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OPTIMAL DIVIDEND POLICY IN DISCRETE TIME

Abstract. A problem of optimal dividend policy for a firm with a bank loan is considered. A regularity of a value function is established. A numerical example of calculating value function is given.

1. Introduction

A firm which net income is uncertain must choose between paying dividends and creating cash reserves in order to maximize the expected present value of all payoffs until bankruptcy time. In influential papers [DV] and [JS] some continuous time problems were formulated and studied. In [DV], Décamps and Villeneuve considered a firm with a growth option, which gives the opportunity to invest in a new technology in order to increase its profit rate. They showed when it is optimal to postpone dividend payoffs, to accumulate cash and to invest at a certain date.

This work deals with the optimal dividend and growth option problem, where expenses are covered by a loan. We are concerned with a discrete time model.

2. Formulation of the problem

Consider a firm whose activities generate at moment n an income Y_n and a cash process X_n. Moreover, the firm took out a loan that have to be paid in constant rates C at each moment n. That may cause bankruptcy if the cash process is too low. The manager of the firm acts in the best interest of shareholders and at any moment n may decide to pay out dividends. The aim of this paper is to maximize the expected discounted satisfaction of payoffs paid out till bankruptcy time.

The mathematical formulation of this problem is as follows. Assume we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence $\xi_1, \xi_2, \ldots$ of independent

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identically distributed random variables such that

$$\mathbb{P}(\xi_1 > -1) = 1 \quad \text{and} \quad \mathbb{E}|\xi_1|^p < \infty, \quad \text{for some } p > 1.$$

Let $r \geq 0$ be a risk free rate of return. Define

$$F(x, y, b, \xi) := y(1+\xi)\, h(b, x),$$
$$G(x, y, a, b) := (x - C)(1+r)(1-a-b) + y,$$

where function h is positive, increasing and bounded from above by a constant $M > 0$. Starting with initial values $(X_0, Y_0) = (x, y)$, we have the following recursive formula

$$Y_{n+1} = F(X_n, Y_n, b_n, \xi_{n+1}),$$
$$X_{n+1} = G(X_n, Y_{n+1}, a_n, b_n),$$

where a_n and b_n represent respectively dividend paid out and investment at time n. A control policy $\pi = (a_n^\pi, b_n^\pi)_{n\in\mathbb{N}}$ is said to be admissible if and only if (a_n^π, b_n^π) is adapted to the filtration generated by the process (X_n, Y_n) and $(a_n^\pi, b_n^\pi) \in U$, for all $n \in \mathbb{N}$, where

$$U = \{(a, b) : a \geq 0, b \geq 0 \text{ and } a + b \leq 1\}.$$

Define the bankruptcy time τ_0 by

$$\tau_0 := \inf\{n \geq 0 \colon X_n - C < 0\}$$

and performance functional

$$J^\pi(x, y) = \mathbb{E}^{x,y}\Big[\sum_{n=0}^{\infty} \gamma^n g\left(a_n^\pi (X_n - C)\right) \mathbb{I}_{\{\tau_0 > n\}}\Big],$$

where $\gamma \in (0,1)$ is a given discount factor and g is a utility function from the class

$$\mathcal{K} := \Big\{ f \colon \mathbb{R} \to [0, \infty) : f(x) = 0 \text{ for all } x \leq 0,$$
$$f \text{ is continuous and } \|f\|_{\mathcal{K}} := \sup_{x\in\mathbb{R}} \frac{|f(x)|}{1+|x|} < \infty \Big\}.$$

The objective is to calculate the value function v_∞ and an optimal dividend policy $\pi^\infty = (a_n^\infty, b_n^\infty)_{n\in\mathbb{N}}$, that is v_∞ and π^∞ such that

$$v_\infty(x, y) := \sup J^\pi(x, y) = J^{\pi^\infty}(x, y),$$

where supremum is taken over all admissible strategies. Notice that since the time of bankruptcy is included in the function J^π, we need to extend a state space putting $E := \{\delta\} \cup E_1 \cup E_2$, where

$$E_1 := \{(x, y) \in [0, \infty)^2 \colon x - C < 0\},$$
$$E_2 := \{(x, y) \in [0, \infty)^2 \colon x - C \geq 0\},$$
$$\delta \in \mathbb{R}^2 \backslash [0, \infty)^2.$$

