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DIRECT ESTIMATE FOR SOME OPERATORS OF DURRMEYER TYPE IN EXPONENTIAL WEIGHTED SPACE

Abstract. In the present paper, we investigate the convergence and the approximation order of some Durrmeyer type operators in exponential weighted space. Furthermore, we obtain the Voronovskaya type theorem for these operators.

1. Introduction

O. Agratini considered in [1] the operator

$$L_n(f)(x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right), \quad x \geq 0,$$

where $(a)_0 = 1$, $(a)_k = a(a+1)\cdots(a+k-1)$, $k \in \mathbb{N} = \{1, 2, \ldots\}$, $a \in \mathbb{R}$. Author established an asymptotic formula and some quantitative estimates for the rate of convergence of the operator L_n.

In the present paper, we introduce two operators of Durrmeyer type related to the operator L_n ([2]). We define the operator $M_{\alpha,t}$ by

$$M_{\alpha,t}(f;x) = M_\alpha(f;t,x) = \int_0^\infty f(s) K_\alpha(t,x,s)\, ds, \quad x \geq 0,\ t > 0,\ \alpha \geq 0,$$

where

$$K_\alpha(t,x,s) = \frac{1}{t} \sum_{k=0}^{\infty} p_k\left(\frac{x}{t}\right) q_k^\alpha\left(\frac{s}{t}\right),$$

$$p_k(u) = 2^{-u}(u)_k \frac{1}{2^k k!}, \qquad q_k^\alpha(u) = \frac{e^{-u} u^{\alpha+k}}{\Gamma(\alpha+k+1)},$$

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$k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $u \geq 0$, $\alpha \geq 0$, and the operator $S_{\alpha,t}$ by

$$S_{\alpha,t}(f; x) = S_\alpha(f; t, x) = f(0)2^{-\frac{x}{t}} + \int_0^\infty f(s)H_\alpha(t, x, s)\, ds,$$

where

$$H_\alpha(t, x, s) = \frac{1}{t}\sum_{k=1}^{\infty} p_k\left(\frac{x}{t}\right) q_{k-1}^{\alpha}\left(\frac{s}{t}\right), \quad x \geq 0,\ t > 0,\ \alpha \geq 0.$$

It is clear that these operators are linear and positive. The similar operators for the Szász-Mirakjan type operator ([4], [5]) was considered by S. M. Mazhar and V. Totik in [3] and by E. Wachnicki in [6].

The aim of this paper is to study the approximation properties of $M_{\alpha,t}$ and $S_{\alpha,t}$ in the set of all continuous functions on $[0, +\infty)$ such that $f(x) = O\left(e^{px}\right)$, $p \geq 0$. In our considerations $\frac{1}{t}$ plays a role of n and we consider approximation when $t \to 0^+$.

It would be interesting to investigate approximation properties of operators $M_{\alpha,t}$, $S_{\alpha,t}$ in the space L^p.

2. Auxiliary results

In this section, we give some lemmas and properties which will be useful later in proofs of the main results.

We will consider the set E_p, $p \geq 0$, of functions f defined and continuous in $[0, +\infty)$ such that $|f(x)| \leq C_f\, e^{px}$ for $x \in [0, +\infty)$, where C_f is a constant depending on f. In E_p we consider the norm:

$$\|f\|_{E_p} = \sup_{x \in [0,+\infty)} \left|e^{-px} f(x)\right|.$$

Note that if $p \leq q$ then $E_p \subset E_q$ and $\|f\|_{E_q} \leq \|f\|_{E_p}$.

We introduce the weighted modulus of continuity of function $f \in E_p$. The first order modulus of continuity $\omega_1(f, p, \delta)$:

$$\omega_1(f, p, \delta) = \sup_{\substack{x,u,v \geq 0 \\ |u-v| \leq \delta}} e^{-px}|f(x + u) - f(x + v)|$$

and the second order modulus of continuity (modulus of smoothness) $\omega_2(f, p, \delta)$:

$$\omega_2(f, p, \delta) = \sup_{\substack{x,u,v \geq 0 \\ |u-v| \leq \delta}} e^{-px}|f(x + 2u) - 2f(x + u + v) + f(x + 2v)|, \quad \delta \geq 0.$$

When $p = 0$, we write $\omega_1(f, \delta)$, $\omega_2(f, \delta)$. The following properties hold:

$$\sup_{\substack{x \geq 0 \\ 0 \leq h \leq \delta}} e^{-px}|f(x + h) - f(x)| \leq \omega_1(f, p, \delta), \tag{1}$$

$$\sup_{\substack{x\ge 0\\ 0\le u+v\le\delta}} e^{-px}|f(x+2(u+v))-2f(x+u+v)+f(x)| \le \omega_2(f,p,\delta), \tag{2}$$

$$\omega_1(f,p,\lambda\delta) \le (1+\lambda)\omega_1(f,p,\delta), \quad \delta\ge 0,\ \lambda\ge 0. \tag{3}$$

