DE GRUYTER
OPEN

DEMONSTRATIO MATHEMATICA
Vol. XLVII No 2 2014

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SEMIDIRECT PRODUCT OF GROUPOIDS AND ASSOCIATED ALGEBRAS

Abstract. One of the pressing problems in mathematical physics is to find a generalized Poincaré symmetry that could be applied to nonflat space-times. As a step in this direction, we define the semidirect product of groupoids $\Gamma_0 \rtimes \Gamma_1$ and investigate its properties. We also define the crossed product of a bundle of algebras with the groupoid Γ_1 and prove that it is isomorphic to the convolutive algebra of the groupoid $\Gamma_0 \rtimes \Gamma_1$. We show that families of unitary representations of semidirect product groupoids in a bundle of Hilbert spaces are random operators. An important example is the Poincaré groupoid defined as the semidirect product of the subgroupoid of generalized Lorentz transformations and the subgroupoid of generalized translations.

1. Introduction

One of the main stumbling blocks in combining general relativity with quantum mechanics into a suitable generally covariant quantum field theory is the fact that the Poincaré group does not act on a curved space-time as a group of motions (e.g., [12, p. 360]). There are strong reasons to believe that to make a synthesis of general relativity and quantum physics would require a suitable generalization of the Poincaré group so as it would be able to enter into a fruitful interaction with geometry of curved manifolds (see, for instance, [3, 5, 7]).

The paper is based on the paper [11] of Pysiak on groupoids. Natural generalization of group symmetries are groupoid symmetries. In the present work, we propose such a generalization. First, we define the semidirect product of groupoids $\Gamma_0 \rtimes \Gamma_1$ where Γ_0 and Γ_1 are subgroupoids of the same groupoid Γ (Sect. 2). Then, in Sect. 3, we define the action of the subgroupoid Γ_1 on a bundle A of algebras related to the subgroupoid Γ_0. This allows us to define the crossed product of a bundle A of algebras with Γ_1, $A \rtimes \Gamma_1$, which we prove to be isomorphic to the convolutive algebra of the

2010 *Mathematics Subject Classification*: 22A22, 22A30, 47L65.
Key words and phrases: groupoid, semidirect product, crossed product algebra.

DOI: 10.2478/dema-2014-0023

groupoid $\Gamma_0 \rtimes \Gamma_1$. In Sect. 4, we show that families of unitary representations of the semidirect product groupoids in a bundle of Hilbert spaces form an algebra of random operators (in the sense of Connes [4]), which can be completed to a von Neumann algebra. Finally, in Sect. 5, we construct, as an example of the developed theory, the Poincaré groupoid being a semidirect product of the subgroupoid of generalized Lorentz transformations and the subgroupoid of generalized translations.

To fix notation, we briefly describe the groupoid structure (for the definition see [2, p. 85] or [8, p. 269]). A groupoid Γ over X, or a groupoid with base X, is a 7-tuple $(\Gamma, X, d, r, \epsilon, \iota, m)$ consisting of the following elements: (1) sets Γ and X, (2) surjections $(d, r) : \Gamma \to X$, called source and target map, respectively, (3) injection $\epsilon : X \to \Gamma$, $x \mapsto \epsilon(x)$, called identity section or simply identity, (4) map $\iota : \Gamma \to \Gamma$, $\gamma \mapsto \iota(\gamma) = \gamma^{-1}$, called inversion map. Moreover, a composition law is defined $m : \Gamma^{(2)} \to \Gamma$, $(\gamma, \xi) \mapsto m(\gamma, \xi) = \gamma \circ \xi$, with the domain $\Gamma^{(2)} := \{(\gamma, \xi) \in \Gamma \times \Gamma : r(\xi) = d(\gamma)\}$ such that the following axioms are satisfied: (i) (associativity law) for arbitrary γ, ξ, $\eta \in \Gamma$ the triple product $(\gamma \circ \xi) \circ \eta$ is defined iff $\gamma \circ (\xi \circ \eta)$ is defined. In such a case, we have $(\gamma \circ \xi) \circ \eta = \gamma \circ (\xi \circ \eta)$, (ii) (identities) for each $\gamma \in \Gamma$, $\epsilon(r(\gamma)) \circ \gamma = \gamma \circ \epsilon(d(\gamma)) = \gamma$, (iii) (inverses) for each $\gamma \in \Gamma, \gamma \circ \iota(\gamma) = \epsilon(r(\gamma))$, $\iota(\gamma) \circ \gamma = \epsilon(d(\gamma))$.

