DE GRUYTER
OPEN

DEMONSTRATIO MATHEMATICA
Vol. XLVII No 1 2014

Agnieszka M. Gdula, Andrzej Krajka

ON THE COMPLETE CONVERGENCE OF RANDOMLY WEIGHTED SUMS OF RANDOM FIELDS

Abstract. Let $\{X_{\underline{n}}, \underline{n} \in V \subseteq \mathbb{N}^d\}$ be a d-dimensional random field indexed by some subset V of lattice $\mathbb{N}^d$, which are stochastically dominated by a random variable X. Let $\{a_{\underline{n},\underline{i}}, \underline{n}, \underline{i} \in V\}$ be a $2d$-dimensional random field independent of $\{X_{\underline{n}}, \underline{n} \in V\}$ and such that $|a_{\underline{n},\underline{i}}| < M, \underline{n}, \underline{i} \in V$ for some constant M. In this paper, we give conditions under which the following series

$$\sum_{\underline{n}\in V} |\underline{n}|^t P\left[\frac{|\sum_{\underline{i}\in V} a_{\underline{n},\underline{i}} X_{\underline{i}}|}{|\underline{n}|^{1/p}} > \epsilon\right],$$

is convergent for some real t, some fixed $p > 0$ and all $\epsilon > 0$. Here $|\underline{n}|$ is used for $\prod_{i=1}^d n_i$. The randomly indexed sums of field $\{X_{\underline{n}}, \underline{n} \in V\}$ are considered too.

1. Introduction

Let $\mathbb{N}^d$ denotes the positive integer d-dimensional lattice points and let $\{X_{\underline{n}}, \underline{n} \in V \subseteq \mathbb{N}^d\}$ be a set of random variables indexed by some subset V of lattice $\mathbb{N}^d$. We use the notation $\underline{n} < \underline{m}$ if $n_i < m_i, i = 1, 2, \ldots d$, and similarly $\underline{n} \leq \underline{m}$. The symbol $|\underline{n}|$ denotes $\prod_{i=1}^d n_i$. The divergence $\underline{n} \to \infty$ is interpreted as $n_i \to \infty, i = 1, 2, \ldots d$. Let $\{a_{\underline{n},\underline{i}}, \underline{n}, \underline{i} \in V \subseteq \mathbb{N}^d\}$ be a $2d$-dimensional random field independent of $\{X_{\underline{n}}, \underline{n} \in V\}$ and such that $|a_{\underline{n},\underline{i}}| < M,$ a.s. $\underline{n}, \underline{i} \in V$ for some constant M. We say that the random field $\{X_{\underline{n}}, \underline{n} \in V\}$ is stochastically dominated by the random variable X iff

$$(1) \qquad P[|X_{\underline{n}}| > x] \leq CP[|X| > x], \text{ for all } x > 0, \text{ and all } \underline{n} \in V.$$

For $\underline{n} \in V$ we will write $S_{\underline{n}}(V) = \sum_{\underline{i}\in V, \underline{i}\leq\underline{n}} X_{\underline{i}}$ and $S_{(n)}(V) = \sum_{\underline{i}\in V, |\underline{i}|\leq n} X_{\underline{i}}$. If it doesn't lead to any misunderstanding we will omit argument and write short $S_{\underline{n}}$ and $S_{(n)}$, respectively.

2010 *Mathematics Subject Classification*: 60F15, 60G50.

Key words and phrases: complete convergence, rate of convergence, sums of random fields, multidimensional index.

DOI: 10.2478/dema-2014-0018

In this paper, we consider the following complete convergence of random moving averages of random fields

$$\forall_{\epsilon>0} \sum_{\underline{n}\in V} |\underline{n}|^t P\left[\frac{|\sum_{\underline{i}\in V} a_{\underline{n},\underline{i}} X_{\underline{i}}|}{|\underline{n}|^{1/p}} > \epsilon\right] < \infty, \tag{2}$$

for some fixed $p > 0, t \in \mathbb{R}$.

