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DE MORGAN FUNCTIONS AND FREE DE MORGAN ALGEBRAS

Abstract. It is commonly known that the free Boolean algebra on n free generators is isomorphic to the Boolean algebra of Boolean functions of n variables. The free bounded distributive lattice on n free generators is isomorphic to the bounded lattice of monotone Boolean functions of n variables. In this paper, we introduce the concept of De Morgan function and prove that the free De Morgan algebra on n free generators is isomorphic to the De Morgan algebra of De Morgan functions of n variables. This is a solution of the problem suggested by B. I. Plotkin.

1. Introduction and preliminaries

An algebra $(Q; \{+, \cdot, ^-, 0, 1\})$ with two binary, one unary and two nullary operations is called a *De Morgan algebra* if $(Q; \{+, \cdot, 0, 1\})$ is a bounded distributive lattice with least element 0 and greatest element 1 and $(Q; \{+, \cdot, ^-, 0, 1\})$ satisfies the following identities:

$$\overline{x + y} = \overline{x} \cdot \overline{y},$$
$$\overline{\overline{x}} = x,$$

where $\overline{\overline{x}} = \overline{(\overline{x})}$ ([2, 4, 6, 8, 18, 19, 24, 31, 32, 34]; In book [6], the definition of the reduct of De Morgan algebra is given). The standard fuzzy algebra $F = ([0, 1]; max(x, y), min(x, y), 1 - x, 0, 1)$ is an example of a De Morgan algebra.

Except in mathematical logic ([1, 3, 7, 14, 15, 20, 22, 23]) and algebra, De Morgan algebras (and De Morgan bisemilattices) have applications in multi-valued simulations of digital circuits too ([9, 10]).

A De Morgan algebra $\mathcal{F} = (F; \{+, \cdot, ^-, 0, 1\})$ is called a free De Morgan algebra with the system of free generators $X \subseteq F$ if the algebra $\mathcal{F}$ is generated by the subset $X \subseteq F$ and for every De Morgan algebra $\mathfrak{S} = (S; \{+, \cdot, ^-, 0, 1\})$

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and for every mapping $\mu : X \to S$, there exists a unique homomorphism: $\nu : \mathcal{F} \to \mathfrak{S}$ with $\nu|_X = \mu$. The concepts of the free bounded distributive lattice and the free Boolean algebra have similar definitions ([16]). These definitions are special cases of the general concept of the free algebra of variety ([17, 35]).

Let us consider the following De Morgan algebras: $\mathbf{2} = (\{0,1\}; \{+,\cdot,\ ^{-},0,1\})$, $\mathbf{3} = (\{0,a,1\};\{+,\cdot,\ ^{-},0,1\})$, where $\overline{a} = a$, and $\mathbf{4} = (\{0,a,b,1\};\{+,\cdot,\ ^{-},0,1\})$, where $\overline{a} = a,\ \overline{b} = b,\ a+b = 1,\ a\cdot b = 0$. (Here 0 and 1 are respectively the least and greatest elements of distributive lattice and so the other values of operations are defined uniquely. In particular, $\overline{1} = 0$, $\overline{0} = 1$, for those three algebras.)

Let us recall the following result that makes clear our approach for the definition of the concept of De Morgan functions.

Theorem 1.1. ([18]) *Every non-trivial subdirectly irreducible De Morgan algebra is isomorphic to one of the following algebras:* $\mathbf{2},\mathbf{3},\mathbf{4}$, *where* $\mathbf{2}$ *is the unique non-trivial subdirectly irreducible Boolean algebra.* ∎

As a corollary, we can state that the free n-generated De Morgan algebra is isomorphic to the subalgebra of $\mathbf{4}^{\mathbf{4}^n}$ generated by the projections. Elements of this subalgebra are functions from $\mathbf{4}^n$ into $\mathbf{4}$.

Let $B = \{0,1\}$. Define the operations $+,\cdot,^{-}$ on B by the following way: $0+0 = 0, 0+1 = 1+0 = 1+1 = 1, 0\cdot 1 = 0\cdot 0 = 1\cdot 0 = 0, 1\cdot 1 = 1, \overline{0} = 1, \overline{1} = 0$. We get the Boolean algebra $\mathbf{2} = (B;\{+,\cdot,^{-},0,1\})$ and the bounded distributive lattice $(B;\{+,\cdot,0,1\})$. Let "$\leq$" be its order. For $u = (u_1,\dots,u_n)$, $v = (v_1,\dots,v_n) \in B^n$, we define: $u \leq v$ if and only if $u_i \leq v_i$ for all $i = 1,\dots,n$, where B^n is the set of all n-element sequences of B. Here and afterwards $n \geq 1$ is a positive integer.