We equipped E with a natural topology and assume that if $(X_n, Y_n) \in \{\delta\} \cup E_1$, then $(X_{n+1}, Y_{n+1}) = \delta$. Further, for the admissible strategy $\pi = (a_n^\pi, b_n^\pi)_{n\in\mathbb{N}}$, we have that $(a_n^\pi, b_n^\pi) \in \mathcal{A}(X_n, Y_n)$, for each $n \in \mathbb{N}$, where

$$\mathcal{A}(x, y) = \{(a, b) : a \geq 0, b \geq 0 \text{ and } a + b \leq 1\}, \text{ for } (x, y) \in E_2,$$
$$\mathcal{A}(x, y) = \{(0, 0)\}, \text{ for } (x, y) \in E_1 \cup \{\delta\}.$$

Finally, we will denote by $(X_n^\pi, Y_n^\pi)_{n\in\mathbb{N}}$ the process corresponding to the strategy $\pi = (a_n^\pi, b_n^\pi)_{n\in\mathbb{N}}$. Note that

$$J^\pi(x, y) = \mathbb{E}^{x,y}\Big[\sum_{n=0}^{\infty} \gamma^n g\left(a_n^\pi (X_n - C)\right)\Big].$$

3. Solution by the Bellman Dynamic Programming

For the convenience of the reader, we recall the definitions of upper semi-continuous functions and set-valued mappings. Below (S, τ) and (R, ρ) are topological spaces and 2^R denotes the set of all subsets of R.

Definition 3.1. A function $\phi\colon S \to \mathbb{R}$ is called upper semi-continuous (**u.s.c.**) if for each open set $B \subset \mathbb{R}$, the set $\{s \in S\colon \phi(s) \in B\}$ is open.

Definition 3.2. A set-valued mapping $\psi\colon S \to 2^R$ is called upper semi-continuous (**u.s.c.**) if for each open set $F \subset R$, the set $\{s \in S\colon \psi(s) \subset F\}$ is open.

Let A be a positive constant. We will consider the following set of functions

$$\mathcal{W} := \Bigg\{ f\colon E \to [0, \infty) : f \text{ is u.s.c. on } E,\ \|f\|_{\mathcal{W}} := \sup_{(x,y)\in E} \frac{|f(x, y)|}{w(x, y)} < \infty,$$
$$f(x, y) = 0 \text{ for each } (x, y) \in E_1 \cup \{\delta\} \Bigg\},$$

with the weight function

$$w(x, y) = \begin{cases} 1 + x + Ay, & \text{if} \quad (x, y) \in E \backslash \{\delta\}, \\ 1, & \text{if} \quad (x, y) = \delta. \end{cases}$$

Obviously $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ is a complete metric space. Define an operator T_γ on

$\mathcal{W}$ by the formula

$$T_\gamma f(x,y) := \sup_{(a,b)\in\mathcal{A}(x,y)} \left\{ g(a(x-C)) + \gamma P^{a,b} f(x,y) \right\},$$

where

$$P^{a,b} f(x,y) := \mathbb{E}(f(X_{n+1}, Y_{n+1}) | X_n = x, Y_n = y, a_n = a, b_n = b).$$

Theorem 3.3. *Let*

$$\alpha := \max\{1+r, M(1+\mathbb{E}\xi_1)\}.$$

Suppose that $g \in \mathcal{K}$ and $\gamma\alpha < 1$. Then there exists $A > 0$ such that the value function v^* belongs to $\mathcal{W}$ and is the unique solution to the equation

$$T_\gamma v(x,y) = v(x,y), \qquad v \in \mathcal{W}.$$

Moreover, there are measurable functions $a^* \colon E \to [0,1]$ and $b^* \colon E \to [0,1]$, such that for all $(x,y) \in E$, $(a^*(x,y), b^*(x,y)) \in \mathcal{A}(x,y)$ and

$$T_\gamma v^*(x,y) = g(a^*(x,y)(x-C)) + \gamma P^{a^*(x,y), b^*(x,y)} v^*(x,y),$$

and the optimal strategy $\pi^* = (a_n^*, b_n^*)_{n\in\mathbb{N}}$ is of the form

$$(a_n^*, b_n^*) = \left(a^*\left(X_n^{\pi^*}, Y_n^{\pi^*}\right), b^*\left(X_n^{\pi^*}, Y_n^{\pi^*}\right) \right), \quad n \in \mathbb{N}.$$

To prove Theorem 3.3, we need the following general result on measurable selectors (for proofs see [S1] and [S2]).