The complete convergence of random sequences was introduced and first investigated by Hsu, Robbins [7] and Erdös [1]. Their results were extended by Baum and Katz [2] to establish a rate of convergence in the sense of Marcinkiewicz-Zygmund's strong law of large numbers. This result has rich generalizations, e.g. Hu, Móricz and Taylor [9] extended complete convergence on arrays of rowwise independent random variables. We refer to Gut [3] for a rich survey on complete convergence related to strong laws results. The complete convergence and the convergence rate of randomly indexed partial sums were considered by Gut [4] and additionally for multidimensional indices by Gut [5]. In 1998, Hsu et al. [10] proved the complete convergence theorem for arrays of rowwise independent random variables. This result was generalized by Sung [14], who proved the following theorem:

Theorem 1.1. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which are stochastically dominated by a random variable X satisfying $E|X|^{p(t+\beta+1)} < \infty$, where $p(t+\beta+1) > 0$, $\beta, t \in \mathbb{R}$ and $p > 0$. Let $\{a_{n,i}, n, i \geq 1\}$ be a bounded array of real numbers such that*

$$\sum_{i=1}^{\infty} |a_{n,i}|^q = O(n^\beta),$$

for some $q < p(t+\beta+1)$. Assume that one of the following conditions holds:

(i) $0 < p(t+\beta+1) < 1$,
(ii) $1 \leq p(t+\beta+1) < 2, \{X_n, n \geq 1\}$ *is the sequence of independent random variables with* $EX_n = 0, n \geq 1$,
(iii) $2 \leq p(t+\beta+1), \{X_n, n \geq 1\}$ *is the sequence of independent random variables with* $EX_n = 0, n \geq 1$ *and* $\sum_{i=1}^{\infty} a_{n,i}^2 = O(n^\alpha)$, *for some* $\alpha < \frac{2}{p}$,

then for all $\epsilon > 0$,

$$\sum_{n=1}^{\infty} n^t P\left[\frac{|\sum_{i=1}^{\infty} a_{n,i} X_i|}{n^{1/p}} > \epsilon\right] < \infty.$$

On the other hand, the almost sure and complete convergence for randomly weighted sums of arrays of rowwise independent Banach space valued random elements were investigated by Thanh and Yin [16].

In this paper, we generalize the Theorem 1.1 in the following directions:

(i) We consider the d-dimensional random field of independent random variables.
(ii) In a random field, we consider some cases of summation index V.
(iii) We allow that constants $\{a_{\underline{n},\underline{i}}, \underline{n}, \underline{i} \in V\}$ (cf. Theorem 1.1) are random variables independent of $\{X_{\underline{n}}, \underline{n} \in V\}$. This situation allows us to obtain the complete convergence result for randomly indexed random sums of the field.

The notations, main results and examples of applications are given in section 2, whereas some lemmas and proofs in section 3. In the whole paper C denotes the generic constants different in different places, maybe. By the $I[A]$ we will denote the indicator of event A as well as the characteristic function and by $\log_+ x$ we will understand $\max\{\log x, 1\}$, where $\log x$ is the natural logarithm.

2. Main results

We will consider two kind of subsets $\mathbb{N}^d$ and $V_c = \{\underline{n} : \frac{1}{c} n_i < n_j < c n_i$, for $1 \leq i < j \leq d$, $n_i, n_j \in \mathbb{N}\backslash\{0\}$, $i, j = 1, 2, \ldots, d\}$ for some $c > 1$, of positive integer d-dimensional lattice $\mathbb{N}^d$(for some positive integer d).