Definition 1.2. A Boolean function $f : B^n \to B$ is called *monotone* if

$$x \leq y \Rightarrow f(x) \leq f(y),$$

where $x, y \in B^n$.

If $u = (u_1,\dots,u_n),\ v = (v_1,\dots,v_n) \in B^n$ then we will say $u \preceq v$ if there exists k $(1 \leq k \leq n)$ such that $u_i = v_i$ for all $i \neq k$ and $u_k = 0,\ v_k = 1$. It is easy to see that a Boolean function $f : B^n \to B$ is monotone if and only if

$$x \preceq y \Rightarrow f(x) \leq f(y),\ x, y \in B^n.$$

Denote the set of all monotone Boolean functions of n variables by $\mathcal{M}_n$. We can define $f+g$ and $f\cdot g$ for any two Boolean functions of n variables by the standard way: $(f+g)(x) = f(x)+g(x)$, $(f\cdot g)(x) = f(x)\cdot g(x)$, $x \in B^n$. It is obvious that if f and g are monotone Boolean functions, then $f+g$ and $f\cdot g$

are monotone, too. Thus, we get the algebra $\mathfrak{L}_n = (\mathcal{M}_n; \{+, \cdot, 0, 1\})$ (here 0 and 1 are the constant Boolean functions) which obviously is a bounded distributive lattice. Also, let $m_n = |\mathcal{M}_n|$ be the number of monotone Boolean functions of n variables. (Note that the numbers m_n are called Dedekind's numbers.) For instance, $m_1 = 3, m_2 = 6, m_3 = 20, m_4 = 168, m_5 = 7581, m_6 = 7828354$ ([11, 21]).

Now, let $S \subseteq 2^{\{1,\ldots,n\}}$ be an antichain (or Sperner set [12, 36]) with respect to the order $\subseteq$. This means that S consists of subsets of $\{1, \ldots, n\}$, none of which is contained in any other subset from S. Note that the empty set is also considered an antichain. For an antichain $S \subseteq 2^{\{1,\ldots,n\}}$ define the following monotone Boolean function:

$$f_S(x_1, \ldots, x_n) = \sum_{s \in S} \prod_{i \in s} x_i. \tag{1.1}$$

For $S = \emptyset$ we set $f_\emptyset = 0$, and for $S = \{\emptyset\}$ we set $f_{\{\emptyset\}} = 1$. Notice that f_S does not depend on the order of the elements in the set S. It is easy to see that if $S_1 \neq S_2$ are two antichains then $f_{S_1} \neq f_{S_2}$. To see this without loss of generality suppose that there exists $s \in S_1$ such that $s \notin S_2$. We can also suppose that there does not exist $s' \in S_2$ with $s' \subseteq s$. Otherwise, we would take s' instead of s (in that case $s' \notin S_1$, because S_1 is an antichain). Take the following values of the variables:

$$x_i = \begin{cases} 1, & \text{if } i \in s, \\ 0, & \text{if } i \notin s. \end{cases}$$

For these values of variables, we have: $f_{S_1} = 1$ and $f_{S_2} = 0$.

The form (1.1) is uniquely determined by the antichain $S \subseteq 2^{\{1,\ldots,n\}}$. And conversely, we show next that every monotone Boolean function is obtained in that way.

We prove the following well-known result since we use that proof in Section 3.

Proposition 1.3. ([6, 11, 12, 16, 21, 33, 36]) *For every monotone Boolean function of n variables, there exists a unique antichain $S \subseteq 2^{\{1,\ldots,n\}}$ such that $f = f_S$.*

Proof. For $a = (a_1, \ldots, a_n) \in B^n$ let $s_a = \{i : a_i = 1\}$. Consider the set $A = \{s_a : a \in B^n,\ f(a) = 1\}$. Let S be the subset of A, consisting exactly of all minimal sets in A. Then $S \subseteq 2^{\{1,\ldots,n\}}$ is an antichain. Notice that $f(a_1, \ldots, a_n) = 1$ iff for some $s \in S$, we have $a_i = 1$ for all $i \in s$. The same is valid for f_S. Therefore, $f(a_1, \ldots, a_n) = f_S(a_1, \ldots, a_n)$ for all $(a_1, \ldots, a_n) \in B^n$, and so $f = f_S$. The uniqueness follows from the argument stated above. ∎