For each $\gamma \in \Gamma$, the sets: $\Gamma^x = \{\gamma \in \Gamma : r(\gamma) = x\}$, $x \in X$, and $\Gamma_x = \{\gamma \in \Gamma : d(\gamma) = x\}$, $x \in X$, are defined. If $H \subset \Gamma$ and H is closed under multiplication and inverses, H can be naturally given the structure of the subgroupoid of Γ. A groupoid Γ is called a Lie groupoid if the sets Γ and X are manifolds, the maps d and r are surjective submersions, and the structure maps are smooth [2]. A groupoid Γ is called transitive if for every pair $x, y \in X$, there exists $\gamma \in \Gamma$ such that $d(\gamma) = x$ and $r(\gamma) = y$.

2. Semidirect products of groupoids

Given two groups one can obtain a new group in various ways. The resulting product group could be a direct product, a semidirect product, etc. A direct product of groupoids could be easily defined (see for instance [2]). The concept of a semi-direct product of groupoids has been introduced in [1] in the framework of category theory. The definition we adopt below closely follows [1], although it is slightly less general (we assume the same base space of two groupoids). On the other hand, it is better suited to the crossed-product construction in Section 3 and possible physical applications.

Let Γ be a topological, locally compact, and locally trivial groupoid over X [10]. Let us notice that the local triviality of a groupoid implies its transitivity [9, 11]. With these assumptions, there exists the Haar system $\{\lambda_x\}_{x \in X}$ on Γ [11]. We impose the minimal assumptions necessary for the

convolution (see Section 3) to be well defined. Since our target algebras consist of L^1 functions, we are interested in the measurability properties rather than in topology.

Let us recall the notion of an *isotropy groupoid* of the groupoid Γ

$$\Gamma_0 := \bigcup_{x\in X} \Gamma_x^x,$$

where $\Gamma_x^x = \Gamma_x \cap \Gamma^x$. The subgroupoid Γ_0 is a wide subgroupoid of Γ, i.e., it is the subgroupoid of Γ with the same base space.

Let us now consider a wide subgroupoid Γ_1 of Γ such that $r \times d : \Gamma \to X \times X$, given by

$$(r \times d)(\gamma) = (r(\gamma), d(\gamma)),$$

is a surjection, i.e., the subgroupoid Γ_1 is transitive. Here d and r are source and target mappings in Γ, respectively.

With the same symbol $\{\lambda_x\}_{x\in X}$, we denote the Haar system obtained by the restriction of measures λ_x to the sets $\Gamma_{0,x}$. In the following, we write $d\lambda_x(\gamma_0) = d\gamma_0$ for the measure λ_x on $\Gamma_{0,x}$. We also assume, for simplicity, that the groups $\Gamma_{0,x}$ are unimodular. It is straightforward to see that the restricted measures are right-invariant Haar measures on locally compact groups $\Gamma_{0,x}$ [11].

Definition 1. The *semidirect product* $\Gamma_0 \rtimes \Gamma_1$ of the groupoids Γ_1 and Γ_0 is the groupoid given by the following elements:

- set
$$\bar{\Gamma} = \Gamma_0 * \Gamma_1 = \{(\gamma_0, \gamma_1) \in \Gamma_0 \times \Gamma_1 : d(\gamma_0) = r(\gamma_1)\};$$
- source and target
$$\bar{d}(\gamma_0, \gamma_1) = d(\gamma_1),$$
$$\bar{r}(\gamma_0, \gamma_1) = r(\gamma_0);$$
- multiplication
$$(\gamma_0, \gamma_1) \circ (\gamma_0', \gamma_1') = (\gamma_0 \circ \alpha_{\gamma_1}(\gamma_0'), \gamma_1 \circ \gamma_1'),$$
provided that γ_1 and γ_1' are composable; here we have introduced the following abbreviation (representing the action of Γ_1 on Γ_0)
$$\alpha_{\gamma_1}(\gamma_0) = \gamma_1 \circ \gamma_0 \circ \gamma_1^{-1},$$
for $(\gamma_0, \gamma_1^{-1}) \in \bar{\Gamma}$; it can be easily seen that $\alpha_{\gamma_1} : \Gamma_{0,d(\gamma_1)} \to \Gamma_{0,r(\gamma_1)}$ is an isomorphism of groups;
- identity
$$\epsilon_{\bar{\Gamma}}(x) = (\epsilon_{\Gamma_0}(x), \epsilon_{\Gamma_1}(x)) = (\epsilon(x), \epsilon(x)),$$
where ϵ_{Γ_0} and ϵ_{Γ_1} are identities of Γ_0 and Γ_1, respectively;

