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ENDOMORPHISMS OF IMPLICATION ALGEBRAS

Abstract. In this note we prove that if two implication algebras have isomorphic monoids of endomorphisms then they are isomorphic.

It is known that Boolean algebras (they are the algebraic counterpart of classical propositional logic) are determined by their endomorphism monoids (see [4, 5, 7]). Implication algebras, also called Tarski algebras or semi-Boolean algebras (see [1, 6]), are the algebraic counterpart of the implication fragment of classical propositional logic. Implication algebras form a variety which corresponds to the $\{\rightarrow\}$-subreducts of Boolean algebras. For a given implication algebra $\mathbf{A}$, we denote the monoid of its endomorphisms by $\mathrm{End}(\mathbf{A})$. In [3], we prove that finite implication algebras are determined by their endomorphisms. In this paper, we remove the finiteness hypothesis proving the following result:

Theorem 1. *Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two implication algebras. Then*

$$\mathbf{A}_1 \cong \mathbf{A}_2 \text{ iff } End(\mathbf{A}_1) \cong End(\mathbf{A}_2).$$

In [2] Celani and Cabrer prove that Tarski spaces with T-partial functions as arrows (see definitions below) form a category and this category is dually equivalent to the algebraic category of implication algebras.

A *Tarski space* (T-*space* for short) is a structure $\mathcal{X} = \langle X, \mathcal{K}, \tau \rangle$ such that

(1) $\langle X, \tau \rangle$ is a topological space where $\mathcal{K}$ is a basis for τ consisting of compact open subsets.
(2) If $A, B \in \mathcal{K}$ then $A \cap B^{\complement} \in \mathcal{K}$.
(3) If $x, y \in X$ and $x \neq y$ then there exists $U \in \mathcal{K}$ such that $x \notin U$ and $y \in U$.

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(4) If F is a closed subset and $\{U_i : i \in I\}$ is a directed subfamily of $\mathcal{K}$ such that $F \cap U_i \neq \emptyset$ for all $i \in I$ then $F \cap (\bigcap\{U_i : i \in I\}) \neq \emptyset$.

It follows from conditions (1), (2) and (3) that a T-space is a Hausdorff space and consequently, every compact open set is closed (and hence, clopen). Also, it follows from (2) that $\emptyset \in \mathcal{K}$.

A *T-partial function* between two T-spaces $\mathcal{X}_1$ and $\mathcal{X}_2$ is a partial map $\varphi : X_1 \longrightarrow X_2$ such that $\varphi^{-1}(U) \in \mathcal{K}_1$ for all $U \in \mathcal{K}_2$.

If $\mathcal{X}$ is a T-space, a T-partial function between $\mathcal{X}$ and itself is called a T-partial endomorphism of $\mathcal{X}$. We denote the monoid of T-partial endomorphisms of $\mathcal{X}$ by $\mathrm{Pend}(\mathcal{X})$.

In the light of the equivalence of Celani and Cabrer (see details of this equivalence in [2]), Theorem 1 is equivalent to the following theorem:

Theorem 2. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two Tarski spaces. Then*

$$\mathcal{X}_1 \cong \mathcal{X}_2 \text{ iff } Pend(\mathcal{X}_1) \cong Pend(\mathcal{X}_2).$$

Here, $\mathcal{X}_1 \cong \mathcal{X}_2$ means that there exists a T-function from X_1 onto X_2 (i.e., a T-partial function with domain X_1) which is a bijection and such that its inverse is also a T-function (see Lemma 4.8 in [2]).

So, instead of proving Theorem 1, we prove Theorem 2. With this purpose, we introduce next an important family of T-partial functions: Let $\mathcal{X} := \langle X, \mathcal{K}, \tau \rangle$ be a Tarski space. For each $U \in \mathcal{K}$ and $a \in X$, define the partial function $\varphi_{U,a}$ on X with domain U by the rule $\varphi_{U,a}(x) = a$ if $x \in U$. Clearly, $\varphi_{U,a}$ is a T-partial endomorphism because for $V \in \mathcal{K}$,

$$\varphi^{-1}_{U,a}(V) := \begin{cases} U & \text{if } a \in V\ , \\ \emptyset & \text{otherwise.} \end{cases}$$

We now state and prove various propositions and lemmas from which our main result will follow easily. Our first proposition is an easy consequence of the definition of $\varphi_{U,a}$.

