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INEQUALITIES AND MEANS FROM A CYCLIC DIFFERENTIAL EQUATION

Abstract. We prove that the solution of the cyclic initial value problem

$$u_k' = 1/2 - u_k/(u_{k+1} + u_{k+2}) \quad (k \in \mathbb{Z}/n\mathbb{Z}), \quad u(0) = x$$

is convergent to an equilibrium $\mu(x)(1, \dots, 1)$, and study the properties of the function $x \mapsto \mu(x)$ and its relation to Shapiro's inequality.

1. Introduction

In 1954 H. S. Shapiro [6] raised the question of proving

$$\frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_4} + \dots + \frac{x_{n-1}}{x_n + x_1} + \frac{x_n}{x_1 + x_2} \geq \frac{n}{2}, \tag{1}$$

for all $n \in \mathbb{N}\backslash\{1, 2\}$ and all $x_1, \dots, x_n > 0$. Meanwhile, after the investigations of numerous authors, (1) is known to be false in general for even $n \geq 14$ and all $n \geq 25$. Moreover (1) is always true for even $n \leq 12$ and for odd $n \leq 23$, see [1, p. 233], [3], [5, p. 440], and the references given there.

Inspired by Shapiro's inequalities, we investigated the following corresponding initial value problem, which turns out to have some interesting aspects in its asymptotic behaviour. In the sequel, let always $n \geq 3$, and let all indexes of vectors in $\mathbb{R}^n$ be understood modulo n, that is from $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$. Let

$$D := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_k > 0 \ (k \in \mathbb{Z}_n)\}$$

and let $f : D \to \mathbb{R}^n$ be defined by

$$f(x) = (f_1(x), \dots, f_n(x)) = \left(\frac{1}{2} - \frac{x_k}{x_{k+1} + x_{k+2}}\right)_{k \in \mathbb{Z}_n}.$$

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We consider the initial value problem

$$u'(t) = f(u(t)), \quad u(0) = x \in D. \tag{2}$$

In the sequel, let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^n$ and let $\mathcal{A}(x)$, $\mathcal{G}(x)$ and $\mathcal{Q}(x)$ denote the arithmetic, geometric and quadratic mean of $x \in D$, i.e.

$$\mathcal{A}(x) = \frac{1}{n}\sum_{k\in\mathbb{Z}_n} x_k, \quad \mathcal{G}(x) = \left(\prod_{k\in\mathbb{Z}_n} x_k\right)^{1/n}, \quad \mathcal{Q}(x) = \frac{1}{\sqrt{n}}\|x\|,$$

and note that $\mathcal{G}(x) \le \mathcal{A}(x) \le \mathcal{Q}(x)$ $(x \in D)$.

We will see that the solution $u(t,x)$ of (2) is convergent to an equilibrium point $\mu(x)(1,\dots,1)$ as $t \to \infty$, and we will study the properties of the hereby defined function $\mu : D \to \mathbb{R}$. Among others we show that

$$\mathcal{G}(x) \le \mu(x) \le \mathcal{Q}(x) \quad (x \in D),$$

and that, if (1) is true on D, then even

$$\mu(x) \le \mathcal{A}(x) \quad (x \in D).$$

For $n = 3$, we will see that

$$\mu(x) = \frac{1}{\sqrt{3}}\sqrt{x_1x_2 + x_2x_3 + x_3x_1} \quad (x \in D).$$

2. Preliminaries

First note that $f : D \to \mathbb{R}^n$ is locally Lipschitz continuous, hence (2) is uniquely solvable for each $x \in D$, and that $f(\lambda p) = 0$, $p := (1,\dots,1)$, $\lambda > 0$. Let $u(.,x) : [0,\omega_+(x)) \to D$ denote the solution of (2), non extendable to the right. Next, we consider $\mathbb{R}^n$ as ordered by the natural cone

$$K := \{x = (x_1,\dots,x_n) \in \mathbb{R}^n : x_k \ge 0 \ (k \in \mathbb{Z}_n)\}, \quad x \le y :\Longleftrightarrow y - x \in K.$$