- inverse

$$(\gamma_0, \gamma_1)^{-1} = (\alpha_{\gamma_1^{-1}}(\gamma_0^{-1}), \gamma_1^{-1}).$$

Let $\gamma, \gamma' \in \Gamma$. The elements $\gamma \sim \gamma'$ are equivalent (in the sense of equivalence relation) if there exists $\gamma_0 \in \Gamma_0$, with $d(\gamma_0) = r(\gamma)$, such that $\gamma_0 \circ \gamma = \gamma'$. On the set $\Gamma/\Gamma_0 = \{[\gamma]_\sim : \gamma \in \Gamma\}$, we introduce the groupoid structure [9] by defining

$$\tilde{d}([\gamma]) = d(\gamma), \quad \tilde{r}([\gamma]) = r(\gamma),$$
$$[\gamma] \circ [\gamma'] = [\gamma \circ \gamma']$$

provided that $d(\gamma) = r(\gamma')$,

$$[\gamma]^{-1} = [\gamma^{-1}], \quad \tilde{\epsilon}(x) = [\epsilon(x)].$$

Proposition 1. *The groupoid Γ_1 is isomorphic with the groupoid Γ/Γ_0 if and only if the semidirect product $\Gamma_0 \rtimes \Gamma_1$ is isomorphic with the groupoid Γ.*

Proof. $\Rightarrow$) Let the isomorphism assumed in the Proposition be denoted by $j : \Gamma_1 \to \Gamma/\Gamma_0$. We define $J : \Gamma_0 \rtimes \Gamma_1 \to \Gamma$ by

$$J(\gamma_0, \gamma_1) = \gamma_0 \circ \gamma_1.$$

First, we show that J is a bijection. Let $\gamma \in \Gamma$ and $[\gamma] \in \Gamma/\Gamma_0$. One has $\gamma_1 = j^{-1}([\gamma]) \in \Gamma_1$ and let $\gamma_0 = \gamma \circ \gamma_1^{-1}$. From the isomorphism property of j, we have $d(\gamma_1) = \tilde{d}([\gamma])$, $r(\gamma_1) = \tilde{r}([\gamma])$. But $\tilde{d}([\gamma]) = d(\gamma)$, $\tilde{r}([\gamma]) = r(\gamma)$ and therefore

$$d(\gamma_0) = d(\gamma_1^{-1}) = r(\gamma_1),$$
$$r(\gamma_0) = r(\gamma) = r(\gamma_1).$$

Hence, $(\gamma_0, \gamma_1) \in \Gamma_0 \rtimes \Gamma_1$, $(\gamma_0, \gamma_1) = J^{-1}(\gamma)$.

To show that J is a homomorphism, we apply J to both sides of $(\gamma_0, \gamma_1) \circ (\gamma_0', \gamma_1') = (\gamma_0 \circ \alpha_{\gamma_1}(\gamma_0'), \gamma_1 \circ \gamma_1')$ to obtain

$$J((\gamma_0, \gamma_1) \circ (\gamma_0', \gamma_1')) = J(\gamma_0, \gamma_1) \circ J(\gamma_0', \gamma_1').$$

$\Leftarrow$) Let $I : \Gamma \to \Gamma_0 \rtimes \Gamma_1$ be an isomorphism of groupoids. Then $I(\gamma) = (\gamma_0, \gamma_1)$. We define $i : \Gamma/\Gamma_0 \to \Gamma_1$ by

$$i([\gamma]) = \mathrm{pr}_2(I(\gamma)) = \gamma_1.$$

Let us notice that if $I(\gamma) = (\gamma_0, \gamma_1)$, i.e., $\gamma = \gamma_0 \circ \gamma_1$ then $\gamma_1 \in [\gamma]$. Let now $\gamma' \in [\gamma]$. This means that there exists $\bar{\gamma}_0 \in \Gamma_0$ such that

$$\gamma' = \gamma_0' \circ \gamma_1' = \bar{\gamma}_0 \circ \gamma = \bar{\gamma}_0 \circ \gamma_0 \circ \gamma_1,$$

which gives $(\gamma_0', \gamma_1') = (\bar{\gamma}_0 \circ \gamma_0, \gamma_1)$ under the action of the isomorphism I and, in particular, $\gamma_1' = \gamma_1$. This shows that the mapping i is well defined.