Using properties of the gamma function, we obtain

$$\int_0^\infty s^r e^{ps} e^{-\frac{s}{t}} \left(\frac{s}{t}\right)^{\alpha+k} ds = \frac{t^{r+1}\Gamma(\alpha+k+r+1)}{(1-tp)^{\alpha+k+r+1}}, \tag{4}$$

where $k, r \in \mathbb{N}$, $\alpha \ge 0$ and $0 < t < \frac{1}{p}$ for $p > 0$, and $0 < t < \infty$ for $p = 0$. We have

$$\left(\frac{1}{1-a}\right)^z = \sum_{k=0}^{\infty} \frac{(z)_k}{k!} a^k, \quad |a| < 1,\ z \ge 0. \tag{5}$$

Using this and $(z)_{k+1} = z(z+1)_k$, $k \in \mathbb{N}_0$, we can write

$$\sum_{k=0}^{\infty} \frac{(z)_k}{k!} a^k k^r = \begin{cases} \left(\frac{1}{1-a}\right)^z & \text{for } r=0,\\ \left(\frac{1}{1-a}\right)^{z+1} za & \text{for } r=1,\\ \left(\frac{1}{1-a}\right)^{z+2} \left(z^2a^2+za\right) & \text{for } r=2,\\ \left(\frac{1}{1-a}\right)^{z+3} \left(z^3a^3+3z^2a^2+za^2+za\right) & \text{for } r=3,\\ \left(\frac{1}{1-a}\right)^{z+4} \left(z^4a^4+6z^3a^3+7z^2a^2 \right. & \\ \qquad \left. +za+4z^2a^3+za^3+4za^2\right) & \text{for } r=4. \end{cases} \tag{6}$$

Let $e_r(t) = t^r$, $r \in \mathbb{N}_0$. From (4), (5), (6), we have the following lemmas.

Lemma 2.1. *For $t > 0$, $x \ge 0$ and $\alpha \ge 0$, it holds*

$$\begin{aligned} M_\alpha(e_0;t,x) &= 1,\ \ M_\alpha(e_1;t,x) = x + (\alpha+1)t,\\ M_\alpha(e_2;t,x) &= x^2 + (2\alpha+5)xt + t^2(\alpha+1)(\alpha+2),\\ M_\alpha(e_3;t,x) &= x^3 + (3\alpha+12)x^2t + (3\alpha^2+18\alpha+29)xt^2\\ &\quad + t^3(\alpha+1)(\alpha+2)(\alpha+3),\\ M_\alpha(e_4;t,x) &= x^4 + (4\alpha+22)x^3t + (6\alpha^2+54\alpha+131)x^2t^2\\ &\quad + (4\alpha^3+42\alpha^2+154\alpha+206)xt^3\\ &\quad + t^4(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4). \end{aligned}$$

Lemma 2.2. *For $t > 0$, $x \ge 0$ and $\alpha \ge 0$, we have*

$$S_\alpha(e_0;t,x) = 1, \quad S_\alpha(e_1;t,x) = x + t\alpha\left(1 - 2^{-\frac{x}{t}}\right),$$

$$\begin{aligned}S_\alpha(e_2;t,x) &= x^2+2tx+(2\alpha+1)xt+t^2\alpha(\alpha+1)\left(1-2^{-\frac{x}{t}}\right),\\ S_\alpha(e_3;t,x) &= x^3+(3\alpha+9)x^2t+(3\alpha^2+12\alpha+14)xt^2\\ &\quad+t^3\alpha(\alpha+1)(\alpha+2)\left(1-2^{-\frac{x}{t}}\right),\\ S_\alpha(e_4;t,x) &= x^4+(4\alpha+18)x^3t+(6\alpha^2+42\alpha+83)x^2t^2\\ &\quad+(4\alpha^3+40\alpha^2+82\alpha+90)xt^3\\ &\quad+t^4\alpha(\alpha+1)(\alpha+2)(\alpha+3)\left(1-2^{-\frac{x}{t}}\right).\end{aligned}$$

If we define the function $\phi_{x,r}$ ($r \in \mathbb{N}_0$, $x \geq 0$) by $\phi_{x,r}(t) = (t-x)^r$ then, by Lemma 2.1 and Lemma 2.2, one can get the following results, immediately.