Theorem 3.4. *Let X be a metric space, $\mathcal{U}$ be a separable metric space, and let $U \colon X \to 2^{\mathcal{U}}$ be a u.s.c. set-valued measurable mapping, such that for each $x \in X$, $U(x)$ is nonempty and compact in $\mathcal{U}$. Assume that*

$$\Psi := \{(x,u) : x \in X \text{ and } u \in U(x)\}$$

is measurable with respect to the product topology on $X \times \mathcal{U}$ and that $\psi \colon \Psi \to \mathbb{R}$ is measurable, u.s.c. and bounded from above. Then there exists a measurable function $f \colon X \to \mathcal{U}$ such that for all $x \in X$,

$$f(x) \in U(x) \quad \text{and} \quad \psi^*(x) := \sup_{u\in U(x)} \psi(x,u) = \psi(x, f(x)).$$

Furthermore, ψ^* is measurable, u.s.c. and bounded from above on X.

Proof of Theorem 3.3. Let us first prove that the operator T_γ transforms $\mathcal{W}$ into $\mathcal{W}$. To this end, notice that for any function $f \in \mathcal{W}$, we have

$$f(x,y) \le \|f\|_{\mathcal{W}} w(x,y), \quad \text{for all } (x,y) \in E. \tag{1}$$

Consider two cases:

1. If $(x,y) \in \{\delta\} \cup E_1$ then $\mathcal{A}(x,y) = \{(0,0)\}$ and

$$T_\gamma f(x,y) = g(0) + \gamma \mathbb{E} f(\delta) = 0, \tag{2}$$

$$\sup_{(x,y)\in\{\delta\}\cup E_1} \frac{|T_\gamma f(x,y)|}{w(x,y)} = 0.$$

2. If $(x,y) \in E_2$ then

$$\begin{aligned} T_\gamma f(x,y) &= \sup_{(a,b)\in\mathcal{A}(x,y)} \left\{ g(a(x-C)) + \gamma P^{a,b} f(x,y) \right\} \\ &\le \sup_{(a,b)\in\mathcal{A}(x,y)} \{ g(a(x-C)) \\ &\quad + \gamma \|f\|_{\mathcal{W}} (1 + x(1+r) + y(1+A) M (1+\mathbb{E}\xi_1)) \}. \end{aligned} \tag{3}$$

Since $g \in \mathcal{K}$, we have

$$g(a(x-C)) \le \|g\|_{\mathcal{K}} (1+x), \quad \text{for all } 0 \le a \le 1 \text{ and } x \ge 0. \tag{4}$$

By (3) and (4),

$$T_\gamma f(x,y) \le w(x,y) \left(\|g\|_{\mathcal{K}} + \gamma \left(\alpha + \frac{M(1+\mathbb{E}\xi_1)}{A} \right) \|f\|_{\mathcal{W}} \right).$$

Therefore, in particular we have

$$\sup_{(x,y)\in E_2} \frac{|T_\gamma f(x,y)|}{w(x,y)} < \infty.$$

In conclusion

$$\|T_\gamma f\|_{\mathcal{W}} = \sup_{(x,y)\in E} \frac{|T_\gamma f(x,y)|}{w(x,y)} < \infty, \quad \text{for all } f \in \mathcal{W}. \tag{5}$$

Furthermore, from (2) we have $T_\gamma f(x,y) = 0$, for all $(x,y) \in \{\delta\} \cup E_1$.

Our next claim is that $T_\gamma f$ is u.s.c. for any $f \in \mathcal{W}$. Fix an arbitrary $f \in \mathcal{W}$ and define

$$\phi(x,y) := T_\gamma f(x,y) - \|T_\gamma f\|_{\mathcal{W}} w(x,y) = \sup_{(a,b)\in\mathcal{A}(x,y)} \psi(x,y,a,b),$$

where

$$\psi(x,y,a,b) := g(a(x-C)) + \gamma P^{a,b} f(x,y) - \|T_\gamma f\|_{\mathcal{W}} w(x,y).$$