For $x, z \in \mathbb{R}^n$, $x \le z$ let $[x,z]$ denote the order interval

$$\{y \in \mathbb{R}^n : x \le y \le z\}.$$

Now f is quasimonotone increasing on D with respect to K in the sense of Walter [8, Def. 6.II], that is

$$x, y \in D,\ x \le y,\ x_k = y_k \ \Rightarrow\ f_k(x) \le f_k(y).$$

Since f is locally Lipschitz continuous, it is known that for each pair of functions $v, w \in C^1([a,b), D)$, the implication

$$\begin{cases} v'(t) - f(v(t)) \le w'(t) - f(w(t)) \ (t \in [a,b)), \quad v(a) \le w(a) \\ \Longrightarrow v(t) \le w(t) \ (t \in [a,b)) \end{cases} \tag{3}$$

is valid, see [7, Satz 2]. From (3), we obtain

$$\min\{x_1,\dots,x_n\}p \le u(t,x) \le \max\{x_1,\dots,x_n\}p \quad (t \in [0,\omega_+(x))), \tag{4}$$

and therefore $\omega_+(x) = \infty$ $(x \in D)$. Moreover (3) implies

$$u(t,x) \le u(t,y) \quad (x \le y,\ t \ge 0). \tag{5}$$

Proposition 1. *For $x \in D$ we have*

$$f(x) = 0 \iff \exists \lambda > 0 : x = \lambda p.$$

Proof. If $f(x) = 0$ then $x_k = \frac{x_{k+1}+x_{k+2}}{2}$ for all $k \in \mathbb{Z}_n$. Hence

$$\begin{aligned} x_k - x_{k+1} &= \frac{x_{k+2} - x_{k+1}}{2} = -\frac{1}{2}(x_{k+1} - x_{k+2}) \\ &= \ldots = \frac{(-1)^n}{2^n}(x_{k+n} - x_{k+n+1}) = \frac{(-1)^n}{2^n}(x_k - x_{k+1}) \end{aligned}$$

for all $k \in \mathbb{Z}_n$. Thus all x_k must be equal. ∎

Proposition 2. *Let $u = u(\cdot, x)$ be the solution of (2). Then*

1. *$t \mapsto \mathcal{G}(u(t))$ is monotone increasing on $[0,\infty)$;*
2. *$t \mapsto \mathcal{Q}(u(t))$ is monotone decreasing on $[0,\infty)$;*
3. *$t \mapsto \mathcal{A}(u(t))$ is monotone decreasing on $[0,\infty)$, if (1) holds on D.*

Proof. 1. From (2), we get

$$\sum_{k\in\mathbb{Z}_n} \frac{u_k'(t)}{u_k(t)} = \sum_{k\in\mathbb{Z}_n} \left(\frac{1}{2u_k(t)} - \frac{1}{u_{k+1}(t) + u_{k+2}(t)} \right) \quad (t \ge 0).$$

We use the inequality

$$\frac{1}{\alpha+\beta} \le \frac{1}{4\alpha} + \frac{1}{4\beta} \quad (\alpha, \beta > 0),$$

and obtain

$$\sum_{k\in\mathbb{Z}_n} (\log(u_k))'(t) \ge \sum_{k\in\mathbb{Z}_n} \left(\frac{1}{2u_k(t)} - \frac{1}{4u_{k+1}(t)} - \frac{1}{4u_{k+2}(t)} \right) = 0 \quad (t \ge 0).$$

Thus

$$t \mapsto \log\Big(\prod_{k\in\mathbb{Z}_n} u_k(t) \Big)$$

is monotone increasing on $[0,\infty)$, and the assertion follows.

