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WEAK CONVERGENCE FOR NONSELF NEARLY ASYMPTOTICALLY NONEXPANSIVE MAPPINGS BY ITERATIONS

Abstract. In this paper, we obtain a couple of weak convergence results for nonself nearly asymptotically nonexpansive mappings. Our first result is for the Banach spaces satisfying Opial condition and the second for those whose dual satisfies the Kadec-Klee property.

1. Introduction

Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers. Let E be a real Banach space and C a nonempty subset of E. Let $T : C \to C$ be a mapping then $F(T)$ denotes the set of fixed points of T, I denotes the identity mapping. A self mapping $T : C \to C$ is called asymptotically nonexpansive if for a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, we have $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in C$ and $n \in \mathbb{N}$. Also T is called uniformly L-Lipschitzian if for some $L > 0$, $\|T^n x - T^n y\| \leq L\|x - y\|$ for all $n \in \mathbb{N}$ and all $x, y \in C$.

Fix a sequence $\{a_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} a_n = 0$, then according to Agarwal et al. [1], T is said to be nearly Lipschitzian with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exist constants $k_n \geq 0$, such that $\|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n)$ for all $x, y \in C$. The infimum of constants k_n for which the above inequality holds is denoted by $\eta(T^n)$ and is called nearly Lipschitz constant.

A nearly Lipschitzian mapping T with sequence $\{a_n, \eta(T^n)\}$ is said to be nearly asymptotically nonexpansive if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T^n) = 1$, and nearly uniformly k-Lipschitzian if $\eta(T^n) \leq k$ for all $n \in \mathbb{N}$.

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A subset C of E is called a retract of E if there exists a continuous map $P : E \to C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach space is a retract. A map $P : E \to E$ is said to be a retraction if $P^2 = P$. It follows that if P is a retraction then $Py = y$ for all y in the range of P.

Chidume et al. [2] defined nonself asymptotically nonexpansive mappings as follows: Let $P : E \to C$ be a nonexpansive retraction of E into C. A nonself mapping $T : C \to E$ is called asymptotically nonexpansive if for a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, we have $\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x-y\|$ for all $x, y \in C$ and $n \in \mathbb{N}$. Also T is called uniformly k-Lipschitzian if for some $k > 0$, $\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k\|x - y\|$ for all $n \in \mathbb{N}$ and all $x, y \in C$.

In the light of above, we can define the following.

Fix a sequence $\{a_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} a_n = 0$, then a nonself mapping $T : C \to E$ is said to be nearly Lipschitzian with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exist constants $k_n \geq 0$, such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n(\|x - y\| + a_n),$$

for all $x, y \in C$. The infimum of constants k_n for which the above inequality holds, is denoted by $\eta(T(PT)^{n-1})$ and is called nearly Lipschitz constant.

For $n = 1$, the above definition gives $\|T(PT)^{1-1}x - T(PT)^{1-1}y\| \leq k_1(\|x - y\| + a_1)$, where we have to take a_1 as zero sequence. Thus in this case we have $\|T(PT)^{1-1}x - T(PT)^{1-1}y\| \leq k_1\|x - y\|$.

A nearly Lipschitzian mapping T with sequence $\{a_n, \eta(T(PT)^{n-1})\}$ is said to be nearly asymptotically nonexpansive if $\eta(T(PT)^{n-1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T(PT)^{n-1}) = 1$.

Recently, Agarwal et al. [1] introduced the following iteration process for nearly asymptotically nonexpansive self mappings:

$$\text{(1.1)} \qquad \begin{cases} x_1 = x \in C, \\ x_{n+1} = (1 - \alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n = (1 - \beta_n)\, x_n + \beta_n T^n x_n, \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0, 1)$.