Notice that by (5), the function ψ is non positive. To use Theorem 3.4, we need ψ to be u.s.c. on $\Psi := \{(x,y,a,b) : (x,y) \in E, (a,b) \in \mathcal{A}(x,y)\}$. As g and w are continuous, it is sufficient to prove that $P^{a,b} f(x,y)$ is u.s.c.. Consider an arbitrary sequence $(x_n, y_n, a_n, b_n)_{n\ge1} \subset \Psi$ such that $(x_n, y_n, a_n, b_n) \to (x,y,a,b)$. Applying (1), we obtain

$$|q_n|^p := |f(x_{n+1}, y_{n+1})|^p \le q(1 + |\xi_1|^p), \quad \mathbb{P} - a.s.,$$

for some $q > 0$. Hence, as $\mathbb{E}\,|\xi_1|^p < \infty$, the sequence $\{q_n\}_{n\geq 1}$ is uniformly integrable and by the Fatou lemma, for all $(x_n, y_n, a_n, b_n) \to (x, y, a, b)$, we have

$$\limsup_{n\to\infty} P^{a_n,b_n} f\,(x_n, y_n) \leq P^{a,b} f\,(x, y)\,.$$

Therefore, $P^{a,b} f\,(x, y)$ is u.s.c. on Ψ.

Applying Theorem 3.4, we conclude that ϕ is u.s.c. and there exists a measurable selector $(\hat{a}, \hat{b})$ such that

$$\phi\,(x, y) = \psi\left(x, y, \hat{a}\,(x, y)\,, \hat{b}\,(x, y)\right), \quad \text{for all } (x, y) \in E.$$

Hence, since w is continuous on E, the function $T_\gamma f$ is u.s.c. on E and

$$T_\gamma f\,(x, y) = g\,(\hat{a}\,(x, y)\,(x - C)) + \gamma P^{\hat{a}(x,y),\hat{b}(x,y)} f\,(x, y).$$

Finally, $T_\gamma f \in \mathcal{W}$ for all $f \in \mathcal{W}$.

To show that T_γ is a contraction on $\mathcal{W}$, consider two arbitrary functions f_1, $f_2 \in \mathcal{W}$. Then

$$|T_\gamma f_1\,(x, y) - T_\gamma f_2\,(x, y)| \leq \gamma \sup_{(a,b)\in\mathcal{A}(x,y)} \left|P^{a,b}\left[(f_1 - f_2)\,(x, y)\right]\right|.$$

Using (1) we obtain

$$|T_\gamma f_1\,(x, y) - T_\gamma f_2\,(x, y)| \leq \gamma \,\|f_1 - f_2\|_{\mathcal{W}} \sup_{(a,b)\in\mathcal{A}(x,y)} P^{a,b} w\,(x, y).$$

We need to consider 3 cases:

1. If $(x, y) = \delta$ then since $\mathcal{A}\,(x, y) = \{(0, 0)\}$ and $P^{0,0} w\,(\delta) = 1$, we have

$$\frac{|T_\gamma f_1\,(\delta) - T_\gamma f_2\,(\delta)|}{w\,(\delta)} \leq \gamma\,\|f_1 - f_2\|_{\mathcal{W}}\,.$$

2. If $(x, y) = E_1$ then $\mathcal{A}\,(x, y) = \{(0, 0)\}$ and $P^{0,0} w\,(x, y) = w\,(\delta) = 1$. Moreover, since $w \geq 1$, for all $(x, y) = E_1$, we have

$$\frac{|T_\gamma f_1\,(x, y) - T_\gamma f_2\,(x, y)|}{w\,(x, y)} \leq \frac{\gamma\,\|f_1 - f_2\|_{\mathcal{W}}}{w\,(x, y)} \leq \gamma\,\|f_1 - f_2\|_{\mathcal{W}}\,.$$

3. If $(x, y) \in E_2$ then $\mathcal{A}\,(x, y) = \left\{(a, b) \in [0, 1]^2 : a + b \leq 1\right\}$ and

$$\sup_{(a,b)\in\mathcal{A}(x,y)} P^{a,b} w\,(x, y) \leq 1 + (1 + r)\,x + y\,(1 + A)\,M\,(1 + \mathbb{E}\xi_1)\,.$$