The mapping i is a surjection since, for $\gamma \in \Gamma_1$, one has $i([\gamma]) = \gamma$, and it is also an injection since $(i^{-1})(\gamma_1) = [\gamma_1]$. Being a homomorphism it is, therefore, an isomorphism. ∎

3. Algebras associated with the semidirect product of groupoids

In this section, we define the action of the subgroupoid Γ_1 on a bundle A of algebras related to the subgroupoid Γ_0. The action, denoted by $A \rtimes \Gamma_1$, is called the crossed product of a bundle A of algebras with Γ_1. We also prove that $A \rtimes \Gamma_1$ is isomorphic to the convolutive algebra of the groupoid $\Gamma_0 \rtimes \Gamma_1$. Similar ideas are developed by Goehle [6] in the context of C^*-algebras.

With a semidirect product of groupoids we can associate various algebraic structures. Let us first define the bundle A of algebras $(A_x, \bullet)_{x\in X}$ where $A_x = L^1(\Gamma_{0,x})$ with convolution

$$(a_1 \bullet a_2)(\gamma_0) = \int_{\Gamma_{0,x}} a_1(\gamma_0')a_2(\gamma_0'^{-1} \circ \gamma_0)d\gamma_0'.$$

The action $\alpha : \Gamma_1 \times \Gamma_0 \to \Gamma_0$, as it is defined in Section 2, induces the dual action $\beta : \Gamma_1 \times A \to A$ given by

$$\beta_{\gamma_1} : A_{r(\gamma_1)} \to A_{d(\gamma_1)}, \qquad (\beta_{\gamma_1}(a))(\gamma_0) = a(\alpha_{\gamma_1}(\gamma_0)).$$

This allows us to formulate the following definition

Definition 2. The *crossed product* of the bundle of algebras $A = (A_x, \bullet)_{x\in X}$ and the groupoid Γ_1 is the algebra

$$\mathcal{B} = A \rtimes \Gamma_1 = \{F \in L^1(\Gamma_1, A) : F(\gamma_1) \in A_{r(\gamma_1)}, \forall \gamma_1 \in \Gamma_1\},$$

with twisted convolution as multiplication

$$(F_1 \circledast F_2)(\gamma_1) = \int_{\Gamma_1^{r(\gamma_1)}} F_1(\gamma_1') \bullet \beta_{\gamma_1'^{-1}}(F_2(\gamma_1'^{-1} \circ \gamma_1))d\gamma_1'.$$

By $d\gamma_1'$ we denote the restriction of the suitable Haar measure λ_x on Γ to $\Gamma_1^{r(\gamma_1)}$.

We also define the algebra $\mathcal{A} = (L^1(\Gamma_0 \rtimes \Gamma_1, \mathbb{C}), *)$. The fact that functions $f_1, f_2 \in \mathcal{A}$ are integrable allows us to define their convolution in the following way

$$(f_1 * f_2)(\gamma_0, \gamma_1) = \int_{\Gamma_1^{r(\gamma_1)}} \int_{\Gamma_{0,r(\gamma_1)}} f_1(\gamma_0', \gamma_1') f_2\big((\gamma_0', \gamma_1')^{-1} \circ (\gamma_0, \gamma_1)\big) d\gamma_0' d\gamma_1'.$$

Theorem 1. *The algebra $\mathcal{B}$ is isomorphic with the algebra $\mathcal{A}$; the isomorphism $K : \mathcal{B} \to \mathcal{A}$ is given by*

$$(KF)(\gamma_0, \gamma_1) = (F(\gamma_1))(\gamma_0), \qquad \textit{for all } (\gamma_0, \gamma_1) \in \bar{\Gamma}.$$