Chidume et al. [2] studied the following iteration process for nonself asymptotically nonexpansive mappings:

$$\text{(1.2)} \qquad \begin{cases} x_1 = x \in C, \\ x_{n+1} = P\left(\alpha_n T(PT)^{n-1} x_n + (1 - \alpha_n)\, x_n\right), \ n \in \mathbb{N}. \end{cases}$$

Incorporating the above two, we get the following:

$$(1.3)\qquad \begin{cases} x_1 = x \in C, \\ x_{n+1} = P\left((1-\alpha_n)\,T(PT)^{n-1}x_n + \alpha_n T(PT)^{n-1}y_n\right), \\ y_n = P\left((1-\beta_n)\,x_n + \beta_n T(PT)^{n-1}x_n\right), n \in \mathbb{N}. \end{cases}$$

We intend to prove some results using this process for nearly asymptotically nonexpansive nonself mappings. For this we need the following lemmas.

Lemma 1. [8] *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2. [6] *If $\{r_n\}, \{t_n\}$ and $\{s_n\}$ are sequences of nonnegative real numbers such that $r_{n+1} \leq (1+t_n)\,r_n + s_n$, $\sum_{n=1}^{\infty} t_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$ then $\lim r_n$ exists.*

Recall that a Banach space E is said to satisfy the Kadec-Klee property if for every sequence $\{x_n\}$ in E converging weakly to x together with $\|x_n\|$ converging strongly to $\|x\|$ imply $\{x_n\}$ converges strongly to x. Uniformly convex Banach spaces and Banach spaces of finite dimension are some of the examples of reflexive Banach spaces which satisfy the Kadec-Klee property. Let $\omega_w(\{x_n\})$ denote the set of all weak subsequential limits of a bounded sequence $\{x_n\}$ in E. Then the followings can be found in Falset et al. [3].

Lemma 3. [3] *Let E be a uniformly convex Banach space such that its dual E^* satisfies the Kadec-Klee property. Assume that $\{x_n\}$ is a bounded sequence such that $\lim_{n\to\infty} \|tx_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0,1]$ and for all $p_1,\ p_2 \in \omega_w(\{x_n\})$. Then $\omega_w(\{x_n\})$ is a singleton.*

Lemma 4. [3] *Let C be a convex subset of a uniformly convex Banach space. Then there is a strictly increasing and continuous convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that for every Lipschitzian map $U : C \to C$ with Lipschitz constant $L \geq 1$, the following inequality holds:*

$$\|U(tx + (1-t)y) - (tUx + (1-t)Uy\| \leq Lg^{-1}(\|x-y\| - L^{-1}\|Ux - Uy\|)$$

for all $x, y \in C$ and $t \in [0,1]$.

2. Main result

Lemma 5. *Let E be a uniformly convex Banach space and let C be its closed and convex subset. Let $P : E \to C$ be a nonexpansive retraction of E into C and $T : C \to E$ be a nearly asymptotically nonexpansive nonself mapping with a sequence $\{a_{n,}\eta(T(PT)^{n-1})\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$*

and $\sum_{n=1}^{\infty}\left(\eta(T(PT)^{n-1})-1\right)<\infty$. *Define a sequence* $\{x_n\}$ *in* C *as in* (1.3), *where* $\{\alpha_n\},\{\beta_n\}$ *are in* $[\varepsilon,1-\varepsilon]$ *for all* $n\in\mathbb{N}$ *and for some* ε *in* $(0,1)$. *If* $F(T)\neq\varnothing$, *and* $q\in F(T)$ *then* (*i*) $\lim_{n\to\infty}\|x_n-q\|$ *exists and* (*ii*) $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.

Proof. Let q be a fixed point of T. For the sake of simplicity, set

$$A_nx=P\left((1-\beta_n)x+\beta_nT(PT)^{n-1}x\right)$$

and

$$S_nx=P[(1-\alpha_n)T(PT)^{n-1}x+\alpha_nT(PT)^{n-1}A_nx].$$

Then $y_n=A_nx_n$ and $x_{n+1}=S_nx_n$. Moreover, it is clear that q is a fixed point of S_n for all n. Let $\eta=\sup_{n\in\mathbb{N}}\eta(T(PT)^{n-1})$.