Proposition 1. *For any T-partial endomorphism ψ, any $a \in X$ and any $U \in \mathcal{K}$, the following hold:*

(i) $\varphi_{\emptyset,a} = \emptyset$.

(ii) $\psi \circ \varphi_{U,a} = \begin{cases} \varphi_{U,\psi(a)} & \textit{if } a \textit{ is in the domain of } \psi, \\ \emptyset & \textit{otherwise.} \end{cases}$

(iii) $\varphi_{U,a} \circ \psi = \varphi_{\psi^{-1}(U),a}$.

In what follows, $\mathcal{X}_i := \langle X_i, \mathcal{K}_i, \tau_i \rangle$ $(i = 1, 2)$ will denote a Tarski space and $\Phi : \mathrm{Pend}(\mathcal{X}_1) \longrightarrow \mathrm{Pend}(\mathcal{X}_2)$, a monoid isomorphism. The next lemma asserts that the T-partial endomorphisms $\varphi_{U,a}$ are preserved under monoid isomorphisms.

Lemma 1.

(i) $\Phi(\emptyset) = \emptyset$.

(ii) *For* $U_1 \in \mathcal{K}_1 \setminus \{\emptyset\}$ *and* $a_1 \in X_1$ *there exist* $U_2 \in \mathcal{K}_2$ *and* $a_2 \in X_2$ *such that* $\Phi(\varphi_{U_1,a_1}) = \varphi_{U_2,a_2}$.

Proof. (i) It follows from the following facts: in any Tarski space, $\emptyset = \psi \circ \emptyset = \emptyset \circ \psi$ for each partial endomorphism ψ and any element in a semigroup with such a property is unique.

(ii) Set $\Psi := \Phi^{-1}$ and $\sigma := \Phi(\varphi_{U_1,a_1})$. By injectivity of Φ, $\Phi(\varphi_{U_1,a_1}) \neq \emptyset$. Let $b \in \mathrm{Im}\sigma$ and $c \in X_2$ such that $\sigma(c) = b$. Pick $U \in \mathcal{K}_2$ such that $b \in U$. By Proposition 1 (iii), $\varphi_{U,b} \circ \sigma = \varphi_{\sigma^{-1}(U),b}$. Observe that $\emptyset \neq \sigma^{-1}(U) \in \mathcal{K}_2$ and consequently, $\varphi_{\sigma^{-1}(U),b} \neq \emptyset$. Then, on one hand we have

$$\Psi(\varphi_{U,b}) \circ \varphi_{U_1,a_1} = \Psi(\varphi_{U,b}) \circ \Psi(\sigma) = \Psi(\varphi_{\sigma^{-1}(U),b})$$

and, on the other hand, using Proposition 1 (ii) we have

$$\Psi(\varphi_{U,b}) \circ \varphi_{U_1,a_1} = \varphi_{U_1,d}$$

where $d = \Psi(\varphi_{U,b})(a_1)$ and, as a byproduct, a_1 is in the domain of $\Psi(\varphi_{U,b})$. So,

$$\Phi(\varphi_{U_1,d}) = \varphi_{\sigma^{-1}(U),b}.$$

Now pick $V \in \mathcal{K}_1$ such that $d \in V$. Clearly, for $t \in U_1$ we have

$$\varphi_{V,a_1} \circ \varphi_{U_1,d}(t) = \varphi_{U_1,a_1}(d) = a_1,$$

so, $\varphi_{V,a_1} \circ \varphi_{U_1,d} = \varphi_{U_1,a_1}$ and, by applying Φ on both sides of this equality, we get $\Phi(\varphi_{U_1,a_1}) = \Phi(\varphi_{V,a_1}) \circ \varphi_{\sigma^{-1}(U),b}$ so that, for $t \in \sigma^{-1}(U)$ we have $\Phi(\varphi_{U_1,a_1})(t) = \Phi(\varphi_{V,a_1})(b)$ which means that $\Phi(\varphi_{U_1,a_1}) = \varphi_{\sigma^{-1}(U),e}$ where $e = \Phi(\varphi_{V,a_1})(b)$. ∎

Lemma 2. $\Phi(\varphi_{U,a}) = \varphi_{V,b}$ *and* $\Phi(\varphi_{U,a'}) = \varphi_{V',b'}$ *imply* $V = V'$.