Proof. First, we show that K is an homomorphism of algebras. Indeed,

$$\begin{aligned}
&(KF_1 * KF_2)(\gamma_0,\gamma_1)\\
&\quad= \int_{\Gamma_1^{r(\gamma_1)}}\int_{\Gamma_{0,r(\gamma_1)}} (KF_1)(\gamma_0',\gamma_1')(KF_2)\big((\gamma_0',\gamma_1')^{-1}\circ(\gamma_0,\gamma_1)\big)d\gamma_0'd\gamma_1'\\
&\quad= \int_{\Gamma_1^{r(\gamma_1)}}\int_{\Gamma_{0,r(\gamma_1)}} (KF_1)(\gamma_0',\gamma_1')(KF_2)\big(\alpha_{\gamma_1'^{-1}}(\gamma_0'^{-1}\circ\gamma_0),\gamma_1'^{-1}\circ\gamma_1\big)d\gamma_0'd\gamma_1'\\
&\quad= \int_{\Gamma_1^{r(\gamma_1)}}\int_{\Gamma_{0,r(\gamma_1)}} (F_1(\gamma_1'))(\gamma_0')(F_2(\gamma_1'^{-1}\circ\gamma_1))\big(\alpha_{\gamma_1'^{-1}}(\gamma_0'^{-1}\circ\gamma_0)\big)d\gamma_0'd\gamma_1'\\
&\quad= \int_{\Gamma_1^{r(\gamma_1)}}\int_{\Gamma_{0,r(\gamma_1)}} (F_1(\gamma_1'))(\gamma_0')\big(\beta_{\gamma_1'^{-1}}F_2(\gamma_1'^{-1}\circ\gamma_1)\big)(\gamma_0'^{-1}\circ\gamma_0)d\gamma_0'd\gamma_1'\\
&\qquad\qquad= \big((F_1\circledast F_2)(\gamma_1)\big)(\gamma_0).
\end{aligned}$$

In the second equality, we have used the following formula

$$\begin{aligned}
(\gamma_0',\gamma_1')^{-1}\circ(\gamma_0,\gamma_1) &= (\gamma_1'^{-1}\circ\gamma_0'^{-1}\circ\gamma_1',\gamma_1'^{-1})\circ(\gamma_0,\gamma_1)\\
&= (\gamma_1'^{-1}\circ\gamma_0'^{-1}\circ\gamma_1'\circ\alpha_{\gamma_1'^{-1}}(\gamma_0),\gamma_1'^{-1}\circ\gamma_1)\\
&= (\alpha_{\gamma_1'^{-1}}(\gamma_0'^{-1}\circ\gamma_0),\gamma_1'^{-1}\circ\gamma_1).
\end{aligned}$$

It remains to show that K is a bijection. Let $F\in L^1(\Gamma_1,A)$ and suppose that $KF=0$, i.e.,

$$(KF)(\gamma_0,\gamma_1) = (F(\gamma_1))(\gamma_0) = 0,$$

for almost all $(\gamma_0,\gamma_1)\in\bar\Gamma$ in the sense of L^1. Therefore, $F(\gamma_1)=0$ for almost all $\gamma_1\in\Gamma_1$, and $(F(\gamma_1))(\gamma_0)=0$ for almost all $\gamma_0\in\Gamma_{0,r(\gamma_1)}$. Thus $F=0$ which shows that K is an injection.

It is also a surjection since for every $f\in L^1(\Gamma_0\rtimes\Gamma_1,\mathbb{C})$ there exists $F\in L^1(\Gamma_1,A)$, namely $(F(\gamma_1))(\gamma_0)=f(\gamma_0,\gamma_1)$. ∎

4. Unitary representations and random operators

Let us now investigate the unitary representations of groupoids and the associated algebras.

Definition 3. [10] Unitary representation of a groupoid Γ over X is a pair $(\mathcal{U},\mathcal{H})$ where $\mathcal{H}$ is a bundle of Hilbert spaces over X, $\mathcal{H}=\{H_x\}_{x\in X}$, and $\mathcal{U}=\{\mathcal{U}(\gamma):\gamma\in\Gamma\}$ is a family of unitary transformations $\mathcal{U}(\gamma):H_{d(\gamma)}\to H_{r(\gamma)}$ such that

1. $\mathcal{U}(\epsilon(x))=\mathrm{id}_{H_x}$ for $x\in X$,
2. $\mathcal{U}(\gamma_1\circ\gamma_2)=\mathcal{U}(\gamma_1)\mathcal{U}(\gamma_2)$ for $(\gamma_1,\gamma_2)\in\Gamma^{(2)}$,
3. $\mathcal{U}(\gamma^{-1})=\mathcal{U}(\gamma)^{-1}$, for almost all $\gamma\in\Gamma$ with respect to the measure as it is defined in [10, p. 92],

4. for every $\varphi, \psi \in L^2(X, \mathcal{H})$ the function
$$\Gamma \ni \gamma \mapsto \big(\mathcal{U}(\gamma)\varphi(d(\gamma)), \psi(r(\gamma))\big)_{r(\gamma)} \in \mathbb{C}$$
is measurable on Γ.