Lemma 2.3. *For $t > 0$, $x \geq 0$ and $\alpha \geq 0$, it holds*

$$\begin{aligned}M_\alpha(\phi_{x,0};t,x) &= 1, \quad M_\alpha(\phi_{x,1};t,x) = (\alpha+1)t,\\ M_\alpha(\phi_{x,2};t,x) &= 3xt+t^2(\alpha+1)(\alpha+2),\\ M_\alpha(\phi_{x,4};t,x) &= 27x^2t^2+(6\alpha^2+110\alpha+182)xt^3\\ &\quad+t^4(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4).\end{aligned}$$

Lemma 2.4. *For $t > 0$, $x \geq 0$ and $\alpha \geq 0$, we have*

$$\begin{aligned}S_\alpha(\phi_{x,0};t,x) &= 1, \quad S_\alpha(\phi_{x,1};t,x) = t\alpha\left(1-2^{-\frac{x}{t}}\right),\\ S_\alpha(\phi_{x,2};t,x) &= \left(3+2\alpha 2^{-\frac{x}{t}}\right)xt+t^2\alpha(\alpha+1)\left(1-2^{-\frac{x}{t}}\right),\\ S_\alpha(\phi_{x,4};t,x) &= 4\alpha 2^{-\frac{x}{t}}x^3t+\left(27-6\alpha(\alpha+1)2^{-\frac{x}{t}}\right)x^2t^2\\ &\quad+\left(28\alpha^2+74\alpha+90+4\alpha(\alpha+1)(\alpha+2)2^{-\frac{x}{t}}\right)xt^3\\ &\quad+t^4\alpha(\alpha+1)(\alpha+2)(\alpha+3)\left(1-2^{-\frac{x}{t}}\right).\end{aligned}$$

Observe that operators M_α, S_α preserve the constant functions. If $\alpha = 0$ then the operator S_α preserves the linear functions.

In the sequel, the following functions will be meaningful:

$$f_r(t) = e_r(t)e^{pt} = t^re^{pt}, \;\; \psi_{x,r}(t) = \phi_{x,r}(t)e^{pt} = (t-x)^re^{pt}, \;\; r \in \mathbb{N}_0, \;\; p, x \geq 0.$$

Now, we find operators M_α, S_α for the function f_r and $\psi_{x,r}$ for $r = 0,1,2$ and $p > 0$. In the case $p = 0$, it is clear that $f_r = e_r$, $\psi_{x,r} = \phi_{x,r}$.

At this point, we assume $0 < t < \frac{1}{2p}$. Then $0 < \frac{1}{2(1-tp)} < 1$. From the definitions of M_α, S_α and from (4), we obtain

$$M_\alpha(f_r;t,x) = \frac{t^r2^{-\frac{x}{t}}}{(1-tp)^{\alpha+r+1}}\sum_{k=0}^{\infty}\frac{\left(\frac{x}{t}\right)_k}{2^kk!(1-tp)^k}\cdot\frac{\Gamma(\alpha+k+r+1)}{\Gamma(\alpha+k+1)},$$

$$S_\alpha(f_r;t,x) = 2^{-\frac{x}{t}} f_r(0) + \frac{t^r 2^{-\frac{x}{t}}}{(1-tp)^{\alpha+r}} \sum_{k=1}^{\infty} \frac{\left(\frac{x}{t}\right)_k}{2^k k!(1-tp)^k} \cdot \frac{\Gamma(\alpha+k+r)}{\Gamma(\alpha+k)}$$

for $r \in \mathbb{N}_0$.

From this and (6), we have the following lemmas, which we shall apply to the proofs of the main theorems.

Lemma 2.5. *Let $\alpha \geq 0$, $x \geq 0$, $0 < t < \frac{1}{2p}$. Let $A = \left(\frac{1-tp}{1-2tp}\right)^{x/t}$. Then*

$$M_\alpha(f_0;t,x) = \frac{A}{(1-tp)^{\alpha+1}}, \quad M_\alpha(f_1;t,x) = \frac{A(x+t(\alpha+1)(1-2tp))}{(1-tp)^{\alpha+2}(1-2tp)},$$

$$M_\alpha(f_2;t,x) = \frac{A}{(1-tp)^{\alpha+3}(1-2tp)^2}\left(x^2 + (2\alpha+3-4\alpha tp - 8tp)xt \right.$$
$$\left. + t^2(\alpha+1)(\alpha+2)(1-2tp)^2\right),$$

$$M_\alpha(\psi_{x,0};t,x) = M_\alpha(f_0;t,x) = \frac{A}{(1-tp)^{\alpha+1}},$$

$$M_\alpha(\psi_{x,1};t,x) = \frac{A\left(3ptx - 2t^2p^2x + t(\alpha+1)(1-2tp)\right)}{(1-tp)^{\alpha+2}(1-2tp)},$$

$$M_\alpha(\psi_{x,2};t,x) = \frac{At}{(1-tp)^{\alpha+3}(1-2tp)^2}\left(x^2tp^2(2tp-1)^2\right.$$
$$+ x(3+8tp-(\alpha+1)(6tp+16t^2p^2-8t^3p^3))$$
$$\left. + t(\alpha+1)(\alpha+2)(1-2tp)^2\right).$$