2. First, consider $y \in D$. The Cauchy–Schwarz inequality leads to

$$\begin{aligned} \Big(\sum_{k\in\mathbb{Z}_n} y_k\Big)^2 &= \Big(\sum_{k\in\mathbb{Z}_n} \frac{y_k}{\sqrt{y_{k+1}+y_{k+2}}} \sqrt{y_{k+1}+y_{k+2}} \Big)^2 \\ &\le \sum_{k\in\mathbb{Z}_n} \frac{y_k^2}{y_{k+1}+y_{k+2}} \sum_{k\in\mathbb{Z}_n} (y_{k+1}+y_{k+2}) \\ &= \sum_{k\in\mathbb{Z}_n} \frac{y_k^2}{y_{k+1}+y_{k+2}} \sum_{k\in\mathbb{Z}_n} 2y_k, \end{aligned}$$

hence

$$\sum_{k\in\mathbb{Z}_n} y_k \le \sum_{k\in\mathbb{Z}_n} \frac{2y_k^2}{y_{k+1}+y_{k+2}},$$

and therefore

$$\Big(\sum_{k\in\mathbb{Z}_n} u_k^2\Big)'(t) = \sum_{k\in\mathbb{Z}_n}\Big(u_k(t) - \frac{2u_k(t)^2}{u_{k+1}(t)+u_{k+2}(t)}\Big) \le 0 \quad (t\ge 0).$$

3. Validity of (1) on D leads to

$$\Big(\sum_{k\in\mathbb{Z}_n} u_k\Big)'(t) = \frac{n}{2} - \sum_{k\in\mathbb{Z}_n} \frac{u_k(t)}{u_{k+1}(t)+u_{k+2}(t)} \le 0 \quad (t\ge 0). \ \blacksquare$$

3. Main results

Theorem 1. *For each $x\in D$ there exists $\mu(x)>0$ such that*

$$\lim_{t\to\infty} u(t,x) = \mu(x)p.$$

The function $\mu : D\to(0,\infty)$ has the following properties:

1. *μ is monotone increasing with respect to the order defined by K;*
2. *μ is continuous;*
3. *$\mu((x_1,\dots,x_n)) = \mu((x_2,\dots,x_n,x_1))$ $(x=(x_1,\dots,x_n)\in D)$;*
4. *$\mu(\lambda x) = \lambda\mu(x)$ $(x\in D,\ \lambda>0)$;*
5. *$\mathcal{G}(x)\le\mu(x)\le\mathcal{Q}(x)$ $(x\in D)$;*
6. *$\mu(x)\le\mathcal{A}(x)$ $(x\in D)$, if (1) holds on D.*

Proof. Fix $x\in D$ and let $u(t)=u(t,x)$ $(t\ge 0)$. We consider $h:[0,\infty)\to\mathbb{R}$ defined by $h(t)=\|u(t)\|^2$. Let

$$I := [\min\{x_1,\dots,x_n\}p, \max\{x_1,\dots,x_n\}p],$$

and note that I is a compact subset of D. From (2) and (4), we get

$$u([0,\infty))\subseteq I, \quad u'([0,\infty))\subseteq f(I), \quad u''([0,\infty))\subseteq (f'\cdot f)(I).$$

We conclude that u, u' and u'' are bounded on $[0,\infty)$, hence h, h' and h'' are bounded, too, and in particular, h and h' are Lipschitz continuous on $[0,\infty)$. Moreover, h is decreasing on $[0,\infty)$ according to Proposition 2, and clearly h is bounded below. Thus

$$\lim_{t\to\infty} h'(t) = 0.$$

Now, let y be an ω-limit point of u, that is there is a sequence (t_j) in $[0,\infty)$ such that

$$t_j\to\infty \ \ (j\to\infty), \quad u(t_j)\to y \ \ (j\to\infty).$$

Clearly $y \in I$. From $h'(t_j) \to 0$ $(j \to \infty)$ we obtain

$$\sum_{k \in Z_n} 2y_k \left(\frac{1}{2} - \frac{y_k}{y_{k+1} + y_{k+2}}\right) = 0 \quad \Rightarrow \quad \sum_{k \in Z_n} y_k = \sum_{k \in Z_n} \frac{2y_k^2}{y_{k+1} + y_{k+2}}.$$