Let now $(\mathcal{U}_0, \mathcal{H})$ be a unitary representation of the isotropy groupoid Γ_0 in a Hilbert bundle $\mathcal{H} = \{H_x\}_{x\in X}$ over X, and $\{i_x^y\}_{x,y\in X}$ a family of isomorphisms of Hilbert spaces, $i_x^y : H_x \to H_y$.

Definition 4. The simple extension of the representation $(\mathcal{U}_0, \mathcal{H})$ of the groupoid Γ_0 to the groupoid $\Gamma = \Gamma_0 \rtimes \Gamma_1$ is the representation $(\mathcal{U}, \mathcal{H})$ of Γ given by
$$\mathcal{U}(\gamma_0, \gamma_1) = \mathcal{U}_0(\gamma_0) \circ i_x^y,$$
for $\gamma_0 \in \Gamma_{0,y}, \gamma_1 \in \Gamma_{1,x}$ such that $x = d(\gamma_1), y = r(\gamma_1)$.

It is easy to see that the family $\{i_x^y\}_{x,y\in X}$ determines the unitary representation of the groupoid Γ_1 in the Hilbert bundle $\{H_x\}_{x\in X}$.

Let us denote $\mathcal{U}_1(\gamma_1)h_x = i_x^y(h_x)$, $h_x \in H_x$ for $\gamma_1 \in (\Gamma_1)_x^y$, and let us notice that the following commutation relation is satisfied
$$\mathcal{U}_1(\gamma_1)\mathcal{U}_0(\gamma_0)\mathcal{U}_1(\gamma_1^{-1}) = \mathcal{U}_0(\alpha_{\gamma_1}(\gamma_0)),$$
for γ_1, γ_0 such that $d(\gamma_1) = d(\gamma_0)$.

We now recall the definition of a random operator [4].

Definition 5. A *random operator* in a Hilbert bundle $\mathcal{H} = \{H_x\}_{x\in M}$ is defined to be a map
$$M \ni x \mapsto b_x \in \mathcal{B}(H_x),$$
where $\mathcal{B}(H_x)$ denotes the algebra of bounded operators on H_x. It must satisfy the following conditions:

1. if $\{\psi_i\}_{i=1}^{\infty}$ is a measurable field of bases in $\mathcal{H}$ then the function
$$M \ni x \mapsto (B_x\psi_i(x), \psi_j(x))_x, \quad \text{for } i, j = 1, 2, \ldots$$
is measurable,
2. the family of operators B_x, $x \in M$, is essentially bounded in the operator norm, i.e., $\operatorname{ess\,sup}_{x\in X}\|B_x\| < \infty$.

For $a \in L^1(\Gamma_0)$, we define the operators $\mathcal{U}_{0,x}(a) : H_x \to H_x$ by
$$\mathcal{U}_{0,x}(a)(h_x) := \int_{\Gamma_{0,x}} a(\gamma)\mathcal{U}_0(\gamma)h_x d\lambda_x,$$
for $h_x \in H_x$; $\{\lambda_x\}_{x\in X}$ is here the Haar system on the groupoid Γ_0 defined by the Haar measure on the group G.

Proposition 2. *The family of operators $r_a := \{\mathcal{U}_{0,x}(a)\}_{x\in X}$ forms a random operator in the Hilbert bundle $\mathcal{H}$.*

Proof. Since $a \in L^1(\Gamma_0)$, it is enough to prove that the condition (2) is satisfied. Indeed,

$$\begin{aligned}\|\mathcal{U}_{0,x}(a)\| &= \sup_{\|h_x\|\leq 1} \|\mathcal{U}_{0,x}(a)h_x\| = \sup_{\|h_x\|\leq 1} \left\| \int_{\Gamma_{0,x}} a(\gamma)\mathcal{U}_0(\gamma)h_x d\lambda_x \right\| \\ &\leq \sup_{\|h_x\|\leq 1} \int_{\Gamma_{0,x}} |a(\gamma)| \, \|\mathcal{U}_0(\gamma)h_x\| d\lambda_x = \sup_{\|h_x\|\leq 1} \int_{\Gamma_{0,x}} |a(\gamma)| \, \|h_x\| d\lambda_x \\ &\leq \int_{\Gamma_{0,x}} |a(\gamma)| d\lambda_x \leq \mathrm{vol}(\mathrm{supp}\, a) \cdot \sup |a(\gamma)| < \infty. \ \blacksquare\end{aligned}$$