Lemma 2.6. *Let $\alpha \geq 0$, $x \geq 0$, $0 < t < \frac{1}{2p}$. Let $A = \left(\frac{1-tp}{1-2tp}\right)^{x/t}$. Then*

$$S_\alpha(f_0;t,x) = 2^{-\frac{x}{t}} + \frac{1}{(1-tp)^\alpha}\left(A - 2^{-\frac{x}{t}}\right),$$

$$S_\alpha(f_1;t,x) = \frac{1}{(1-tp)^{\alpha+1}(1-2tp)}\left(Ax + \alpha t(1-2tp)\left(A - 2^{-\frac{x}{t}}\right)\right),$$

$$S_\alpha(f_2;t,x) = \frac{1}{(1-tp)^{\alpha+2}(1-2tp)^2}\left(Ax^2 + A(3-4tp+2\alpha(1-2tp)xt\right.$$
$$\left. + t^2\alpha(\alpha+1)(1-2tp)^2\left(A-2^{-\frac{x}{t}}\right)\right),$$

$$S_\alpha(\psi_{x,0};t,x) = S_\alpha(f_0;t,x) = 2^{-\frac{x}{t}} + \frac{1}{(1-tp)^\alpha}\left(A - 2^{-\frac{x}{t}}\right),$$

$$S_\alpha(\psi_{x,1};t,x) = -2^{-\frac{x}{t}}x + \frac{1}{(1-tp)^{\alpha+1}(1-2tp)}\left(3Atpx - 2At^2p^2x\right.$$
$$\left. + 2^{-\frac{x}{t}}x(1-tp)(1-2tp) + \alpha t(1-2tp)\left(A - 2^{-\frac{x}{t}}\right)\right),$$

$$\begin{aligned}S_\alpha(\psi_{x,2};t,x) = {} & \frac{1}{(1-tp)^{\alpha+2}(1-2tp)^2}\left[Ax^2(4t^4p^4 - 12t^3p^3 + 9t^2p^2)\right.\\ & + A(3-4tp+2\alpha(1-2tp))xt \\ & + t^2\alpha(\alpha+1)(1-2tp)^2\left(A - 2^{-\frac{x}{t}}\right) \\ & - 2\alpha xt(1-2tp)^2(1-tp)\left(A-2^{-\frac{x}{t}}\right) \\ & \left. - 2^{-\frac{x}{t}}x^2(1-tp)^2(1-2tp)^2\right] + x^2 2^{-\frac{x}{t}}.\end{aligned}$$

Now, we prove the auxiliary inequalities.

Lemma 2.7. *If $0 < p < q$ then for*

$$0 < t < \frac{1}{4p}\left(3 - \sqrt{1+\frac{8p}{q}}\right) < \frac{1}{2p}, \tag{7}$$

the inequality

$$A \le e^{qx} \text{ for } x \ge 0$$

holds.

Proof. Let $I = \left(0, \frac{1}{4p}\left(3-\sqrt{1+\frac{8p}{q}}\right)\right)$. We observe that if $0 < p < q$ then $3 - \sqrt{1+8p/q} > 0$ and I is not empty. If $t \in I$ then

$$(1-tp)(1-2tp) > \frac{p}{q}$$

and consequently

$$\frac{p}{(1-tp)(1-2tp)} < q. \tag{8}$$

Let $f(t) = \ln\frac{1-tp}{1-2tp} - tq$ for $0 \le t < \frac{1}{2p}$. According to the properties of f, in view of (8), we conclude that f is decreasing in $I \cup \{0\}$. Therefore, $f(t) < f(0)$ for $t \in I$. From this, it follows that

$$e^{\frac{x}{t}\ln\frac{1-tp}{1-2tp}} \le e^{qx},$$

for $x \ge 0$ and $t \in I$, which completes the proof of Lemma 2.7. ∎

Similarly, we can prove the following lemma.

Lemma 2.8. *If $p > 0$ and $\alpha \ge 0$ then for $0 < t < \frac{1}{2p}$, the inequality*

$$e^{px} \le A \le \frac{A}{(1-tp)^\alpha}$$

holds.

3. Approximation theorems

In this section, we state the Voronovskaya type theorem. Then we compute the rate of convergence of the operators M_α, S_α. To achieve this, we use the first and the second order modulus of continuity.

At the beginning of this section, we give some properties of the operator norm of M_α and S_α, $\alpha \geq 0$.

Theorem 3.9. *Let* $0 < p < q$. *For* $t \in I$, *the operator* $M_{\alpha,t}$ *maps* E_p *into* E_q *and*

$$||M_{\alpha,t}(f)||_{E_q} \leq 2^{\alpha+1}\|f\|_{E_p}. \tag{9}$$

In the case $p = 0$, *the operator* $M_{\alpha,t}$ *maps* E_0 *into itself and*

$$||M_{\alpha,t}(f)||_{E_0} \leq \|f\|_{E_0}, \quad t > 0. \tag{10}$$

Proof. Let $0 < p < q$. We have

$$\begin{aligned} |M_{\alpha,t}(f;x)| = |M_\alpha(f;t,x)| &\leq M_\alpha(|f|;t,x) \leq \|f\|_{E_p} M_\alpha(f_0;t,x) \\ &= \|f\|_{E_p} \frac{A}{(1-tp)^{\alpha+1}}. \end{aligned}$$

Since $0 < \frac{1}{1-tp} < 2$ for $0 < t < \frac{1}{2p}$, we obtain (9), by Lemma 2.7. Using $M_\alpha(e_0;t,x) = 1$ (Lemma 2.1), we check at once that (10) holds. ∎

In the similar fashion, we prove the following inequalities.

