$. Then T has a unique fixed point at $\frac{1}{2}$, but every even iterate of T is the identity map, which has every point of $[0,1]$ as a fixed point. On the other hand, if $X = [0,\pi]$, $Tx = \cos x$, then every iterate of T has the same fixed point as T ([10], [15]) . Rhoades and Abbas [18] considered mappings satisfying a contractive condition of integral type for which fixed points and periodic points coincide.

Definition 3.1. (Property P [15]) Let T be a self-mapping of metric space with fixed point set $F(T) \neq \emptyset$. Then T is said to have property P if $F(T^n) = F(T)$, for each $n \in \mathbb{N}$. Equivalently, a mapping has property P if every periodic point is a fixed point.

Theorem 3.2. *Let (X, d) be a complete metric space and f be a self mapping of X into itself such that the range $f(X)$ is closed. If*

$$\psi(d(fx, fy)) \leq \psi(M(x, y)) - \varphi(M(x, y)),$$

for each $x, y \in X$, $\varphi \in \Phi$, $\psi \in \Psi$, where

$$M(x, y) = \theta \begin{pmatrix} d(x, y), & d(fx, x), \\ d(fy, y), & d(x, fy), d(fx, y) \end{pmatrix},$$

for every $\theta \in \Theta$, then f has property P.

Proof. From Corollary 2.8, f has a unique fixed point. Therefore, $F(f^m) \neq \emptyset$, for each positive integer $m \geq 1$. Fix a positive integer $n > 1$ and let $z \in F(f^n)$. Using (2.1) and the definition of P,

$$\psi(d(z, fz)) = \psi(d(f^n z, f^{n+1} z)) \leq \psi(M(f^{n-1} z, f^n z)) - \phi(M(f^{n-1} z, f^n z)),$$

where

$$\begin{aligned} M(f^{n-1} z, f^n z) &= \theta \begin{pmatrix} d(f^{n-1} z, f^n z), & d(f^n z, f^{n-1} z), \\ d(f^{n+1} z, f^n z), & d(f^{n-1} z, f^{n+1} z), d(f^n z, f^n z) \end{pmatrix} \\ &= \theta \begin{pmatrix} d(f^{n-1} z, z), & d(z, f^{n-1} z), \\ d(fz, z), & d(f^{n-1} z, fz), d(z, z) \end{pmatrix}. \end{aligned}$$

Either (1) $d(f^{n-1} z, z) > d(z, fz)$, or (2) $d(f^{n-1} z, z) \leq d(z, fz)$. Suppose that (1) is true. Then from above inequality, we have

$$\begin{aligned} \psi(d(z, fz)) &= \psi(d(f^n z, f^{n+1} z)) \\ &< \psi(d(f^{n-1} z, z)) = \psi(d(f^{n-1} z, f^n z)) \\ &\vdots \\ &< \psi(d(z, fz)), \end{aligned}$$

which is a contradiction. Therefore, (2) is true, and above inequality becomes

$$\psi(d(z, fz)) \leq \psi(d(z, fz)) - \phi(d(z, fz)),$$

which implies that $\phi(d(z, fz)) \leq 0$, or that $z = fz$. Therefore, f has property P. ∎

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