Consider

$$\begin{aligned}\|A_nx-A_ny\| &= \left\|\begin{array}{l}P\left[(1-\beta_n)x+\beta_nT(PT)^{n-1}x\right]\\ -P\left[(1-\beta_n)y+\beta_nT(PT)^{n-1}y\right]\end{array}\right\|\\ &\le \left\|(1-\beta_n)(x-y)+\beta_n\left[T(PT)^{n-1}x-T(PT)^{n-1}y\right]\right\|\\ &\le (1-\beta_n)\|x-y\|+\beta_n\eta(T(PT)^{n-1})(\|x-y\|+a_n)\\ &\le \left(1-\beta_n)\eta(T(PT)^{n-1}\right)\|x-y\|\\ &\quad +\beta_n\eta(T(PT)^{n-1})(\|x-y\|+a_n)\\ &= \left(1-\beta_n)\eta(T(PT)^{n-1}\right)\|x-y\|\\ &\quad +\beta_n\eta(T(PT)^{n-1})\|x-y\|+\beta_na_n\eta\left(T(PT)^{n-1}\right)\\ &\le \eta\left(T(PT)^{n-1}\right)\|x-y\|+\beta_na_n\eta\left(T(PT)^{n-1}\right).\end{aligned}\tag{2.1}$$

Choosing $x=x_n$ and $y=q$, we get

$$\|y_n-q\|\le\eta\left(T(PT)^{n-1}\right)\|x_n-q\|+\beta_na_n\eta\left(T(PT)^{n-1}\right).\tag{2.2}$$

Next consider,

$$\begin{aligned}\|S_nx-S_ny\| &= \left\|\begin{array}{l}P[(1-\alpha_n)T(PT)^{n-1}x+\alpha_nT(PT)^{n-1}A_nx]-\\ P[(1-\alpha_n)T(PT)^{n-1}y+\alpha_nT(PT)^{n-1}A_ny)]\end{array}\right\|\\ &\le \left\|\begin{array}{l}\left[(1-\alpha_n)T(PT)^{n-1}x+\alpha_nT(PT)^{n-1}A_nx\right]\\ -[(1-\alpha_n)T(PT)^{n-1}y+\alpha_nT(PT)^{n-1}A_ny)]\end{array}\right\|\\ &\le \eta(T(PT)^{n-1})(1-\alpha_n)\left(\|x-y\|+a_n\right)\\ &\quad +\alpha_n\eta(T(PT)^{n-1})\left(\|A_nx-A_ny\|+a_n\right)\\ &= \eta(T(PT)^{n-1})[(1-\alpha_n)\|x-y\|+(1-\alpha_n)a_n\\ &\quad +\alpha_n\|A_nx-A_ny\|+\alpha_na_n].\end{aligned}$$

Now using (2.1) , we get

$$\begin{aligned}
\|S_n x - S_n y\| &\leq \eta(T(PT)^{n-1})[(1-\alpha_n)\|x-y\| \\
&\quad + a_n + \alpha_n\{(1-\beta_n)\eta(T(PT)^{n-1})\ \|x-y\| \\
&\quad + \beta_n \eta(T(PT)^{n-1})(\|x-y\| + a_n)\}] \\
&\leq \eta(T(PT)^{n-1})[(1-\alpha_n)\eta(T(PT)^{n-1})\|x-y\| + a_n \\
&\quad + \alpha_n\{(1-\beta_n)\eta(T(PT)^{n-1})\ \|x-y\| \\
&\quad + \beta_n \eta(T(PT)^{n-1})(\|x-y\| + a_n)\}] \\
&\leq \eta(T(PT)^{n-1})[\eta(T(PT)^{n-1})\|x-y\| \\
&\quad + a_n\left(1 + \alpha_n\beta_n\eta(T(PT)^{n-1})\right)] \\
&\leq \eta(T(PT)^{n-1})[\eta(T(PT)^{n-1})\|x-y\| \\
&\quad + \left(1 + \eta(T(PT)^{n-1})\ a_n\right)] \\
&= \left(\eta(T(PT)^{n-1})\right)^2 \|x-y\| \\
&\quad + \eta(T(PT)^{n-1})\left(1 + \eta(T(PT)^{n-1})\ a_n\right) \\
&\leq \left(\eta(T(PT)^{n-1})\right)^2 \|x-y\| + \eta\,(1+\eta)\, a_n \\
&= c_n\|x-y\| + b_n,
\end{aligned}$$

where $c_n = \left(\eta(T(PT)^{n-1})\right)^2$ and $b_n = \eta(\eta+1)a_n$. Clearly, $\sum_{n=1}^{\infty}(c_n - 1) < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$.