Theorem 3.10. *If* $0 < p < q$ *then for* $t \in I$, *the operator* $S_{\alpha,t}$ *maps* E_p *into* E_q *and*

$$||S_{\alpha,t}(f)||_{E_q} \leq 2^{\alpha}\|f\|_{E_p}. \tag{11}$$

In the case $p = 0$, *the operator* $S_{\alpha,t}$ *maps* E_0 *into itself and*

$$||S_{\alpha,t}(f)||_{E_0} \leq \|f\|_{E_0}, \quad t > 0. \tag{12}$$

Proof. Let $0 < p < q$. By Lemma 2.6, we can write

$$\begin{aligned} |S_{\alpha,t}(f;x)| = |S_\alpha(f;t,x)| &\leq S_\alpha(|f|;t,x) \leq \|f\|_{E_p} S_\alpha(f_0;t,x) \\ &= \|f\|_{E_p}\left[2^{-\frac{x}{t}} + \frac{A - 2^{-\frac{x}{t}}}{(1-tp)^\alpha}\right]. \end{aligned}$$

Analogously to the proof of Theorem 3.9, we have

$$\begin{aligned} 2^{-\frac{x}{t}} + \frac{A - 2^{-\frac{x}{t}}}{(1-tp)^\alpha} &\leq 2^{-\frac{x}{t}} + 2^\alpha\left(e^{qx} - 2^{-\frac{x}{t}}\right) \\ &= 2^\alpha e^{qx} + 2^{-\frac{x}{t}}\left(1 - 2^\alpha\right) \leq 2^\alpha e^{qx}, \end{aligned}$$

since $2^{-\frac{x}{t}}\left(1 - 2^\alpha\right) \leq 0$. This gives the result. ∎

Next, we are interested in some approximation theorems.

Theorem 3.11. *Let $f \in E_p$, $p \geq 0$. Then*

$$\lim_{t\to 0^+} M_\alpha(f;t,x) = f(x), \quad \lim_{t\to 0^+} S_\alpha(f;t,x) = f(x)$$

and the convergence is uniform on every compact subset of $[0,+\infty)$.

Proof. Let $\varepsilon > 0$ and let $x \geq 0$. Since f is continuous at the point x, there exists $\delta > 0$ such that

$$|f(s) - f(x)| < \varepsilon \quad \text{for} \quad |s-x| < \delta.$$

Let $p > 0$. If $|s-x| \geq \delta$ then

$$|f(s) - f(x)| \leq \|f\|_{E_p} \left(e^{ps} + e^{px}\right) \leq \|f\|_{E_p} \left(e^{ps} + e^{px}\right) \frac{(s-x)^2}{\delta^2}.$$

Therefore

$$|f(s) - f(x)| \leq \varepsilon + \|f\|_{E_p} \left(e^{ps} + e^{px}\right) \frac{(s-x)^2}{\delta^2},$$

for an arbitrary $s \in [0,+\infty)$ and $\delta > 0$. From this, it follows

$$\begin{aligned} (13) \quad |M_\alpha(f;t,x) - f(x)| &\leq M_\alpha\left(|f - f(x)|;t,x\right) \leq \varepsilon M_\alpha(e_0;t,x) \\ &\quad + \frac{\|f\|_{E_p}}{\delta^2}\left(M_\alpha\left(\psi_{x,2};t,x\right) + e^{px} M_\alpha\left(\phi_{x,2};t,x\right)\right). \end{aligned}$$

By Lemma 2.3, we have

$$M_\alpha\left(\phi_{x,2};t,x\right) = t(3x + t(\alpha+1)(\alpha+2)),$$

thus $\lim_{t\to 0^+} M_\alpha\left(\phi_{x,2};t,x\right) = 0$. Using Lemma 2.5, we get

$$\lim_{t\to 0^+} M_\alpha\left(\psi_{x,2};t,x\right) = 0.$$

Hence, in view of (13), we obtain

$$(14) \qquad \lim_{t\to 0^+} M_\alpha(f;t,x) = f(x).$$

Let $0 \leq a < b$. If $x \in [a,b]$ and $0 < t < \frac{1}{2p}$ then $3x + t(\alpha+1)(\alpha+2)$ is bounded. Similarly,

$$\begin{aligned} \frac{A}{(1-tp)^{\alpha+3}(1-2tp)^2} &\left[x^2tp^2(2tp-1)^2 + x\left(3 + 8tp - (\alpha+1)\right.\right. \\ &\left.\left.\times\left(6tp + 16t^2p^2 - 8t^3p^3\right)\right) + t(\alpha+1)(\alpha+2)(1-2tp)^2\right] \end{aligned}$$

is bounded for $x \in [a,b]$ and $t \in I$. By Lemma 2.3, Lemma 2.5 and (13), we can deduce that (14) holds uniformly on $[a,b]$.