Considering the results, which are true for both of them we will write shortly V. We note, in all paper,

$$\tau = \tau(V, d) = \begin{cases} 1, & \text{if } V = V_c, \\ d, & \text{if } V = \mathbb{N}^d. \end{cases} \tag{3}$$

The geometry of lattice V plays the big role in the evaluation of the complete convergence. Following Smythe [13], we introduce some notions:

Definition 2.1. For arbitrary subset V of d-dimensional lattice $\mathbb{N}^d$ we define

(i) $d_V(j) = card\{\underline{n} \in V : |\underline{n}| = j\}$, $j = 1, 2, 3, \ldots$.
(ii) $M_V(x) = \begin{cases} \sum_{j \leq x} d_V(j), & x \geq 1,\ x \in \mathbb{R}, \\ 1, & 0 \leq x < 1. \end{cases}$

For the additional information on M_V and d_V cf. Lemmas 3.5 and 3.6.

Theorem 2.1. *Let $\{X_{\underline{n}}, \underline{n} \in V\}$ be a field of random variables which are stochastically dominated by a random variable X such that*

$$E|X|^{p(t+\beta+1)}(\log_+|X|)^{\tau-1} < \infty,$$

where $p(t + \beta + 1) > 0$, β, $t \in \mathbb{R}$ and $p > 0$. In the case $p(t + \beta + 1) \geq 1$, we always assume that $EX_{\underline{n}} = 0$, $\underline{n} \in V$, and that $\{X_{\underline{n}}, \underline{n} \in V\}$ is the field *of independent random variables. Let* $\{a_{\underline{n},\underline{i}},\ \underline{i},\underline{n} \in V\}$ *be a field of random variables independent of field* $\{X_{\underline{n}}, \underline{n} \in V\}$ *and such that*

$$|a_{\underline{n},\underline{i}}| = O(1),\ \ \underline{i},\underline{n} \in V,\ a.s., \tag{4}$$

$$\sum_{\underline{i}\in V} E|a_{\underline{n},\underline{i}}|^q = O(|\underline{n}|^\beta), \tag{5}$$

for some $q < p(t+\beta+1)$.

(i) *If* $0 < p(t+\beta+1) < 2$, $p(t+\beta+1) \neq 1$, *then* (2) *holds.*

(ii) *If* $p(t+\beta+1) = 1$ *and one of the following conditions is satisfied:*

(a) $\sum_{\underline{i}\in V} |a_{\underline{n},\underline{i}}|^q = O(|\underline{n}|^\beta)$, *a.s.*

(b) $\sum_{\underline{i}\in V} E|a_{\underline{n},\underline{i}} - Ea_{\underline{n},\underline{i}}| = O(\frac{|\underline{n}|^\beta}{\log_+^{\tau+\gamma}|\underline{n}|})$, *for some* $\gamma > 0$,

(c) $E|X|(\log_+^{\tau+\gamma}|X|) < \infty$, *for some* $\gamma > 0$,

then (2) *holds.*

(iii) *if* $p(t+\beta+1) \geq 2$ *and*

$$E\Big(\sum_{\underline{i}\in V} a^2_{\underline{n},\underline{i}}\Big)^J = O(|\underline{n}|^{\alpha J}), \tag{6}$$

for some $J > 2$, $\alpha < 2/p$ *and* $J(2/p - \alpha) - t > 1$, *then* (2) *holds.*

Corollary 2.1. *Let* $\{X_{\underline{n}}, \underline{n} \in V\}$ *and constants* p, t, β *be such as in Theorem* 2.1.

(a) *Let* $\{N_{\underline{n}}, \underline{n} \in V\}$ *be a field of random indices independent of the field* $\{X_{\underline{n}}, \underline{n} \in V\}$ *and such that* $EM_V(N_{\underline{n}}) = O(|\underline{n}|^\beta)$.