Choosing $x = x_n$ and $y = q$ in the calculations done above, we get

$$\begin{aligned}
\|x_{n+1} - q\| &= \|S_n x_n - q\| \\
&\leq c_n\|x_n - q\| + b_n.
\end{aligned}$$

Applying Lemma 2, $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F(T)$. This proves (i). Now we prove (ii). Let $\lim_{n\to\infty}\|x_n - p\| = c$. Then $c > 0$ otherwise there is nothing to prove.

Now (2.2) implies that

$$\limsup_{n\to\infty}\|y_n - q\| \leq c. \tag{2.3}$$

Also

$$\|T(PT)^{n-1}x_n - q\| \leq \eta(T(PT)^{n-1})\left(\|x_n - q\| + a_n\right),$$

for all $n = 1, 2, \ldots$, so

$$\limsup_{n\to\infty}\|T(PT)^{n-1}x_n - q\| \leq c. \tag{2.4}$$

Next,

$$\|T(PT)^{n-1}y_n - q\| \leq \eta(T(PT)^{n-1})\left(\|y_n - q\| + a_n\right)$$

gives by virtue of (2.3) that

$$\limsup_{n\to\infty} \|T(PT)^{n-1}y_n - q\| \le c.$$

Moreover, $c = \lim_{n\to\infty} \|x_{n+1} - q\|$ means that

$$\begin{aligned} c &= \lim_{n\to\infty} \|P\left((1-\alpha_n)\,T(PT)^{n-1}x_n + \alpha_n T(PT)^{n-1}y_n\right) - Pq\| \\ &\le \lim_{n\to\infty} \|\,(1-\alpha_n)\left(T(PT)^{n-1}x_n - q\right) + \alpha_n \left(T(PT)^{n-1}y_n - q\right)\| \\ &\le \lim_{n\to\infty} \Big[(1-\alpha_n) \limsup_{n\to\infty} \left\|\left(T(PT)^{n-1}x_n - q\right)\right\| \\ &\qquad\qquad + \alpha_n \limsup\nolimits_{n\to\infty} \left\|\left(T(PT)^{n-1}y_n - q\right)\right\|\Big] \\ &\le \lim_{n\to\infty} \left[(1-\alpha_n)\,c + \alpha_n c\right] \\ &= c. \end{aligned}$$

Thus

$$\lim_{n\to\infty} \|\,(1-\alpha_n)\left(T(PT)^{n-1}x_n - q\right) + \alpha_n \left(T(PT)^{n-1}y_n - q\right)\| = c. \tag{2.5}$$

Applying Lemma 1,

$$\lim_{n\to\infty} \|T(PT)^{n-1}x_n - T(PT)^{n-1}y_n\| = 0. \tag{2.6}$$

Now

$$\begin{aligned} \|x_{n+1} - q\| &= \|P\left((1-\alpha_n)\,T(PT)^{n-1}x_n + \alpha_n T(PT)^{n-1}y_n\right) - Pq\| \\ &\le \|\,(1-\alpha_n)\,T(PT)^{n-1}x_n + \alpha_n \left(T(PT)^{n-1}y_n\right) - q\| \\ &\le \left\|T(PT)^{n-1}x_n - q\right\| + \alpha_n \left\|T(PT)^{n-1}y_n - T(PT)^{n-1}x_n\right\| \end{aligned}$$

yields that

$$c \le \liminf_{n\to\infty} \left\|T(PT)^{n-1}x_n - q\right\|$$

so that by (2.4), we get

$$\lim_{n\to\infty} \left\|T(PT)^{n-1}x_n - q\right\| = c. \tag{2.7}$$

Next,

$$\begin{aligned} \left\|T(PT)^{n-1}x_n - q\right\| &\le \left\|T(PT)^{n-1}x_n - T(PT)^{n-1}y_n\right\| + \left\|T(PT)^{n-1}y_n - q\right\| \\ &\le \left\|T(PT)^{n-1}x_n - T(PT)^{n-1}y_n\right\| \\ &\quad + \eta(T(PT)^{n-1})\left(\|y_n - q\| + a_n\right). \end{aligned}$$