Define the Boolean functions:

$$\delta_n^i(x_1, \ldots, x_n) = x_i, \; i = 1, \ldots, n.$$

Theorem 1.4. ([6, 12, 16, 33, 36]) *The algebra $\mathfrak{L}_n$ is the free bounded distributive lattice with the system of free generators: $\Delta = \{\delta_n^1, \ldots, \delta_n^n\}$. Hence every free n-generated bounded distributive lattice is isomorphic to the bounded distributive lattice $\mathfrak{L}_n$.* ∎

The problem of similar characterization of the n-generated free De Morgan algebra is suggested by B. I. Plotkin (in algebraic conference, St Petersburg, Russia, 1981). In this paper, we introduce the concept of De Morgan function and prove that the free De Morgan algebra on n free generators is isomorphic to the De Morgan algebra of De Morgan functions of n variables.

Now, let us establish some further properties of monotone Boolean functions, which are used in the third section.

If $i \in \{1, \ldots, 2n\}$, then denote:

$$\sigma(i) = \begin{cases} i+n, & \text{if } 1 \le i \le n, \\ i-n, & \text{if } n+1 \le i \le 2n. \end{cases}$$

For a monotone Boolean function $f(x_1, \ldots, x_{2n}) = \sum_{s \in S} \prod_{i \in s} x_i$ consider the function:

$$f'(x_1, \ldots, x_{2n}) = \prod_{s \in S} \sum_{i \in s} x_{\sigma(i)}.$$

Clearly f' is also a monotone Boolean function.

Lemma 1.5. *For any monotone Boolean function f of $2n$ variables, the following equality holds for all $u_1, \ldots, u_n, v_1, \ldots, v_n \in B$:*

$$\overline{f(u_1, \ldots, u_n, \overline{v}_1, \ldots, \overline{v}_n)} = f'(v_1, \ldots, v_n, \overline{u}_1, \ldots, \overline{u}_n). \tag{1.2}$$

Proof. For $i = 1, \ldots, 2n$ define:

$$t_i = \begin{cases} u_i, & \text{if } 1 \le i \le n, \\ \overline{v}_{i-n}, & \text{if } n+1 \le i \le 2n. \end{cases}$$

Then we have:

$$\overline{f(u_1, \ldots, u_n, \overline{v}_1, \ldots, \overline{v}_n)} = \overline{\sum_{s \in S} \prod_{i \in s} t_i} = \prod_{s \in S} \sum_{i \in s} \overline{t}_i =$$

$$f'(\overline{t}_{\sigma(1)}, \ldots, \overline{t}_{\sigma(n)}, \overline{t}_{\sigma(n+1)}, \ldots, \overline{t}_{\sigma(2n)}) = f'(v_1, \ldots, v_n, \overline{u}_1, \ldots, \overline{u}_n).$$

The lemma is proved. ∎

2. De Morgan functions

Denote $D = B \times B = \{(0,0), (1,0), (0,1), (1,1)\} = \{0, a, b, 1\}$, where $0 = (0,0)$, $a = (1,0)$, $b = (0,1)$, $1 = (1,1)$. Defining $0 + x = x + 0 = x$ and

$1 \cdot x = x \cdot 1 = x$ for all $x \in D$, and $a + b = b + a = 1$, $a \cdot b = b \cdot a = 0$, $\overline{0} = 1$, $\overline{1} = 0$, $\overline{a} = a$, $\overline{b} = b$, we get the De Morgan algebra $\mathbf{4} = (D; \{+, \cdot, ^{-}, 0, 1\})$. Notice that

$$\begin{aligned} \overline{(u, v)} &= (\overline{v}, \overline{u}), \\ (u_1, v_1) + (u_2, v_2) &= (u_1 + u_2, v_1 + v_2), \\ (u_1, v_1) \cdot (u_2, v_2) &= (u_1 \cdot u_2, v_1 \cdot v_2), \end{aligned}$$

(here the operations on the right hand side are the operations of the Boolean algebra $\mathbf{2}$). For $x \in D$ let

$$x^* = \begin{cases} x, & \text{if } x = 0, 1, \\ a, & \text{if } x = b, \\ b, & \text{if } x = a. \end{cases}$$

Also for $c = (c_1, \ldots, c_n)$, $d = (d_1, \ldots, d_n) \in D^n$ we say that d is a *permitted modification* of c if for some k $(1 \leq k \leq n)$, we have $d_i = c_i$ for all $1 \leq i \leq n$, $i \neq k$ and

$$d_k = \begin{cases} a, & \text{if } c_k = 0, \\ 1, & \text{if } c_k = b. \end{cases}$$

Definition 2.1. A function $f : D^n \to D$ is called a *De Morgan function* if the following conditions hold:

(1) if $x_i \in \{0, 1\}$, $i = 1, \ldots, n$ then $f(x_1, \ldots, x_n) \in \{0, 1\}$,
(2) if $x_i \in D$, $i = 1, \ldots, n$ then $f(x_1^*, \ldots, x_n^*) = (f(x_1, \ldots, x_n))^*$,
(3) if $x, y \in D^n$ with $f(x) \neq b$ and y is a permitted modification of x then $f(y) \in \{f(x), a\}$.