Proceeding as in the proof of Proposition 2, we get

$$\Big(\sum_{k \in \mathbb{Z}_n} y_k\Big)^2 \leq \sum_{k \in \mathbb{Z}_n} \frac{2y_k^2}{y_{k+1} + y_{k+2}} \sum_{k \in \mathbb{Z}_n} y_k = \Big(\sum_{k \in \mathbb{Z}_n} y_k\Big)^2,$$

and have equality in the Cauchy–Schwarz inequality. Therefore, the vectors

$$\left(\frac{y_k}{\sqrt{y_{k+1} + y_{k+2}}}\right) \quad \text{and} \quad \big(\sqrt{y_{k+1} + y_{k+2}}\big)$$

are linearly dependent. Since both vectors are from D there is some $\alpha > 0$ such that

$$y_k = \alpha(y_{k+1} + y_{k+2}) \quad (k \in \mathbb{Z}_n).$$

Taking the sum, we obtain

$$\sum_{k \in \mathbb{Z}_n} y_k = \alpha \sum_{k \in \mathbb{Z}_n} (y_{k+1} + y_{k+2}) = 2\alpha \sum_{k \in \mathbb{Z}_n} y_k \Rightarrow \alpha = \frac{1}{2}.$$

Therefore $f(y) = 0$, and Proposition 1 proves $y = \lambda p$ for some $\lambda > 0$. Thus every ω-limit point of u is an equilibrium of the differential equation in (2). Now assume that $\lambda_1 p$ and $\lambda_2 p$ be ω-limit points of u with $\lambda_1 < \lambda_2$ and let $\delta := (\lambda_1 + \lambda_2)/2$. We can find some $t_0 \geq 0$ such that $\delta p \leq u(t_0)$. We apply (3) to $v, w : [t_0, \infty) \to \mathbb{R}$ defined by $v(t) = \delta p$, $w(t) = u(t)$. We have

$$v'(t) - f(v(t)) = 0 = w'(t) - f(w(t)) \ (t \geq t_0), \quad v(t_0) \leq w(t_0),$$

hence $v(t) \leq w(t)$, i.e. $\delta p \leq u(t)$ $(t \geq t_0)$. Since $\lambda_1 p$ is an ω-limit point of u, we get the contradiction

$$\frac{\lambda_1 + \lambda_2}{2} \leq \lambda_1.$$

Summing up, there is a single ω-limit point of u and therefore

$$\lim_{t \to \infty} u(t) = \mu p,$$

for some $\mu > 0$.

We now prove the asserted properties of μ: From (5) we get 1. as $t \to \infty$.

To prove 2. first observe that for each fixed $t \geq 0$ the function

$$x \mapsto u(t, x)$$

is continuous on D, [4, Ch. V, Th. 2.1]. Hence

$$x \mapsto \mathcal{G}(u(t, x)), \quad x \mapsto \mathcal{Q}(u(t, x))$$

are continuous functions on D. According to Proposition 2, $t \mapsto \mathcal{G}(u(t, x))$ is increasing and $t \mapsto \mathcal{Q}(u(t, x))$ is decreasing, both with limit $\mu(x)$ as $t \to \infty$

for $x \in D$. Therefore μ is lower and upper semicontinuous, hence continuous on D.

For the proof of 3. let $x \in D$, $y = (x_{k+1})_{k\in\mathbb{Z}_n}$ and $v_k = u_{k+1}(\cdot, x)$ ($k \in \mathbb{Z}_n$). Then v solves $v'(t) = f(v(t))$ and $v(0) = y$. Since (2) is uniquely solvable, we have $v = u(\cdot, y)$. As $t \to \infty$ we get $\mu(x) = \mu(y)$.