In the case $p = 0$ and for the operator S_α, the proof follows similarly. ■

The next step is to establish the Voronovskaya type theorem.

Theorem 3.12. *If $f \in E_p$, $p \geq 0$, is twice differentiable at some point $x \geq 0$ then*

$$\lim_{t\to 0^+} \frac{M_\alpha(f;t,x)-f(x)}{t} = (\alpha+1)f'(x) + \frac{3}{2}xf''(x), \tag{15}$$

$$\lim_{t\to 0^+} \frac{S_\alpha(f;t,x)-f(x)}{t} = \alpha f'(x) + \frac{3}{2}xf''(x). \tag{16}$$

Proof. We use the Taylor expansion

$$f(s) = f(x) + (s-x)f'(x) + \frac{1}{2}(s-x)^2 f''(x) + \varepsilon(s,x)(s-x)^2,$$

where $\lim_{s\to x}\varepsilon_x(s) = \lim_{s\to x}\varepsilon(s,x) = 0$ and $\varepsilon_x \in E_p$. Hence, by Lemma 2.3, we can write

$$\begin{aligned} M_\alpha(f;t,x) &= M_\alpha\left(\varepsilon_x\phi_{x,2};t,x\right) + f(x) + (\alpha+1)tf'(x) \\ &\quad + \frac{1}{2}f''(x)\big(3tx + 2t^2(\alpha+1)(\alpha+2)\big). \end{aligned}$$

In order to prove (15), it is sufficient to obtain

$$\lim_{t\to 0^+} \frac{M_\alpha\left(\varepsilon_x\phi_{x,2};t,x\right)}{t} = 0. \tag{17}$$

Recalling the Cauchy-Schwarz inequality, we can infer

$$\left|M_\alpha\left(\varepsilon_x\phi_{x,2};t,x\right)\right| \leq \sqrt{M_\alpha\left(\varepsilon_x^2;t,x\right)M_\alpha\left(\phi_{x,4};t,x\right)}.$$

Using Lemma 2.3, we get

$$\begin{aligned} \frac{M_\alpha\left(\phi_{x,4};t,x\right)}{t^2} &= 27x^2 + \left(6\alpha^2 + 110\alpha + 182\right)xt \\ &\quad + t^2(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4). \end{aligned}$$

In view of Theorem 3.11, we conclude that $\lim_{t\to 0^+} M_\alpha\left(\varepsilon_x^2;t,x\right) = 0$. Therefore, we have (17).

The proof of (16) follows similarly. ∎

Corollary 3.13. *If f satisfies the assumptions of Theorem 3.12 then*

$$M_\alpha\left(f;t,x\right) = f(x) + O(t), \quad S_\alpha\left(f;t,x\right) = f(x) + O(t)$$

as $t \to 0^+$.

In the sequel, we state the approximation order of the operators M_α and S_α, $\alpha \geq 0$.

Theorem 3.14. *If $f \in E_p$ then for $p \geq 0$, it holds*

$$|M_\alpha(f;t,x) - f(x)| \leq 2e^{px}\omega_1\left(f;p,\delta\right), \tag{18}$$

$$|S_\alpha(f;t,x) - f(x)| \leq 2e^{px}\omega_1\left(f;p,\delta_1\right), \tag{19}$$

for $t > 0$, $x \geq 0$, *where*

$$\delta = \sqrt{3tx + t^2(\alpha+1)(\alpha+2)},$$
$$\delta_1 = \sqrt{\left(3 + 2\alpha 2^{-\frac{x}{t}}\right) xt + t^2\alpha(\alpha+1)\left(1 - 2^{-\frac{x}{t}}\right)}.$$

Proof. Using the property of modulus of continuity, we can write

$$|f(s) - f(x)| \leq e^{px}\left(1 + \frac{(s-x)^2}{\delta^2}\right)\omega_1(f;p,\delta),$$

for arbitrary $s, x \in [0, +\infty)$ and an arbitrary $\delta > 0$. Therefore,

$$|M_\alpha(f;t,x) - f(x)| \leq e^{px}\omega_1(f;p,\delta)\left(M_\alpha(e_0;t,x) + \frac{1}{\delta^2}M_\alpha(\phi_{x,2};t,x)\right).$$

Hence, by Lemma 2.1 and Lemma 2.3, we have

$$|M_\alpha(f;t,x) - f(x)| \leq e^{px}\omega_1(f;p,\delta) \times \left(1 + \frac{3xt + t^2(\alpha+1)(\alpha+2)}{\delta^2}\right).$$

Choosing $\delta = \sqrt{3xt + t^2(\alpha+1)(\alpha+2)}$, we obtain (18).