Notice that Condition (1) is a consequence of Condition (2), however it is convenient to write it as a separate condition.

Note that it follows from Condition (1) that every De Morgan function is an extension of some Boolean function. And notice that the constant functions $f = 1$ and $f = 0$ are De Morgan functions, but the constant functions $f = a$ and $f = b$ are not. This means that 0 and 1 are the only constant De Morgan functions. Further examples of De Morgan functions are $f(x) = x$, $g(x) = \overline{x}$, $h(x, y) = x \cdot y$, $q(x, y) = x + y$, where the operations on the right hand side are the operations of the De Morgan algebra $\mathbf{4}$. We can straightforwardly verify that those functions satisfy the Conditions (1) – (3) of Definition 2.1, but it also follows from the results of the next section.

As Boolean functions, De Morgan functions (and also all functions $D^n \to D$) can be given by tables. Also note that there is an algorithm which, for a given table of a function $f : D^n \to D$, determines whether f is a De Morgan function. Let us find the complexity of this algorithm depending on the number of the rows of the table. We denote that number by k (obviously, $k = 4^n$). To test whether a function f is a De Morgan function we should check whether Condition (2) and (3) are satisfied for f (as we mentioned above, Condition (1) is a consequence of Condition (2)). And it is easy to see that to check Condition (2) we need no more than $O(k)$ operations, and to verify Condition (3) we do no more than $O(k^2)$ operations, as we should consider the pairs of rows to decide whether one of them is a permitted modification of the other (we define $O(k)$ to be a function $p(k)$ such that the ratio $\frac{p(k)}{k}$ is bounded). And thus the complexity of algorithm is polynomial (more precisely, it is not greater than $C \cdot k^2$ for some constant C).

For $x_i \in D$, we denote by (u_i, v_i) the pair from $B \times B$ which is equal to x_i, i.e. $x_i = (u_i, v_i)$. (Often we will consider $B = \{0, 1\}$ as a subset of D.)

Definition 2.2. The function $f : D^n \to D$ is called a *quasi-De Morgan function*, if there exists a Boolean function $\varphi : B^{2n} \to B$ such that

$$f(x_1, \ldots, x_n) = (\varphi(u_1, \ldots, u_n, \overline{v}_1, \ldots, \overline{v}_n), \varphi(v_1, \ldots, v_n, \overline{u}_1, \ldots, \overline{u}_n)), \tag{2.1}$$

for all $x_1, \ldots, x_n \in D$.

Proposition 2.3. *The function $f : D^n \to D$ is a quasi-De Morgan function if and only if it satisfies Conditions* (1) *and* (2) *of Definition* 2.1.

Proof. Let f be a quasi-De Morgan function. If $x_i \in \{0, 1\}$, then $u_i = v_i$ and $\varphi(u_1, \ldots, u_n, \overline{v}_1, \ldots, \overline{v}_n) = \varphi(v_1, \ldots, v_n, \overline{u}_1, \ldots, \overline{u}_n)$. Hence, $f(x_1, \ldots, x_n) \in B$. Thus, Condition (1) holds for f. Now let us check Condition (2). To do this recall that $(u, v)^* = (v, u)$. Hence,

$$\begin{aligned} f(x_1^*, \ldots, x_n^*) &= (\varphi(v_1, \ldots, v_n, \overline{u}_1, \ldots, \overline{u}_n), \varphi(u_1, \ldots, u_n, \overline{v}_1, \ldots, \overline{v}_n)) \\ &= (\varphi(u_1, \ldots, u_n, \overline{v}_1, \ldots, \overline{v}_n), \varphi(v_1, \ldots, v_n, \overline{u}_1, \ldots, \overline{u}_n))^* = (f(x_1, \ldots, x_n))^*. \end{aligned}$$