Next, 4. follows by setting

$$v(t) = \lambda u(t/\lambda, x) \quad (t \geq 0).$$

Then $v(0) = \lambda x$ and

$$v'(t) = u'(t/\lambda, x) = f(u(t/\lambda, x)) = f((1/\lambda)v(t)) = f(v(t)) \ \ (t \geq 0).$$

Thus

$$u(t, \lambda x) = v(t) = \lambda u(t/\lambda, x) \quad (t \geq 0),$$

and as $t \to \infty$ we get $\mu(\lambda x) = \lambda\mu(x)$.

Finally 5. and 6. directly follow by Proposition 2 as $t \to \infty$. ∎

Now, set

$$\gamma_n := \left(\sqrt{2\cos\left(\frac{2\pi}{n}\right) - 1}\right)^{-1} \quad (n \geq 3).$$

It is known [2] that for arbitrary n, Shapiro's inequality (1) is valid, provided that

$$x \in C_n := \bigcup_{\lambda>0} \lambda\left[\frac{1}{\gamma_n}p, \gamma_n p\right].$$

Theorem 2. *Let $x \in D$ and let $u = u(\cdot, x)$ be the solution of (2). Then*

1. *$t \mapsto \mathcal{A}(u(t))$ is eventually monotone decreasing on $[0, \infty)$;*
2. *$\mu(x) \leq \mathcal{A}(x)$ ($x \in C_n$).*

Proof. First note that $\gamma_n > 1$. According to Theorem 1, we have $u(t) \to \mu(x)p$ as $t \to \infty$. Hence, there exists $t_0 \geq 0$ such that

$$\frac{\mu(x)}{\gamma_n}p \leq u(t) \leq \mu(x)\gamma_n p \quad (t \geq t_0).$$

So $u(t) \in C_n$ ($t \geq t_0$), and as in the proof of Proposition 2, we find

$$\Big(\sum_{k\in\mathbb{Z}_n} u_k\Big)'(t) \leq 0 \quad (t \geq t_0).$$

This proves 1. If in addition $x \in C_n$, that is

$$\exists \lambda > 0 : \quad \frac{\lambda}{\gamma_n}p \leq x \leq \lambda\gamma_n p,$$

then by (4)

$$\frac{\lambda}{\gamma_n}p \le \min\{x_1,\dots,x_n\}p \le u(t) \le \max\{x_1,\dots,x_n\}p \le \lambda\gamma_n p \quad (t \ge 0).$$

Again $t \mapsto A(u(t))$ is monotone decreasing on $[0,\infty)$ and we get 2. ∎

The next result gives an implicit representation of the function μ.

Theorem 3. *Let $x \in D$ and let $u = u(\cdot, x)$ be the solution of (2). Then the integral*

$$\Gamma := \int_0^\infty \sum_{k\in\mathbb{Z}_n} \left(u_k(t)\left(1 - \frac{u_{k-1}(t)+u_{k-2}(t)}{u_{k+1}(t)+u_{k+2}(t)}\right)\right) dt$$

is convergent and

$$\mu(x) = \frac{1}{\sqrt{2n}}\sqrt{\sum_{k\in\mathbb{Z}_n} x_k(x_{k+1}+x_{k+2}) + \Gamma}.$$

Proof. We have

$$\sum_{k\in\mathbb{Z}_n} u_k'(t)(u_{k+1}(t)+u_{k+2}(t)) = \sum_{k\in\mathbb{Z}_n}\left(\frac{u_{k+1}(t)+u_{k+2}(t)}{2} - u_k(t)\right) = 0$$