The proof of (19) follows similarly. ■

In the following, we are going to prove another estimate. We will use the second order modulus of smoothness.

Lemma 3.15. *Let* $f \in E_p$ *be a function of* C^2 *in* $[0, +\infty)$ *and let* $f', f'' \in E_p$. *If* T *is a linear and positive operator, which maps* E_p *into* E_q, $0 \leq p \leq q$, *such that* $T(e_0) = e_0$, *then*

$$|T(f)(x) - f(x)| \leq |f'(x)||T(\phi_{x,1})(x)| + \|f''\|_{E_p}\left(\frac{1}{p^2}T(f_0)(x) - \frac{1}{p^2}e^{px} - \frac{1}{p}e^{px}T(\phi_{x,1})(x)\right), \tag{20}$$

if $p > 0$ *and*

$$|T(f)(x) - f(x)| \leq |f'(x)||T(\phi_{x,1})(x)| + \frac{1}{2}\|f''\|_{E_0}T(\phi_{x,2})(x), \tag{21}$$

if $p = 0$.

Proof. We can write

$$f(s) = f(x) + (s-x)f'(x) + \int_x^s \int_x^z f''(u)\,du\,dz.$$

Therefore,

$$|T(f)(x) - f(x)| \le |f'(x)||T(\phi_{x,1})(x)| + \|f''\|_{E_p} T(\lambda_x)(x),$$

where

$$\lambda_x(s) = \int_x^s \int_x^z e^{pu}\,du\,dz = \begin{cases} \frac{1}{p^2}\left(e^{ps} - e^{px}\right) - \frac{1}{p} e^{px}(s-x), & \text{for } p > 0, \\ \frac{1}{2}(s-x)^2, & \text{for } p = 0. \end{cases}$$

Hence we get (20) and (21). ■

Now, we give the following results which will be used in the proof of the next theorem. We will use the Steklov mean (see [3])

$$f_h(x) = \frac{4}{h^2} \int_0^{\frac{h}{2}} \int_0^{\frac{h}{2}} \left[2f(x+u+v) - f(x+2(u+v))\right] du\,dv.$$

Let $p \ge 0$. Note that if $f \in E_p$ then $f_h \in E_p$. Therefore

$$f_h(x) - f(x) = \frac{4}{h^2} \int_0^{\frac{h}{2}} \int_0^{\frac{h}{2}} \left[2f(x+u+v) - f(x+2(u+v)) - f(x)\right] du\,dv.$$

Using this and (2), we obtain

$$e^{-px}\,|f_h(x) - f(x)| \le \omega_2(f; p, h).$$

Hence

$$||f_h - f||_{E_p} \le \omega_2(f; p, h). \tag{22}$$

We observe that if f is continuous then

$$f_h'(x) = \frac{4}{h^2}\Bigg[2\int_0^{\frac{h}{2}} \left(f\left(x+v+\frac{h}{2}\right) - f(x+v)\right) dv - \frac{1}{2}\int_0^{\frac{h}{2}} \left(f(x+2v+h) - f(x+2v)\right) dv\Bigg].$$

From this, we have

$$|f_h'(x)| \le \frac{5}{h} e^{px} \omega_1(f, p, h) \tag{23}$$

and

$$f_h'(x) = \frac{4}{h^2}\Bigg[2\int_{x+\frac{h}{2}}^{x+h} f(z)\,dz - 2\int_x^{x+\frac{h}{2}} f(z)\,dz - \frac{1}{4}\int_{x+h}^{x+2h} f(z)\,dz + \frac{1}{4}\int_x^{x+h} f(z)\,dz\Bigg].$$

Note that f_h is the function of C^2 in $[0, +\infty)$ and

$$f_h''(x) = \frac{1}{h^2}\Bigg[8f(x+h) - 16f\left(x + \frac{h}{2}\right) + 8f(x) - f(x+2h) \\ + 2f(x+h) - f(x)\Bigg].$$

We observe that if $f \in E_p$ then $f_h'' \in E_p$ and

$$||f_h''||_{E_p} \leq \frac{9}{h^2}\omega_2(f;p,h). \tag{24}$$

Theorem 3.16. *If $f \in E_p$, $0 < p < q$, then*

$$\|M_{\alpha,t}(f) - f\|_{E_q} \leq C\omega_2(f,p,\sqrt{t}) + 5(\alpha+1)\sqrt{t}\,\omega_1(f,p,\sqrt{t}), \tag{25}$$

$$\|S_{\alpha,t}(f) - f\|_{E_q} \leq C\omega_2(f,p,\sqrt{t}) + 5\alpha\sqrt{t}\,\omega_1(f,p,\sqrt{t}), \tag{26}$$

for $t \in I$, where I is the interval defined in the proof of Lemma 2.7, C denotes some positive constant depending only on α and p.