Now suppose Conditions (1) and (2) hold for f, and let us prove that there exists a Boolean function φ with condition (2.1). First we prove that there are at most 2^{4^n} functions for which Conditions (1) and (2) hold. To see this notice that there are 2^n n-tuples $(u_1, \ldots, u_n) \in B^n$. For such n-tuples f can take only two values (by Condition (1)). Further, if the n-tuple $(v_1, \ldots, v_n) \in D^n$ contains a or b, then $(v_1^*, \ldots, v_n^*) \neq (v_1, \ldots, v_n)$ and $f(v_1^*, \ldots, v_n^*)$ is uniquely determined by $f(v_1, \ldots, v_n)$ (by Condition (2)). There are $4^n - 2^n$ such n-tuples. Thus, the number of such functions does not exceed $2^{2^n} \cdot 4^{\frac{4^n - 2^n}{2}} = 2^{2^n} \cdot 2^{4^n - 2^n} = 2^{4^n}$. It is clear that for a quasi-De Morgan function f, there exists exactly one Boolean function φ with condition (2.1). Therefore, there are $2^{4^n} = 2^{2^{2n}}$ quasi-De Morgan functions. And all quasi-De Morgan functions satisfy Conditions (1) and (2). Hence, all functions $f : D^n \to D$ satisfying (1) and (2) are quasi-De Morgan functions. ∎

This proposition makes clear why those functions are called quasi-De Morgan functions. As we mentioned in the proof, for a quasi-De Morgan function $f : D^n \to D$, there exists a unique Boolean function $\varphi : B^{2n} \to B$ which satisfies (2.1). To emphasize that φ is the unique Boolean function corresponding to f, we denote it by φ_f.

Theorem 2.4. *The function $f : D^n \to D$ is a De Morgan function if and only if it is a quasi-De Morgan function and φ_f is a monotone Boolean function.*

Proof. If f is a De Morgan function then by Proposition 2.3, it is a quasi-De Morgan function. Let us prove that φ_f is monotone. Let $u = (u_1, \dots, u_{2n})$, $v = (v_1, \dots, v_{2n}) \in B^{2n}$ and for some k $(1 \leq k \leq 2n)$ $u_i = v_i$, if $i \neq k$, $u_k = 0$, $v_k = 1$. We show that $\varphi_f(u) \leq \varphi_f(v)$. Suppose it is not true, i.e. $\varphi_f(u) = 1$, $\varphi_f(v) = 0$. For $1 \leq i \leq n$ denote:

$$c_i = \begin{cases} (u_i, \overline{u_{n+i}}), & \text{if } 1 \leq k \leq n, \\ (\overline{u_{n+i}}, u_i), & \text{if } n+1 \leq k \leq 2n, \end{cases}$$

and

$$d_i = \begin{cases} (v_i, \overline{v_{n+i}}), & \text{if } 1 \leq k \leq n, \\ (\overline{v_{n+i}}, v_i), & \text{if } n+1 \leq k \leq 2n. \end{cases}$$

Suppose $1 \leq k \leq n$. Then $d = (d_1, \dots, d_n)$ is a permitted modification of $c = (c_1, \dots, c_n)$.

$$\begin{aligned} f(c) &= (\varphi_f(u_1, \dots, u_n, u_{n+1}, \dots, u_{2n}), \varphi_f(\overline{u_{n+1}}, \dots, \overline{u_{2n}}, \overline{u}_1, \dots, \overline{u}_n)) \\ &= (1, \varphi_f(\overline{u_{n+1}}, \dots, \overline{u_{2n}}, \overline{u}_1, \dots, \overline{u}_n)) \neq b. \end{aligned}$$

Analogously:

$$f(d) = (0, \varphi_f(\overline{v_{n+1}}, \dots, \overline{v_{2n}}, \overline{v}_1, \dots, \overline{v}_n)).$$

By Condition (3), we have $f(d) = f(c)$ or $f(d) = a$. This gives a contradiction with the above equalities.

Now suppose $n+1 \leq k \leq 2n$. Then c is a permitted modification of d. We have:

$$\begin{aligned} f(d) &= (\varphi_f(\overline{v_{n+1}}, \dots, \overline{v_{2n}}, \overline{v}_1, \dots, \overline{v}_n), \varphi_f(v_1, \dots, v_n, v_{n+1}, \dots, v_{2n})) \\ &= (\varphi_f(\overline{v_{n+1}}, \dots, \overline{v_{2n}}, \overline{v}_1, \dots, \overline{v}_n), 0) \neq b. \end{aligned}$$

And also

$$f(c) = (\varphi_f(\overline{u_{n+1}}, \dots, \overline{u_{2n}}, \overline{u}_1, \dots, \overline{u}_n), 1).$$

Again, by Condition (3), we have $f(c) = f(d)$ or $f(c) = a$, which is a contradiction.