Therefore,

$$\begin{aligned}
&\frac{|T_\gamma f_1(x,y) - T_\gamma f_2(x,y)|}{1+x+Ay} \\
&\quad\le \gamma \|f_1 - f_2\|_{\mathcal{W}} \sup_{(x,y)\in E_2} \frac{1+(1+r)x + y(1+A)M(1+\mathbb{E}\xi_1)}{1+x+Ay} \\
&\quad\le \gamma \|f_1 - f_2\|_{\mathcal{W}} \left(\alpha + \sup_{y\ge 0} \frac{yM(1+\mathbb{E}\xi_1)}{1+Ay}\right) \\
&\quad\le \gamma \|f_1 - f_2\|_{\mathcal{W}} \left(\alpha + \frac{M(1+\mathbb{E}\xi_1)}{A}\right) \\
&\quad= \gamma \left(\alpha + \frac{M(1+\mathbb{E}\xi_1)}{A}\right) \|f_1 - f_2\|_{\mathcal{W}}\,.
\end{aligned}$$

Summarizing, we have

$$\|T_\gamma f_1 - T_\gamma f_2\|_{\mathcal{W}} \le \gamma\eta \|f_1 - f_2\|_{\mathcal{W}}\,,$$

where

$$\eta := \left(\alpha + \frac{M(1+\mathbb{E}\xi_1)}{A}\right).$$

Since we assumed that $\gamma\alpha < 1$, therefore $\gamma\eta < 1$ for A large enough.

This proves that T_γ is a contraction on $\mathcal{W}$. Thus by the Banach fixed point theorem, T_γ has a unique fixed point v^* in $\mathcal{W}$. It remains to prove that

$$\lim_{N\to\infty} \gamma^N \mathbb{E}^{x,y} v^*(X_N^\pi, Y_N^\pi) = 0,$$

for all $(x,y) \in E$ and all admissible strategies π. Notice that for any $(x,y) \in E_2$ and $(a,b) \in \mathcal{A}(x,y)$,

$$P^{a,b} w(x,y) \le 1 + (1+r)x + y(1+A)M(1+\mathbb{E}\xi) \le \eta w(x,y). \tag{6}$$

For $(x,y) \in E_1 \cup \{\delta\}$, $\mathcal{A}(x,y) = \{(0,0)\}$ and

$$P^{0,0} w(x,y) = w(\delta) = 1 \le \eta w(x,y). \tag{7}$$

Combining (6) and (7), we obtain

$$P^{a,b} w(x,y) \le \eta w(x,y), \quad \text{for all } (x,y) \in E,\ (a,b) \in \mathcal{A}(x,y).$$

Going recursively backwards, we get that for any initial state $(x,y) \in E$ and admissible strategy $\pi = (a_n^\pi, b_n^\pi)_{n\in\mathbb{N}}$,

$$\mathbb{E}^{x,y} w(X_n^\pi, Y_n^\pi) \le \eta^n w(x,y). \tag{8}$$

We proved that $v^* \in \mathcal{W}$. Therefore, combining (1) and (8) gives

$$\mathbb{E}^{x,y} v^*(X_N^\pi, Y_N^\pi) \le \|v\|_{\mathcal{W}} \mathbb{E}^{x,y} w(X_N^\pi, Y_N^\pi) \le \|v\|_{\mathcal{W}} \eta^N w(x,y).$$

Hence, we have

$$\lim_{N\to\infty} \gamma^N \mathbb{E}^{x,y} v^* \left(X_N^\pi, Y_N^\pi\right) = 0,$$

and we can use the arguments from the proof of Theorem 8.3.6 in [HL]. ∎

4. Numerical example

By Theorem 3.3, the value function can be obtained by iteration of T_γ. In this section, we give a numerical example. Suppose that the state space is of the form $E = \{\delta\} \cup E_2$, where

$$\delta := (0,0), \qquad E_2 := [C, L] \times [0, S] \quad \text{and} \quad 0 < C < L,\ 0 < S.$$

In order to make numerical approximations, we need to discretize the state space. We shall consider a sequence $(\Delta_m)_{m\in\mathbb{N}}$ of partitions of E_2 defined as follows

$$C = x_1 < x_2 < \ldots < x_{l(m)} = L,$$
$$0 < y_1 < y_2 < \ldots < y_{s(m)} = S,$$

with the convention

$$\|\Delta_m\| := \sup_{k=1,\ldots,l(m)-1} (x_{k+1} - x_k) + \sup_{k=1,\ldots,s(m)-1} (y_{k+1} - y_k),$$

$$\lim_{m\to+\infty} \|\Delta_m\| = 0.$$

Therefore, we replace the original state space by the discrete one

$$E^{(m)} = \{(0,0)\} \cup \left\{(x,y)\colon x \in \left\{0, x_1, \ldots, x_{l(m)}\right\}, y \in \left\{0, y_1, \ldots, y_{s(m)}\right\}\right\}.$$