Proof. We can write

$$|M_\alpha(f;t,x) - f(x)| \leq M_\alpha\left(|f - f_h|;t,x\right) + |M_\alpha\left(f_h - f_h(x);t,x\right)| \\ + |f_h(x) - f(x)|.$$

Using Lemma 3.15, (23), (24) and

$$M_\alpha(\phi_{x,1};t,x) = (\alpha+1)t,$$

$$M_\alpha(f_0;t,x) = \frac{A}{(1-tp)^{\alpha+1}},$$

we get

$$|M_\alpha\left(f_h - f_h(x);t,x\right)| \leq \frac{5}{h}e^{px}(\alpha+1)t\,\omega_1(f,p,h) + \frac{9}{h^2}\omega_2(f,p,h) \\ \times\left[\frac{1}{p^2}\left(\frac{A}{(1-tp)^{\alpha+1}} - e^{px}\right) - \frac{1}{p}e^{px}(\alpha+1)t\right],$$

for $x \geq 0$, $h > 0$ and $0 < t < \frac{1}{2p}$. From Theorem 3.9 and from (22), we have

$$M_\alpha\left(|f - f_h|;t,x\right) \leq 2^{\alpha+1}e^{qx}\|f - f_h\|_{E_p} \leq 2^{\alpha+1}e^{qx}\omega_2(f,p,h).$$

Hence, we obtain

$$|M_\alpha(f;t,x) - f(x)| \leq \Bigg[e^{px} + 2^{\alpha+1}e^{qx} + \frac{9}{h^2}\Bigg(-\frac{1}{p}e^{px}(\alpha+1)t \\ + \frac{1}{p^2}\left(\frac{A}{(1-tp)^{\alpha+1}} - e^{px}\right)\Bigg)\Bigg]\omega_2(f,p,h) \\ + \frac{5}{h}e^{px}(\alpha+1)t\,\omega_1(f,p,h),$$

for $x \geq 0$, $h > 0$ and $0 < t < \frac{1}{2p}$. Choosing $h = \sqrt{t}$, we have

$$|M_\alpha(f;t,x) - f(x)| \leq \Big[e^{px} + 2^{\alpha+1}e^{qx} + \frac{9}{t}\Big(-\frac{1}{p}e^{px}(\alpha+1)t + \frac{1}{p^2}\Big(\frac{A}{(1-tp)^{\alpha+1}} - e^{px}\Big)\Big)\Big]\omega_2(f,p,\sqrt{t}) + 5e^{px}(\alpha+1)\sqrt{t}\,\omega_1(f,p,\sqrt{t}),$$

for $x \geq 0$, $0 < t < \frac{1}{2p}$.

Note that there exists $C_1 > 0$ such that

$$\frac{e^{-qx}}{tp^2}\left(\frac{A}{(1-tp)^{\alpha+1}} - e^{px}\right) - \frac{1}{p}e^{-(q-p)x}(\alpha+1) \leq C_1,$$

for $t \in I$, $x \geq 0$. Therefore,

$$\|M_{\alpha,t}(f) - f\|_{E_q} \leq C\omega_2(f,p,\sqrt{t}) + 5(\alpha+1)\sqrt{t}\,\omega_1(f,p,\sqrt{t}),$$

for $t \in I$, where $C = C_1 + 1 + 2^{\alpha+1}$.

Similarly we can prove (26). ∎

From (26), we obtain the next result at once.

Corollary 3.17. *If $f \in E_p$, $0 < p < q$, then*

$$\|S_{0,t}(f) - f\|_{E_q} \leq C\omega_2(f;p,\sqrt{t}), \quad \textit{for } t \in I.$$

Now, we discuss the case $p = 0$. Using the same method as in the proof of Theorem 3.16, we have the following estimations.

Theorem 3.18. *If $f \in E_0$ then*

$$|M_\alpha(f;t,x) - f(x)| \leq \left(2 + \frac{27}{2}x + \frac{9t}{2}(\alpha+1)(\alpha+2)\right)\omega_2(f,\sqrt{t}) + 5(\alpha+1)\sqrt{t}\,\omega_1(f,\sqrt{t}),$$

$$|S_\alpha(f;t,x) - f(x)| \leq \left(2 + \frac{9}{2}(3+2\alpha)x + \frac{9t}{2}\alpha(\alpha+1)\right)\omega_2(f,\sqrt{t}) + 5\alpha\sqrt{t}\,\omega_1(f,\sqrt{t}),$$

for $x \geq 0$, $t > 0$.

Corollary 3.19. *If $f \in E_0$ then*

$$|S_0(f;t,x) - f(x)| \leq \left(2 + \frac{27}{2}x\right)\omega_2(f,\sqrt{t}), \quad \textit{for } x \geq 0,\ t > 0.$$

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