Taking lim inf on both sides, we get

$$c \le \liminf_{n\to\infty} \|y_n - q\|. \tag{2.8}$$

By (2.3) and (2.8), we obtain

$$\lim_{n\to\infty} \|y_n - q\| = c. \tag{2.9}$$

On the lines similar to (2.5), we can prove that $\lim_{n\to\infty} \| (1-\beta_n)(x_n-q) + \beta_n \left(T(PT)^{n-1}x_n - q\right) \| = c$.

Again by Lemma 1, we get

$$\lim_{n\to\infty} \left\|T(PT)^{n-1}x_n - x_n\right\| = 0. \tag{2.10}$$

Now

$$\begin{aligned}\|y_n - x_n\| &= \left\|P\left[\beta_n T(PT)^{n-1}x_n + (1-\beta_n)x_n\right] - Px_n\right\| \\ &\le \left\|\beta_n T(PT)^{n-1}x_n + (1-\beta_n)x_n - x_n\right\| \\ &= \beta_n \left\|T(PT)^{n-1}x_n - x_n\right\|.\end{aligned}$$

Hence by (2.10),

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{2.11}$$

Also note that

$$\begin{aligned}\|x_{n+1} - x_n\| &= \left\|P\left((1-\alpha_n)T(PT)^{n-1}x_n + \alpha_n T(PT)^{n-1}y_n\right) - Px_n\right\| \\ &\le \left\|(1-\alpha_n)T(PT)^{n-1}x_n + \alpha_n T(PT)^{n-1}y_n - x_n\right\| \\ &\le \left\|T(PT)^{n-1}x_n - x_n\right\| + \alpha_n \left\|T(PT)^{n-1}y_n - T(PT)^{n-1}x_n\right\| \\ &\to 0 \text{ as } n\to\infty,\end{aligned}$$

so that

$$\begin{aligned}\|x_{n+1} - y_n\| &\le \|x_{n+1} - x_n\| + \|y_n - x_n\| \\ &\to 0 \text{ as } n\to\infty.\end{aligned} \tag{2.12}$$

Furthermore, from

$$\begin{aligned}\left\|x_{n+1} - T(PT)^{n-1}y_n\right\| \le\ & \|x_{n+1} - x_n\| + \left\|x_n - T(PT)^{n-1}x_n\right\| \\ &+ \|T(PT)^{n-1}x_n - T(PT)^{n-1}y_n\|,\end{aligned}$$

we obtain

$$\lim_{n\to\infty} \left\|x_{n+1} - T(PT)^{n-1}y_n\right\| = 0. \tag{2.13}$$

Finally, we make use of the fact that every nearly asymptotically nonexpansive mapping is nearly k-Lipschitzian combined with (2.10), (2.12) and (2.13) to reach at

$$\begin{aligned}\|x_n - Tx_n\| &\le \left\|x_n - T(PT)^{n-1}x_n\right\| + \left\|T(PT)^{n-1}x_n - T(PT)^{n-1}y_{n-1}\right\| \\ &\quad + \left\|T(PT)^{n-1}y_{n-1} - Tx_n\right\| \\ &= \left\|x_n - T(PT)^{n-1}x_n\right\| + \left\|T(PT)^{n-1}x_n - T(PT)^{n-1}y_{n-1}\right\| \\ &\quad + \left\|T(PT)^{1-1}(PT)^{n-1}y_{n-1} - T(PT)^{1-1}x_n\right\| \\ &\le \left\|x_n - T(PT)^{n-1}x_n\right\| + \left\|T(PT)^{n-1}x_n - T(PT)^{n-1}y_{n-1}\right\| \\ &\quad + k_1 \left\|(PT)^{n-1}y_{n-1} - x_n\right\|\end{aligned}$$