In both cases, we arrived at a contradiction. Consequently, φ_f is a monotone Boolean function.

Thus, the "only if"-part of the theorem is proved. Now, let us prove the "if"-part.

Suppose that f is a quasi-De Morgan function and φ_f is a monotone Boolean function. We verify that Condition (3) holds for f. To see this, let $d = (d_1, \ldots, d_n) \in D^n$ be a permitted modification of $c = (c_1, \ldots, c_n) \in D^n$. This means that for some k $(1 \le k \le n)$ $c_i = d_i$ if $i \ne k$ and

$$d_k = \begin{cases} a, & \text{if } c_k = 0, \\ 1, & \text{if } c_k = b. \end{cases}$$

Let $c_i = (u_i, v_i)$, $d_i = (p_i, q_i)$. Then $u_i \le p_i$ and $v_i = q_i$ for all $i = 1, \ldots, n$. Therefore, $(u_1, \ldots, u_n, \overline{v}_1, \ldots, \overline{v}_n) \le (p_1, \ldots, p_n, \overline{q}_1, \ldots, \overline{q}_n)$ and $(v_1, \ldots, v_n, \overline{u}_1, \ldots, \overline{u}_n) \ge (q_1, \ldots, q_n, \overline{p}_1, \ldots, \overline{p}_n)$. Hence,

$$\varphi_f(u_1, \ldots, u_n, \overline{v}_1, \ldots, \overline{v}_n) \le \varphi_f(p_1, \ldots, p_n, \overline{q}_1, \ldots, \overline{q}_n)$$

and

$$\varphi_f(v_1, \ldots, v_n, \overline{u}_1, \ldots, \overline{u}_n) \ge \varphi_f(q_1, \ldots, q_n, \overline{p}_1, \ldots, \overline{p}_n).$$

Thus, the first coordinate of $f(c)$ is less than (or equal to) the first coordinate of $f(d)$ and the second coordinate of $f(c)$ is greater than or equal to the second coordinate of $f(d)$. Thus, if $f(c) = 0$ then $f(d) \in \{0, a\}$; if $f(c) = a$ then $f(d) = a$; and if $f(c) = 1$ then $f(d) \in \{1, a\}$. ∎

Corollary 2.5. *There are m_{2n} De Morgan functions of n variables.*

3. Free De Morgan algebras

Denote the set of all De Morgan functions of n variables by $\mathcal{D}_n$. For the functions $f, g : D^n \to D$ define $f + g$, $f \cdot g$ and $\overline{f}$ by the standard way, i.e. $(f+g)(x) = f(x)+g(x)$, $(f \cdot g)(x) = f(x) \cdot g(x)$, $\overline{f}(x) = \overline{f(x)}$, $x \in D^n$, where the operations on the right hand side are the operations of De Morgan algebra **4**. We claim that $\mathcal{D}_n$ is closed under those operations, i.e. if $f, g \in \mathcal{D}_n$, then $f + g, f \cdot g, \overline{f} \in \mathcal{D}_n$. We can verify it straightforwardly, using the definition of De Morgan function. But it is easier to prove it, using Theorem 2.4. If $f, g \in \mathcal{D}_n$ then

$$f(x_1, \ldots, x_n) = (\varphi_f(u_1, \ldots, u_n, \overline{v}_1, \ldots, \overline{v}_n), \varphi_f(v_1, \ldots, v_n, \overline{u}_1, \ldots, \overline{u}_n)),$$

and

$$g(x_1, \ldots, x_n) = (\varphi_g(u_1, \ldots, u_n, \overline{v}_1, \ldots, \overline{v}_n), \varphi_g(v_1, \ldots, v_n, \overline{u}_1, \ldots, \overline{u}_n)).$$

Hence:

$$\begin{aligned}(f + g)&(x_1, \ldots, x_n) \\ &= ((\varphi_f + \varphi_g)(u_1, \ldots, u_n, \overline{v}_1, \ldots, \overline{v}_n), (\varphi_f + \varphi_g)(v_1, \ldots, v_n, \overline{u}_1, \ldots, \overline{u}_n))\end{aligned}$$

and

$$(f \cdot g)(x_1, \ldots, x_n) \\ = ((\varphi_f \cdot \varphi_g)(u_1, \ldots, u_n, \overline{v}_1, \ldots, \overline{v}_n), (\varphi_f \cdot \varphi_g)(v_1, \ldots, v_n, \overline{u}_1, \ldots, \overline{u}_n)).$$

As φ_f, φ_g are monotone boolean functions, $\varphi_f \cdot \varphi_g$ and $\varphi_f + \varphi_g$ are monotone, as well. So by Theorem 2.4 $f + g$ and $f \cdot g$ are De Morgan functions.