on $[0,\infty)$. Thus

$$\begin{aligned}\frac{d}{dt}\Big(\sum_{k\in\mathbb{Z}_n} u_k(t)(u_{k+1}(t)+u_{k+2}(t))\Big) &= \sum_{k\in\mathbb{Z}_n} u_k(t)(u_{k+1}'(t)+u_{k+2}'(t))\\
&= \sum_{k\in\mathbb{Z}_n} u_k'(t)(u_{k-1}(t)+u_{k-2}(t))\\
&= \sum_{k\in\mathbb{Z}_n}\left(\frac{1}{2} - \frac{u_k(t)}{u_{k+1}(t)+u_{k+2}(t)}\right)(u_{k-1}(t)+u_{k-2}(t))\\
&= \sum_{k\in\mathbb{Z}_n}\frac{u_{k-1}(t)+u_{k-2}(t)}{2} - \sum_{k\in\mathbb{Z}_n} u_k(t)\frac{u_{k-1}(t)+u_{k-2}(t)}{u_{k+1}(t)+u_{k+2}(t)}\\
&= \sum_{k\in\mathbb{Z}_n}\left(u_k(t)\left(1 - \frac{u_{k-1}(t)+u_{k-2}(t)}{u_{k+1}(t)+u_{k+2}(t)}\right)\right) \quad (t \ge 0).\end{aligned}$$

Integration yields

$$\begin{aligned}\sum_{k\in\mathbb{Z}_n} u_k(t)(u_{k+1}(t)+u_{k+2}(t)) &= \sum_{k\in\mathbb{Z}_n} x_k(x_{k+1}+x_{k+2})\\
&\quad + \int_0^t \sum_{k\in\mathbb{Z}_n}\left(u_k(s)\left(1 - \frac{u_{k-1}(s)+u_{k-2}(s)}{u_{k+1}(s)+u_{k+2}(s)}\right)\right) ds \quad (t \ge 0).\end{aligned}$$

As $t \to \infty$, we obtain

$$2n\mu(x)^2 = \sum_{k \in \mathbb{Z}_n} x_k(x_{k+1} + x_{k+2}) + \Gamma. \ \blacksquare$$

More precise details on $\mu(x)$ are possible if x is periodic with period 1, 2 or 3. Clearly period 1 means $x = \lambda p$ for some $\lambda > 0$. Then $\mu(x) = \lambda$.

Theorem 4. *Let $x \in D$. If x has period $m \in \{2, 3\}$ then*

$$\mu(x) = \begin{cases} \dfrac{x_1 + x_2}{2} & \text{if } m = 2, \\ \dfrac{1}{\sqrt{3}}\sqrt{x_1x_2 + x_2x_3 + x_3x_1} & \text{if } m = 3. \end{cases}$$

Proof. First note that if $m \in \mathbb{N}$ is a divisor of n and

$$C := \{x \in D : x \text{ is } m \text{ periodic}\},$$

then C is invariant for (2), that is if $x \in C$ then $u(t, x) \in C$ $(t \geq 0)$. If $m = 2$ then $u_k = u_{k+2}$ $(k \in \mathbb{Z}_n)$, thus (2) implies

$$u_1'(t) = \frac{1}{2} - \frac{u_1(t)}{u_2(t) + u_1(t)}, \quad u_2'(t) = \frac{1}{2} - \frac{u_2(t)}{u_1(t) + u_2(t)},$$

hence

$$\frac{u_1'(t) + u_2'(t)}{2} = 0 \quad (t \geq 0).$$

Therefore $\mu(x) = (x_1 + x_2)/2$.

In case $m = 3$, we have $u_{k+2} = u_{k-1}$ $(k \in \mathbb{Z}_n)$, hence $\Gamma = 0$ in Theorem 3. Therefore

$$\mu(x) = \frac{1}{\sqrt{2n}}\sqrt{\sum_{k \in \mathbb{Z}_n} x_k(x_{k+1} + x_{k+2})} = \frac{1}{\sqrt{2n}}\sqrt{2\sum_{k \in \mathbb{Z}_n} x_kx_{k+1}}$$

$$= \frac{1}{\sqrt{2n}}\sqrt{2\frac{n}{3}(x_1x_2 + x_2x_3 + x_3x_1)} = \frac{1}{\sqrt{3}}\sqrt{x_1x_2 + x_2x_3 + x_3x_1}. \ \blacksquare$$

Remark. Combining Theorem 4 and Theorem 1, we obtain the well known inequalities

$$(x_1x_2x_3)^{1/3} \leq \frac{1}{\sqrt{3}}\sqrt{x_1x_2 + x_2x_3 + x_3x_1} \leq \frac{x_1 + x_2 + x_3}{3} \quad (x_1, x_2, x_3 \geq 0).$$