Proof. Pick $W \in \mathcal{K}_1$ such that $a' \in W$. Then, for $t \in U$, $\varphi_{W,a} \circ \varphi_{U,a'}(t) = \varphi_{W,a}(a') = a$; so $\varphi_{W,a} \circ \varphi_{U,a'} = \varphi_{U,a}$ and, by taking the images under Φ on both sides of this equality we get $\Phi(\varphi_{W,a}) \circ \varphi_{V',b'} = \varphi_{V,b}$. From another side, for $t \in V'$, $\Phi(\varphi_{W,a}) \circ \varphi_{V',b'}(t) = \Phi(\varphi_{W,a})(b')$, that is, $\Phi(\varphi_{W,a}) \circ \varphi_{V',b'} = \varphi_{V',c}$, where $c = \Phi(\varphi_{W,a})(b')$. So, we have shown that $\varphi_{V,b} = \varphi_{V',c}$ and this means that $V = V'$ (and $b = c$). ∎

Lemma 3. *If* $U \neq \emptyset \neq U'$ *then,* $\Phi(\varphi_{U,a}) = \varphi_{V,b}$ *and* $\Phi(\varphi_{U',a}) = \varphi_{V',b'}$ *imply* $b = b'$.

Proof. Pick $W \in \mathcal{K}_1$ such that $a \in W$ and, by virtue of Lemma 1, we may set $\Phi(\varphi_{W,a}) = \varphi_{W',c}$ for some $W' \in \mathcal{K}_2$ and $c \in X_2$. Then, as $\varphi_{W,a} \circ \varphi_{U,a} = \varphi_{U,a}$, by taking the images under Φ on both sides of this equality, we get $\varphi_{W',c} \circ \varphi_{V,b} = \varphi_{V,b}$. But for $t \in V$ we have $\varphi_{W',c} \circ \varphi_{V,b}(t) = \varphi_{W',c}(b) = \varphi_{V,b}(t) = b$ from which it follows that $c = b$; so, $\Phi(\varphi_{W,a}) = \varphi_{W',b}$. With the same argument but applied now to $\varphi_{U',a}$ we conclude that $\varphi_{W',b} \circ \varphi_{V',b'} = \varphi_{V',b'}$. Since for $t \in V'$ we have that $\varphi_{V',b'}(t) = b'$ then $b' = \varphi_{W',b} \circ \varphi_{V',b'}(t) = \varphi_{W',b}(b') = b$ (and, as a byproduct. $b' \in W'$). ■

Define $\phi : X_1 \longrightarrow X_2$ by the rule

$\phi(x) = y$ iff for every $V \in \mathcal{K}_2$, there exists $U \in \mathcal{K}_1$ such that$\Phi(\varphi_{U,x}) = \varphi_{V,y}$.

By Lemma 3, ϕ is well defined and by Lemma 2, ϕ is one to one. Indeed, if $\Psi := \Phi^{-1}$ and we define $\psi : X_2 \longrightarrow X_1$ by the rule

$\psi(y) = x$ iff for every $U \in \mathcal{K}_1$, there exists $V \in \mathcal{K}_2$ such that
$$\Psi(\varphi_{V,y}) = \varphi_{U,x}.$$

It is easy to check that $\psi = \phi^{-1}$ so that ϕ is a bijection.

Proposition 2. *ϕ and ϕ^{-1} are T-functions.*

Proof. Since ϕ and ϕ^{-1} are defined symmetrically using Φ, it is enough to prove the result just for ϕ. Let $V \in \mathcal{K}_2$ and $x \in \phi^{-1}(V)$. Let $U \in \mathcal{K}_1$ such that

$$\Phi(\varphi_{U,x}) = \varphi_{V,\phi(x)}. \tag{1}$$

Since $\phi(x) \in V$, $\varphi_{V,\phi(x)} \circ \varphi_{V,\phi(x)} = \varphi_{V,\phi(x)}$. It follows from this that $x \in U$ (if $x \notin U$ then $\varphi_{U,x} \circ \varphi_{U,x} = \emptyset$ and by taking the images under Φ we have $\varphi_{V,\phi(x)} \circ \varphi_{V,\phi(x)} = \emptyset$, a contradiction) and this proves that $\phi^{-1}(V) \subseteq U$. Conversely, suppose that $x \in U$. Then, again from (1) it follows that $\phi(x) \in V$. So, $\phi^{-1}(V) = U$ and this prove that ϕ is a T-function. ■

Conclusion: Theorem 2 follows at once from the previous results and Theorem 1 follows from the equivalence of Celani and Cabrer.

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