Let us notice that for $\gamma_0 \in \Gamma_{0,x}$, $\gamma \in \Gamma_x^y$, we have $\alpha_\gamma(\gamma_0) \in \Gamma_{0,y}$. Let us also define

$$(\alpha_\gamma^*(a))(\gamma_0) = a(\alpha_\gamma(\gamma_0))),$$

for $a \in L^1(\Gamma_0)$ and $\gamma_0 \in \Gamma_0$, $\gamma \in \Gamma_x^y$.

The following transformation rules for the operators r_a hold

$$\begin{aligned}\mathcal{U}_0(\gamma_0)\mathcal{U}_{0,x}(a)\mathcal{U}_0(\gamma_0^{-1}) &= \mathcal{U}_{0,x}(\alpha_{\gamma_0}^*(a)), \\ \mathcal{U}_1(\gamma_1)\mathcal{U}_{0,x}(a)\mathcal{U}_1(\gamma_1^{-1}) &= \mathcal{U}_{0,y}(\alpha_{\gamma_1}^*(a)).\end{aligned}$$

If an operator r_a, $a \in L^1(\Gamma_0)$ satisfies the above conditions, it is said to be equivariant. Let us notice, that the second rule allows us to call the representation of $L^1(\Gamma_0)$ in $\mathcal{H}$ a *covariant representation* with respect to the induced groupoid action α^*.

Let $\mathcal{M}_0$ denote the algebra of operators of the form r_a, $a \in L^1(\Gamma_0)$. Then the algebra $\mathcal{M} = (\mathcal{M}_0)''$ is a von Neumann algebra. The bicommutant $(\mathcal{M}_0)''$ is considered here in the Hilbert space $\int_\oplus H_x d\mu(x)$. We call $\mathcal{M}$ the von Neumann algebra of the groupoid Γ_0. In a forthcoming publication we will describe the von Neumann algebra of the semidirect product of groupoids $\Gamma_0 \rtimes \Gamma_1$.

5. An example—the Poincaré groupoid

In this section, we consider a special case of the above constructions which may have direct applications to physics. Let E be a frame bundle over M (we can think of M as of space-time). We define

$$\tilde{\Gamma} = \{(p_1, p_2) : p_1, p_2 \in E\},$$

and introduce the following equivalence relation in $\tilde{\Gamma}$

$$(p_1, p_2) \sim (p_1', p_2') \Leftrightarrow \exists g \in G,\ p_1' = p_1 g, p_2' = p_2 g.$$

To simplify notation, the equivalence class of the element (p_1, p_2) will be denoted by $[p_1, p_2]$. We also denote $\tilde{\Gamma}/\sim \ = \Gamma = E \times_G E$. We introduce the groupoid structure in Γ in the following way

- composition $[p_1, p_2] \circ [p_3, p_4] = [p_1, p_4 g^{-1}]$ is defined only if there exists $g \in G$ such that $p_3 = p_2 g$,
- source and target
$$d([p_1, p_2]) = \pi_M(p_2),$$
$$r([p_1, p_2]) = \pi_M(p_1),$$
- identity $\epsilon(x) = [p, p]$, $\pi_M(p) = x$,
- inverse $[p_1, p_2]^{-1} = [p_2, p_1]$.

The following sets are naturally defined

$$\Gamma^x = \{[p_1, p_2] : \pi_M(p_1) = x\},$$
$$\Gamma_y = \{[p_1, p_2] : \pi_M(p_2) = y\}.$$

In the literature groupoid Γ is called *gauge groupoid*. It is a transitive groupoid. Indeed, for any $x, y \in M$ one finds $[p_1, p_2] \in \Gamma$ such that $\pi_M(p_1) = x$, $\pi_M(p_2) = y$.