Open Problem. Numerical experiments indicate that in the general case $\Gamma \leq 0$ in Theorem 3, i.e.

$$\mu(x) \leq \frac{1}{\sqrt{2n}}\sqrt{\sum_{k \in \mathbb{Z}_n} x_k(x_{k+1} + x_{k+2})} \quad (x \in D),$$

and that

$$\mu(x) \leq \mathcal{A}(x) \quad (x \in D).$$

For $n = 4, 5, 6$ the first inequality is the better one, since

$$\frac{1}{\sqrt{2n}}\sqrt{\sum_{k\in\mathbb{Z}_n} x_k(x_{k+1}+x_{k+2})} \le \mathcal{A}(x) \quad (x \in D) \iff n \in \{3,4,5,6\},$$

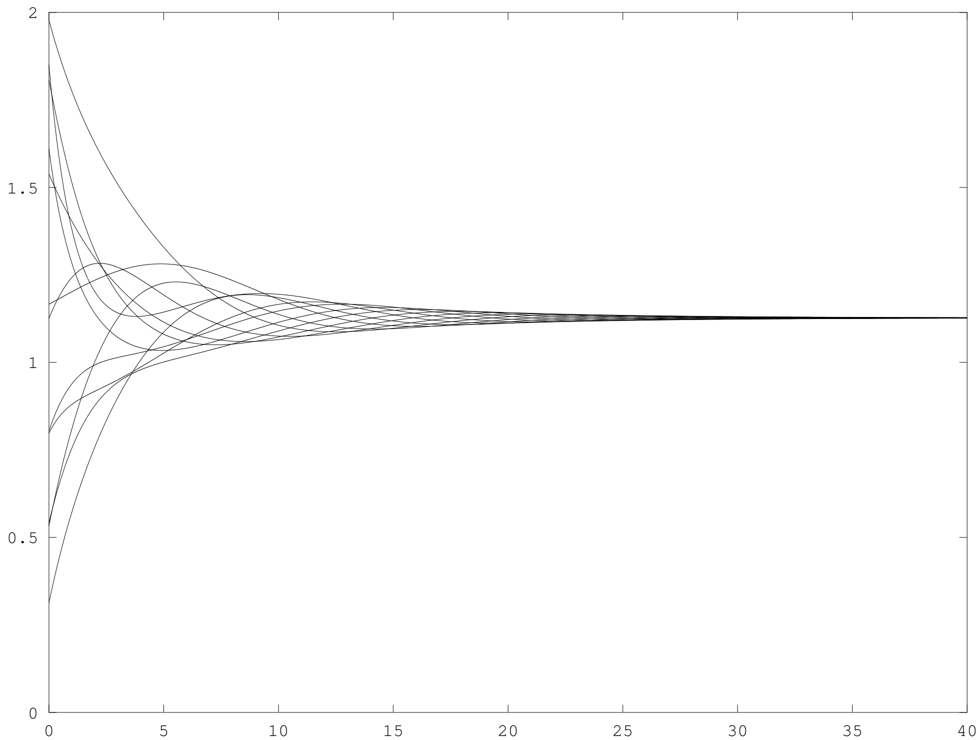
see [3, p. 19].

The following picture shows the solution of (2) in case $n = 12$ for the random initial value

$$x = (1.53842, 1.80684, 0.54198, 1.61010, 0.80408, 0.79786, \\ 1.85042, 0.31325, 1.16632, 0.53282, 1.97749, 1.12554).$$

Numerically we have $\mathcal{G}(x) = 1.0186$, $\mu(x) = 1.1271$, $\mathcal{A}(x) = 1.1721$ and

$$\frac{1}{\sqrt{24}}\sqrt{\sum_{k\in\mathbb{Z}_{12}} x_k(x_{k+1}+x_{k+2})} = 1.1570.$$



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