Moreover, we assume that

$$\mathcal{A}(x,y) = \{(a,b) : a, b \in \{0, 0.1, \ldots, 1\} \text{ and } a + b \le 1\}, \text{ for } (x,y) \in E_2,$$
$$\mathcal{A}(x,y) = \{(0,0)\}, \text{ for } (x,y) = \delta.$$

Then for all $n \in \mathbb{N}$, we have

$$Y_{n+1}^S := \min\{F(X_n, Y_n, b_n, \xi), S\},$$
$$X_{n+1}^L := \min\{G(X_n, Y_{n+1}, a_n, b_n), L\},$$

and the value function is of the form

$$V^{(L,S)}(x,y) := \sup_{(a,b)\in\mathcal{A}(x,y)} \mathbb{E}^{x,y}\Big[\sum_{n=0}^{\infty} \gamma^n g\left(a_n^\pi\left(X_n^L - C\right)\right)\Big].$$

First, we estimate the value function $V^{(L,S)}(x,y)$ from below. Define

$$\underline{\Pi}_m^X(x) := \begin{cases} 0, & \text{if } \ x < x_1, \\ x_i, & \text{if } \ x_i \le x < x_{i+1}, \quad \text{where } i \in \{1, \ldots, l(m)-1\}, \\ L, & \text{if } \ x \ge L, \end{cases}$$

$$\underline{\Pi}_m^Y(y) := \begin{cases} 0, & \text{if} \quad y < y_1, \\ y_j, & \text{if} \quad y_j \le y < y_{j+1}, \quad \text{where } j \in \{1, \dots, s(m)-1\}, \\ S, & \text{if} \quad y \ge S, \end{cases}$$

and consider a new control problem. Namely, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \underline{Y}_{n+1} &:= \underline{F}_m\left(\underline{X}_n, \underline{Y}_n, b_n, \xi\right) := F\left(\underline{\Pi}_m^X\left(\underline{X}_n\right), \underline{\Pi}_m^Y\left(\underline{Y}_n\right), b_n, \xi\right), \\ \underline{X}_{n+1} &:= \underline{G}_m\left(\underline{X}_n, \underline{Y}_{n+1}, a_n, b_n\right) := G\left(\underline{\Pi}_m^X\left(\underline{X}_n\right), \underline{Y}_{n+1}, a_n, b_n\right), \end{aligned}$$

with an initial condition

$$\left(\underline{X_0}, \underline{Y_0}\right) := \left(\underline{\Pi}_m^X\left(X_0\right), \underline{\Pi}_m^Y\left(Y_0\right)\right)$$

and the value function

$$\underline{V}_m^{(L,S)}(x,y) := \sup_{(a,b)\in\mathcal{A}(x,y)} \mathbb{E}^{x,y}\Big[\sum_{n=0}^{\infty} \gamma^n g\left(a_n^\pi\left(\underline{X}_n - C\right)\right)\Big].$$

To estimate the value function $V^{(L,S)}(x,y)$ from above, we need to consider one more problem. Define

$$\overline{\Pi}_m^X(x) := \begin{cases} 0, & \text{if} \quad x < x_1, \\ x_1, & \text{if} \quad x = x_1, \\ x_{i+1}, & \text{if} \quad x_i < x \le x_{i+1}, \quad \text{where } i \in \{1, \dots, l(m)-1\}, \\ L, & \text{if} \quad x \ge L, \end{cases}$$

$$\overline{\Pi}_m^Y(y) := \begin{cases} 0, & \text{if} \quad y < y_1, \\ y_1, & \text{if} \quad y = y_1, \\ y_{j+1}, & \text{if} \quad y_j < y \le y_{j+1}, \quad \text{where } j \in \{1, \dots, s(m)-1\}, \\ S, & \text{if} \quad y \ge S. \end{cases}$$