$$\leq \left\| x_n - T(PT)^{n-1}x_n \right\| + \eta(T(PT)^{n-1}) \left(\|x_n - y_{n-1}\| + a_n \right)$$
$$+k_1 \left\| PT(PT)^{n-2}y_{n-1} - x_n \right\|$$
$$= \left\| x_n - T(PT)^{n-1}x_n \right\| + \eta(T(PT)^{n-1}) \left(\|x_n - y_{n-1}\| + a_n \right)$$
$$+k_1 \left\| PT(PT)^{n-2}y_{n-1} - Px_n \right\|$$
$$\leq \left\| x_n - T(PT)^{n-1}x_n \right\| + \eta(T(PT)^{n-1}) \left(\|x_n - y_{n-1}\| + a_n \right)$$
$$+k_1 \left\| T(PT)^{n-2}y_{n-1} - x_n \right\|$$

so that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \quad (2.14) \blacksquare$$

Lemma 6. *Under the conditions of Lemma 5 and for any $p_1, p_2 \in F(T)$, $\lim_{n\to\infty} \|tx_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0,1]$.*

Proof. By Lemma 5, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$ and therefore $\{x_n\}$ is bounded. Thus there exists a real number $r > 0$ such that $\{x_n\} \subseteq D \equiv \overline{B_r(0)} \cap C$, so that D is a closed convex nonempty subset of C. Put

$$u_n(t) = \|tx_n + (1-t)p_1 - p_2\|,$$

for all $t \in [0,1]$. Then $\lim_{n\to\infty} u_n(0) = \|p_1 - p_2\|$ and $\lim_{n\to\infty} u_n(1) = \lim_{n\to\infty} \|x_n - p_2\|$ exist. Let $t \in (0,1)$.

Define $S_n : D \to D$ by:

$$S_n x = P[(1-\alpha_n)T(PT)^{n-1}x + \alpha_n T(PT)^{n-1}A_n x],$$
$$A_n x = P\left((1-\beta_n)x + \beta_n T(PT)^{n-1}x\right).$$

Then it follows that $S_n x_n = x_{n+1}$, $S_n p = p$ for all $p \in F(T)$ and, as shown before, $\|S_n x - S_n y\| \leq c_n \|x - y\| + b_n$ for all $x, y \in D$.

Set

$$R_{n,m} = S_{n+m-1}S_{n+m-2}...S_n, \; m \geq 1$$

and

$$v_{n,m} = \|R_{n,m}(tx_n + (1-t)p_1 - (tR_{n,m}x_n + (1-t)p_1)\|.$$

Then $R_{n,m}x_n = x_{n+m}$ and $R_{n,m}p = p$ for all $p \in F(T)$. Also

$$\begin{aligned}
\|R_{n,m}x - R_{n,m}y\| &\leq \|S_{n+m-1}S_{n+m-2}...S_n x - S_{n+m-1}S_{n+m-2}...S_n y\| \\
&\leq c_{n+m-1} \|S_{n+m-2}...S_n x - S_{n+m-2}...S_n y\| + b_{n+m-1} \\
&\leq c_{n+m-1}c_{n+m-2} \|S_{n+m-3}...S_n x - S_{n+m-3}...S_n y\| \\
&\quad + b_{n+m-2} + b_{n+m-1} \\
&\quad \vdots \\
&\leq \prod_{j=n}^{n+m-1} c_j \|x - y\| + \sum_{j=n}^{n+m-1} b_j.
\end{aligned}$$

Applying Lemma 4 with $x = x_n,\ y = p_1, U = R_{n,m}$ and using the facts that $\sum_{k=1}^{\infty} (c_n - 1) < \infty$ and $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$, we obtain $v_{n,m} \to 0$ as $n \to \infty$ and for all $m \geq 1$.