We construct two subgroupoids of Γ: the subgroupoid $\Gamma_0 = \{[p_1, p_1 g] : p_1 \in E, g \in G\}$ of the gauge groupoid Γ consists of equivalence classes of Lorentz transformations

$$(p_1, p_1 g) \sim (p_1 g_0, p_1 g g_0),$$

and the subgroupoid $\Gamma_1 = \{[s(x), s(y)] : x, y \in M\}$, where s is a measurable cross section $s : M \to E$, consisting of generalized translations in M. In fact, there is an isomorphism between Γ_1 and $M \times M$ for each s. Note that since we do not assume that s is continuous, the frame bundle need not to be trivial.

Let us notice that Γ_0 is the isotropy groupoid of the groupoid Γ, i.e., $\Gamma_0 = \{\Gamma_x^x\}_{x \in M}$ where $\Gamma_x^x = \Gamma_x \cap \Gamma^x$.

Proposition 3. *The semidirect product $\Gamma_0 \rtimes \Gamma_1$, for any cross section s, is isomorphic with the gauge groupoid Γ.*

Proof. It is enough to show that there exists an isomorphism $\Gamma_1 \simeq \Gamma/\Gamma_0$. Let $\rho : \Gamma \to \Gamma/\Gamma_0$ be a canonical projection; it is a homomorphism of groupoids. Let further $i : \Gamma/\Gamma_0 \to \Gamma_1$ be as defined in the proof of Proposition 1. We can write

$$i\big(\rho([s(x)g, s(y)])\big) = i\big(\rho([s(x)g, s(x)] \circ [s(x), s(y)])\big) = [s(x), s(y)].$$

Therefore, i is a homomorphism of groupoids. It is also a bijection. Indeed, i is obviously a surjection on Γ_1, and

$$i^{-1}([s(x), s(y)]) = \rho([s(x), s(y)]$$

shows that it is an injection. ∎

Let us notice that translations in E depend on the cross section s; this can be regarded as a kind of gauge fixing. If G is the Lorentz group, the above proposition justifies calling the gauge groupoid the Poincaré groupoid (as a semidirect product of generalized Lorentz transformations and generalized translations).

For the Poincaré groupoid, the Haar system $\{\lambda_x\}_{x\in M}$ on $\Gamma_{0,x}$ is of the form

$$\int_{\Gamma_{0,x}} f(\gamma_0)d\lambda_x(\gamma_0) = \int_G f([p_0, p_0 g])dg,$$

where G is the Lorentz group, dg is a Haar measure on G and p_0 is a selected element of E_x. On Γ_1 one has

$$\int_{\Gamma_{1,y}} f(\gamma_1)d\lambda_y = \int_M f([s(x), s(x)])d\mu(x),$$

where μ is a Lebesgue measure on M.

We have the bundle of algebras $A = \{A_x\}_{x\in M}$ where $A_x = L^1(\Gamma_{0,x})$ and, on the strength of Theorem 1, the crossed product $\mathcal{B} = A \rtimes \Gamma_1$ is isomorphic with the algebra $(\mathcal{A}, *)$ associated with the Poincaré groupoid $\Gamma = \Gamma_0 \rtimes \Gamma_1$. Let us write down the product in $\mathcal{A}$ explicitly.

Let $f_1, f_2 \in \mathcal{A}$ and $[s(x)g, s(y)] \in \Gamma = \Gamma_0 \rtimes \Gamma_1$ then we have

$$\begin{aligned}
&(f_1 * f_2)([s(x)g, s(x)], [s(x), s(y)]) \\
&\quad = \int_{\Gamma_1^x}\int_{\Gamma_{0,x}} f_1([s(x)g', s(x)], [s(x), s(z)]) \\
&\qquad \times f_2(([s(x)g', s(x)], [s(x), s(z)])^{-1} \circ ([s(x)g, s(x)], [s(x), s(y)])dg'd\mu(z) \\
&\quad = \int_{\Gamma_1^x}\int_{\Gamma_{0,x}} f_1([s(x)g', s(x)], [s(x), s(z)]) \\
&\qquad\qquad \times f_2([s(z)g'^{-1}g, s(z)], [s(z), s(y)])dg'd\mu(z).
\end{aligned}$$

In this formula the nonlocal character of the generalized Poincaré symmetry (of the groupoid type) can explicitly be seen.

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Received August 30, 2012; revised version February 11, 2013.
Communicated by Z. Pasternak-Winiarski.