Further, from Lemma 1.5 we get:

$$\begin{aligned} \overline{f}(x_1, \ldots, x_n) &= \overline{(\varphi_f(u_1, \ldots, u_n, \overline{v}_1, \ldots, \overline{v}_n), \varphi_f(v_1, \ldots, v_n, \overline{u}_1, \ldots, \overline{u}_n))} \\ &= (\overline{\varphi_f(v_1, \ldots, v_n, \overline{u}_1, \ldots, \overline{u}_n)}, \overline{\varphi_f(u_1, \ldots, u_n, \overline{v}_1, \ldots, \overline{v}_n)}) \\ &= (\varphi'_f(u_1, \ldots, u_n, \overline{v}_1, \ldots, \overline{v}_n), \varphi'_f(v_1, \ldots, v_n, \overline{u}_1, \ldots, \overline{u}_n)). \end{aligned}$$

And φ_f is monotone; therefore, φ'_f is also monotone. Hence, $\overline{f}$ is a De Morgan function.

Thus, we get an algebra: $\mathfrak{D}_n = (\mathcal{D}_n, \{+, \cdot, ^-, 0, 1\})$ (here 0 and 1 are the constant De Morgan functions), which obviously is a De Morgan algebra. Also, for $f, g \in \mathcal{D}_n$ we have: $\varphi_{f+g} = \varphi_f + \varphi_g$, $\varphi_{f \cdot g} = \varphi_f \cdot \varphi_g$, $\varphi_{\overline{f}} = \varphi'_f$.

Let $a = (a_1, a_2)$, $b = (b_1, b_2) \in 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$. We say that $a \subseteq b$, if $a_1 \subseteq b_1$ and $a_2 \subseteq b_2$. In this way, we get a partially ordered set $2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}(\subseteq)$. For an antichain $S \subseteq 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$, define the function $f_S : D^n \to D$ in the following way:

$$f_S(x_1, \ldots, x_n) = \sum_{s=(s_1,s_2)\in S} \Big(\prod_{i \in s_1} x_i \cdot \prod_{i \in s_2} \overline{x}_i \Big). \tag{3.1}$$

Notice that f_S does not depend on the order of the elements in the set S (cf. [32]).

Note that we set $f_\emptyset = 0$ and $f_{\{(\emptyset,\emptyset)\}} = 1$.

Let us consider the functions

$$\delta_n^i(x_1, \ldots, x_n) = x_i, \ i = 1, \ldots, n,$$

as functions $D^n \to D$. Obviously, δ_n^i is a De Morgan function. And according to (3.1), for any antichain $S \subseteq 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$, we have:

$$f_S = \sum_{s=(s_1,s_2)\in S} \Big(\prod_{i \in s_1} \delta_n^i \cdot \prod_{i \in s_2} \overline{\delta_n^i} \Big).$$

Hence, $f_S \in \mathcal{D}_n$, i.e. f_S is a De Morgan function for any antichain $S \subseteq 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$.

For $s = (s_1, s_2) \in 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$ let $s' = s_1 \cup \{n+i : i \in s_2\} \in 2^{\{1,\ldots,2n\}}$, and for $S \subseteq 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$ let $S' = \{s' : s \in S\} \subseteq 2^{\{1,\ldots,2n\}}$. In this way, we give a bijective mapping from the set of all antichains of $2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}(\subseteq)$ to the set of all antichains of $2^{\{1,\ldots,2n\}}(\subseteq)$. And so the number of all antichains of the partially ordered set $2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}(\subseteq)$ is m_{2n}.

Now, for any De Morgan function $f \in \mathcal{D}_n$ from Proposition 1.3 and Theorem 2.4, we conclude that there exists an antichain $S' \subseteq 2^{\{1,\ldots,2n\}}$ such that:

$$\begin{aligned} f(x_1,\ldots,x_n) &= (\varphi_f(u_1,\ldots,u_n,\overline{v}_1,\ldots,\overline{v}_n),\varphi_f(v_1,\ldots,v_n,\overline{u}_1,\ldots,\overline{u}_n)) \\ &= \Bigg(\sum_{s'\in S'}\Big(\prod_{\substack{i\in s'\\ 1\le i\le n}} u_i\cdot \prod_{\substack{i\in s'\\ n+1\le i\le 2n}} \overline{v}_{i-n}\Big),\sum_{s'\in S'}\Big(\prod_{\substack{i\in s'\\ 1\le i\le n}} v_i\cdot \prod_{\substack{i\in s'\\ n+1\le i\le 2n}} \overline{u}_{i-n}\Big)\Bigg) \\ &= \sum_{s'\in S'}\Big(\prod_{\substack{i\in s'\\ 1\le i\le n}} (u_i,v_i)\cdot \prod_{\substack{i\in s'\\ n+1\le i\le 2n}} (\overline{v}_{i-n},\overline{u}_{i-n})\Big) \\ &= \sum_{s=(s_1,s_2)\in S}\Big(\prod_{i\in s_1} x_i\cdot\prod_{i\in s_2}\overline{x}_i\Big) = f_S(x_1,\ldots,x_n), \end{aligned}$$

where S is the antichain of $2^{\{1,\ldots,n\}}\times 2^{\{1,\ldots,n\}}(\subseteq)$, corresponding to S'.

Moreover, the number of all De Morgan functions of n variables is the same as the number of all antichains of $2^{\{1,\ldots,n\}}\times 2^{\{1,\ldots,n\}}(\subseteq)$. Hence, we get the following result.

Theorem 3.1. *For any De Morgan function f of n variables there exists a unique antichain $S \subseteq 2^{\{1,\ldots,n\}}\times 2^{\{1,\ldots,n\}}$ such that $f = f_S$.* ∎

In particular, $f_{S_1} \neq f_{S_2}$ if $S_1 \neq S_2$.

Thus, every nonconstant De Morgan function can be uniquely presented in the form (3.1). This form is called the canonical form (or disjunctive normal form (or briefly - DNF)) of De Morgan function f. Notice that from Theorem 2.4 and from the proofs of Theorem 3.1 and Proposition 1.3, we get an algorithm which, given a De Morgan function, gives its disjunctive normal form. It is easy to see that the complexity of this algorithm is linear depending on the number of the rows of the table of the given De Morgan function f.

We can also prove that every nonconstant De Morgan function can be uniquely presented in conjunctive normal form (CNF), i.e. in the following form:

$$\prod_{(s_1,s_2)\in S}\Big(\sum_{i\in s_1} x_i + \sum_{i\in s_2}\overline{x}_i\Big).$$

Theorem 3.2. *The algebra $\mathfrak{D}_n$ is the free De Morgan algebra with the system of free generators: $\Delta = \{\delta_n^1,\ldots,\delta_n^n\}$. Hence, every free n-generated De Morgan algebra is isomorphic to the De Morgan algebra $\mathfrak{D}_n$.*

Proof. Let $\mathfrak{F} = (Q;\{+,\cdot,^-,0,1\})$ be a De Morgan algebra and $\mu : \Delta \to Q$ be a mapping. We prove that there exists a unique homomorphism: $\nu : \mathcal{D}_n \to \mathfrak{F}$

with $\nu|_\Delta = \mu$. Any element $f \in \mathcal{D}_n$ can be represented in the form

$$f = \sum_{s=(s_1,s_2)\in S} \Big(\prod_{i\in s_1} \delta_n^i \cdot \prod_{i\in s_2} \overline{\delta_n^i}\Big),$$

for the uniquely determined antichain $S \subseteq 2^{\{1,\dots,n\}} \times 2^{\{1,\dots,n\}}$. Set

$$\nu(f) = \sum_{s=(s_1,s_2)\in S} \Big(\prod_{i\in s_1} \mu(\delta_n^i) \cdot \prod_{i\in s_2} \overline{\mu(\delta_n^i)}\Big).$$

Obviously, ν is a homomorphism and $\nu(\delta_n^i) = \mu(\delta_n^i)$, $i = 1, \dots, n$. Uniqueness of ν is evident too. ∎

A similar result is valid for the finitely generated free algebras of the hypervariety defined by the system of hyperidentities of De Morgan algebras (Boolean algebras, distributive lattices)([25–30, 32]).

4. Conclusion

We give a new characterization of finitely generated free De Morgan algebras. The free De Morgan algebras have been characterized by several authors by describing the canonical forms of their elements ([2, 4, 13–15]) (cf. [27, 28, 32]). Besides, it is commonly known that the free Boolean algebra on n free generators is isomorphic to the Boolean algebra of Boolean functions of n variables. In this paper, we introduce the concept of De Morgan function and prove that the free De Morgan algebra on n free generators is isomorphic to the De Morgan algebra of De Morgan functions of n variables. The advantage of this representation of the free De Morgan algebra is the fact that it gives rise to a new concept of the De Morgan function, which is a new object in discrete mathematics.

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