Finally, from the inequality

$$\begin{aligned} u_{n+m}(t) &= \|tx_{n+m} + (1-t)p_1 - p_2\| \\ &= \|tR_{n,m}x_n + (1-t)p_1 - p_2\| \\ &\leq v_{n,m} + \|R_{n,m}(tx_n + (1-t)p_1) - p_2\| \\ &\leq v_{n,m} + \|tx_n + (1-t)p_1 - p_2\| \\ &\leq v_{n,m} + \prod_{j=n}^{n+m-1} k_j^3 u_n(t), \end{aligned}$$

it follows that

$$\begin{aligned} \limsup_{n\to\infty} u_n(t) &\leq \limsup_{n,m\to\infty} v_{n,m} + \liminf_{n\to\infty} u_n(t) \\ &= \liminf_{n\to\infty} u_n(t). \end{aligned}$$

That is,

$$\limsup_{n\to\infty} u_n(t) \leq \liminf_{n\to\infty} u_n(t).$$

Hence $\lim_{n\to\infty} \|tx_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0,1]$. ∎

Recall that a Banach space E satisfies Opial condition [5] if for any sequence $\{x_n\}$ in $E, x_n \rightharpoonup x$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$. Examples of Banach spaces satisfying this condition are Hilbert spaces and all spaces $l^p (1 < p < \infty)$, whereas $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial's condition.

We now give our weak convergence theorems. Our first theorem deals with the spaces satisfying Opial condition.

Theorem 1. *Let E be a uniformly convex Banach space satisfying the Opial's condition and C, T and $\{x_n\}$ be as taken in Lemma 5. If $F(T) \neq \phi$, then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Let $x^* \in F(T)$. Then as proved in Lemma 5, $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. To prove this, let z_1 and z_2 be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemma 5, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $I - T$ is demiclosed with respect to zero, therefore we obtain $Tz_1 = z_1$. Similarly, $Tz_2 = z_2$. Next, we prove the uniqueness. For this suppose that $z_1 \neq z_2$, then by the Opial's condition

$$\begin{aligned}\lim_{n\to\infty}\|x_n-z_1\| &= \lim_{n_i\to\infty}\|x_{n_i}-z_1\| \\ &< \lim_{n_i\to\infty}\|x_{n_i}-z_2\| \\ &= \lim_{n\to\infty}\|x_n-z_2\| \\ &= \lim_{n_j\to\infty}\|x_{n_j}-z_2\| \\ &< \lim_{n_j\to\infty}\|x_{n_j}-z_1\| \\ &= \lim_{n\to\infty}\|x_n-z_1\|.\end{aligned}$$

This is a contradiction. Hence $\{x_n\}$ converges weakly to a point in $F(T)$. ■

Let us note at this point that there exist uniformly convex Banach spaces which do not satisfy the Opial condition but their duals do have the KK-property.

Example 1. (Example 3.1, [3]) Let us take $X_1 = \mathbb{R}^2$ with the norm denoted by $|x| = \sqrt{\|x_1\|^2 + \|x_2\|^2}$ and $X_2 = L_p[0,1]$ with $1 < p < \infty$ and $p \neq 2$. The Cartesian product of X_1 and X_2 furnished with the l^2-norm is uniformly convex, it does not satisfy the Opial condition but its dual does have the KK-property.

Now we prove a weak convergence theorem for the spaces whose dual have KK-property.

Theorem 2. *Under the conditions of Lemma* 5 *with an additional assumption that dual E^* of E satisfies Kadec-Klee property and $I-T$ is demiclosed at* 0*, $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. By the boundedness of $\{x_n\}$ and reflexivity of E, we have a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to some p in C. By Lemma 5, we have $\lim_{i\to\infty}\|x_{n_i} - Tx_{n_i}\| = 0$. Now applying demiclosedness of $I-T$ at 0, we have $p \in F(T)$. To prove that $\{x_n\}$ converges weakly to p, suppose that $\{x_{n_k}\}$ is another subsequence of $\{x_n\}$ that converges weakly to some q in C. Then $p, q \in W \cap F(T)$ where W is the weak limit set of $\{x_n\}$. Since $\lim_{n\to\infty}\|tx_n + (1-t)p - q\|$ exists for all $t \in [0,1]$ by Lemma 6, therefore $p = q$ by Lemma 3. Consequently, $\{x_n\}$ converges weakly to $p \in F(T)$ and this completes the proof. ■

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