For all $n \in \mathbb{N}$, we have

$$\begin{aligned} \overline{Y}_{n+1} &:= \overline{F}_m\left(\overline{X}_n, \overline{Y}_n, b_n, \xi\right) := F\left(\overline{\Pi}_m^X\left(\overline{X}_n\right), \overline{\Pi}_m^Y\left(\overline{Y}_n\right), b_n, \xi\right), \\ \overline{X}_{n+1} &:= \overline{G}_m\left(\overline{X}_n, \overline{Y}_{n+1}, a_n, b_n\right) := G\left(\overline{\Pi}_m^X\left(\overline{X}_n\right), \overline{Y}_{n+1}, a_n, b_n\right), \end{aligned}$$

with an initial condition

$$\left(\overline{X_0}, \overline{Y_0}\right) := \left(\overline{\Pi}_m^X\left(X_0\right), \overline{\Pi}_m^Y\left(Y_0\right)\right).$$

The value function of this problem is

$$\overline{V}_m^{(L,S)}(x,y) := \sup_{(a,b)\in\mathcal{A}(x,y)} \mathbb{E}^{x,y}\Big[\sum_{n=0}^{\infty} \gamma^n g\left(a_n^\pi\left(\overline{X}_n - C\right)\right)\Big].$$

Notice that, for all (x, y, a, b, ξ)

$$\underline{F}_m \le \underline{F}_{m+1} \le F \le \overline{F}_{m+1} \le \overline{F}_m,$$

and

$$\underline{G}_m \le \underline{G}_{m+1} \le G \le \overline{G}_{m+1} \le \overline{G}_m.$$

Hence

$$\underline{V}_m^{(S,L)} \le \underline{V}_{m+1}^{(S,L)} \le V^{(S,L)} \le \overline{V}_{m+1}^{(S,L)} \le \overline{V}_m^{(S,L)}. \tag{9}$$

From (9), the limits $\lim_{m\to\infty} \underline{V}_m^{(S,L)}$ and $\lim_{m\to\infty} \overline{V}_m^{(S,L)}$ exist. This result raises some natural question. Namely

Question 1. If

$$\lim_{m\to\infty} \underline{V}_m^{(S,L)}(x,y) = V^{(S,L)}(x,y) = \lim_{m\to\infty} \overline{V}_m^{(S,L)}(x,y), \quad \text{for all } (x,y) \in E?$$

Now we present numerical solutions for parameters

$$r = 0.03, \quad C = 1, \quad \gamma = 0.35 \quad \text{and} \quad M = 2.$$

Suppose $S = 4.5$, $L = 14.3$ and the partition of E_2 is of the form

$$\begin{aligned} C = 1 &< 1.1 < 1.2 < 1.3 < 1.8 < 2.3 < 2.8 < 3.3 \\ &< 3.8 < 4.8 < 5.8 < 6.8 < 9.3 < 11.8 < 14.3 = L, \\ 0 &< 0.75 < 1.5 < 3 < 4.5 = S. \end{aligned}$$

Let

$$h(b,x) = 2 - \frac{1}{b(x-C)+1}, \qquad g(z) = \sqrt{z},$$

and the random variable

$$\xi = \begin{cases} 0.1, & \text{where} \quad \mathbb{P}(\xi = 0.1) = 0.6, \\ -0.1, & \text{where} \quad \mathbb{P}(\xi = -0.1) = 0.4. \end{cases}$$

Note that all the assumptions of Theorem 3.3 are satisfied. In the Appendix we show the optimal strategies, the value functions and the difference between the value functions.

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5. Appendix

The optimal consumption $\underline{a}$															
State	State x														
y	0	1.1	1.2	1.3	1.8	2.3	2.8	3.3	3.8	4.8	5.8	6.8	9.3	11.8	14.3
0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	0.8
0.75	0	1	1	1	1	1	1	1	0.8	1	0.9	0.8	0.8	0.8	0.8
1.5	0	1	1	1	0.8	0.9	0.9	0.9	0.9	0.8	0.9	0.8	0.9	0.9	0.9
3	0	1	1	1	1	0.9	0.9	0.9	0.9	0.9	0.8	0.9	0.9	0.9	0.9
4.5	0	1	1	1	1	0.8	1	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9

The optimal investment $\underline{b}$															
State	State x														
y	0	1.1	1.2	1.3	1.8	2.3	2.8	3.3	3.8	4.8	5.8	6.8	9.3	11.8	14.3
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.75	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1.5	0	0	0	0	0.2	0.1	0.1	0.1	0.1	0.1	0	0.1	0.1	0.1	0
3	0	0	0	0	0	0.1	0.1	0.1	0.1	0.1	0.2	0.1	0.1	0.1	0.1
4.5	0	0	0	0	0	0.2	0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1

The optimal consumption $\overline{a}$															
State	State x														
y	0	1.1	1.2	1.3	1.8	2.3	2.8	3.3	3.8	4.8	5.8	6.8	9.3	11.8	14.3
0	0	1	1	1	1	1	1	1	1	1	1	1	0.8	0.8	0.9
0.75	0	1	1	1	0.3	0.6	0.7	0.7	0.7	0.8	0.8	0.8	0.8	0.9	0.9
1.5	0	1	1	1	0.8	0.9	0.9	0.9	0.9	1	1	0.9	0.9	0.9	0.9
3	0	1	1	1	1	1	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9
4.5	0	1	1	1	1	1	1	0.9	1	1	1	1	0.9	0.9	0.9

The optimal investment $\overline{b}$															
State	State x														
y	0	1.1	1.2	1.3	1.8	2.3	2.8	3.3	3.8	4.8	5.8	6.8	9.3	11.8	14.3
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.75	0	0	0	0	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0	0.1	0	0
1.5	0	0	0	0	0.2	0.1	0.1	0.1	0.1	0	0	0.1	0.1	0.1	0
3	0	0	0	0	0	0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
4.5	0	0	0	0	0	0	0	0.1	0	0	0	0	0.1	0.1	0.1

The value function $\underline{V}^{(S,L)}(x,y)$

State y	State x: 0	1.1	1.2	1.3	1.8	2.3	2.8	3.3	3.8	4.8	5.8	6.8	9.3	11.8	14.3
0	0	0.316	0.447	0.548	0.894	1.140	1.342	1.517	1.673	1.949	2.191	2.408	2.881	3.286	3.661
0.75	0	0.316	0.447	0.548	0.894	1.140	1.342	1.517	1.674	1.949	2.238	2.467	2.976	3.409	3.793
1.5	0	0.559	0.690	0.790	1.153	1.434	1.626	1.791	1.940	2.206	2.446	2.696	3.196	3.580	3.961
3	0	0.984	1.115	1.216	1.563	1.837	2.060	2.226	2.374	2.660	2.905	3.154	3.624	4.063	4.405
4.5	0	1.262	1.393	1.493	1.840	2.090	2.287	2.509	2.657	2.919	3.148	3.355	3.878	4.263	4.605

The value function $\overline{V}^{(S,L)}(x,y)$

State y	State x: 0	1.1	1.2	1.3	1.8	2.3	2.8	3.3	3.8	4.8	5.8	6.8	9.3	11.8	14.3
0	0	0.316	0.447	0.548	0.894	1.140	1.342	1.517	1.673	1.949	2.191	2.408	2.890	3.388	3.773
0.75	0	0.316	0.447	0.548	0.946	1.339	1.579	1.739	1.919	2.214	2.479	2.684	3.194	3.613	3.990
1.5	0	0.902	1.033	1.134	1.483	1.764	1.956	2.121	2.270	2.536	2.777	3.002	3.501	3.885	4.249
3	0	1.225	1.356	1.456	1.803	2.049	2.289	2.455	2.604	2.888	3.117	3.323	3.810	4.245	4.587
4.5	0	1.444	1.575	1.675	2.022	2.268	2.469	2.646	2.801	3.077	3.319	3.536	4.086	4.471	4.813

The difference $\overline{V}^{(S,L)}(x,y) - \underline{V}^{(S,L)}(x,y)$

State y	State x: 0	1.1	1.2	1.3	1.8	2.3	2.8	3.3	3.8	4.8	5.8	6.8	9.3	11.8	14.3
0	0	0	0	0	0	0	0	0	0	0	0	0	0.009	0.052	0.112
0.75	0	0	0	0	0.052	0.199	0.237	0.222	0.245	0.264	0.241	0.217	0.218	0.204	0.197
1.5	0	0.344	0.334	0.334	0.330	0.330	0.330	0.330	0.330	0.329	0.331	0.306	0.305	0.305	0.288
3	0	0.241	0.241	0.241	0.241	0.211	0.230	0.230	0.230	0.228	0.212	0.169	0.187	0.182	0.182
4.5	0	0.182	0.182	0.182	0.182	0.178	0.182	0.137	0.144	0.158	0.170	0.181	0.208	0.208	0.208

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