If (i) $0 < p(t+\beta+1) < 2$, $p(t+\beta+1) \neq 1$,

or (ii) $p(t+\beta+1) = 1$, *and*

$EM_V(N_{\underline{n}}) = O(\frac{|\underline{n}|^\beta}{\log_+^{\tau+\gamma}|\underline{n}|})$ *for some* $\gamma > 0$,

or

$E|X|(\log_+^{\tau+\gamma}|X|) < \infty$ *for some* $\gamma > 0$,

or (iii) $p(t+\beta+1) \geq 2$, *and* $E(M_V(N_{\underline{n}}))^J = O(|\underline{n}|^{\alpha J})$ *for some* $\alpha < 2/p$, $J > 2$ *and* $J(2/p - \alpha) - t > 1$,

then

$$\forall_{\epsilon>0} \quad \sum_{\underline{n}\in V} |\underline{n}|^t P[|S_{(N_{\underline{n}})}| > \epsilon|\underline{n}|^{1/p}] < \infty.$$

(b) *Let* $\{\underline{N}_{\underline{n}}, \underline{n} \in V\}$ *be a d-dimensional random field independent of the field* $\{X_{\underline{n}}, \underline{n} \in V\}$ *and such that* $E|\underline{N}_{\underline{n}}| = O(|\underline{n}|^\beta)$. *If the assumptions of* *point* (a) *hold with* $M_V(N_{\underline{n}})$ *replaced by* $|\underline{N}_{\underline{n}}|$, *then*

$$\forall_{\epsilon>0} \quad \sum_{\underline{n}\in V} |\underline{n}|^t P[|S_{\underline{N}_{\underline{n}}}| > \epsilon|\underline{n}|^{1/p}] < \infty.$$

Corollary 2.2. *Let* $\{X_{\underline{n}}, \underline{n} \in V\}$ *and constants* p, t, β *be such as in Theorem* 2.1.

(a) *Let* $\{\varepsilon_{\underline{n}}, \underline{n} \in V\}$ *be a field of random variables such that* $P[\varepsilon_{\underline{n}} = 1] = 1 - P[\varepsilon_{\underline{n}} = 0] = \hat{p}$, $\underline{n} \in V$ *and independent of the field* $\{X_{\underline{n}}, \underline{n} \in V\}$. *Assume that* $\beta > 1$. *Then*

$$\forall_{\epsilon>0} \quad \sum_{\underline{n}\in V} |\underline{n}|^t P[\sum_{\underline{i}\in V, \underline{i}\leq\underline{n}} |\varepsilon_{\underline{i}} X_{\underline{i}}| > \epsilon|\underline{n}|^{1/p}] < \infty.$$

(b) *Let* $\{\varepsilon_{\underline{n},\underline{i}}, \underline{n}, \underline{i} \in V\}$ *be a* $2d$*-dimensional fields of random variables such that* $P[\varepsilon_{\underline{n},\underline{i}} = 1] = 1 - P[\varepsilon_{\underline{n},\underline{i}} = 0] = \hat{p}_{\underline{n},\underline{i}}$, *where* $\{\hat{p}_{\underline{n},\underline{i}}, \underline{n}, \underline{i} \in V\}$ *is* $2d$*-dimensional field of numbers and*

$$\sum_{\underline{i}\in V} \hat{p}_{\underline{n},\underline{i}} = O(|\underline{n}|^{\beta}).$$

If $p(t + \beta + 1) = 1$ *we assume additionally that for some* $\gamma > 0$

$$\sum_{\underline{i}\in V} \hat{p}_{\underline{n},\underline{i}} = O\Big(\frac{|\underline{n}|^{\beta}}{\log_+^{\tau+\gamma} |\underline{n}|}\Big),$$

whereas if $p(t+\beta+1) \geq 2$, *we assume that for integer* $J > 2$ *and* $\alpha < 2/p$ *such that* $J(2/p - \alpha) - t > 1$

$$\Big(\sum_{\underline{i}\in V} \hat{p}_{\underline{n},\underline{i}}\Big)^{J/2} = O(|\underline{n}|^{\alpha J}),$$

then

$$\forall_{\epsilon>0} \sum_{\underline{n}\in V} |\underline{n}|^t P\Big[\sum_{\underline{i}\in V} |\varepsilon_{\underline{n},\underline{i}} X_{\underline{i}}| > \epsilon|\underline{n}|^{1/p}\Big] < \infty.$$

Example 2.1. Let $\{Y_{\underline{n}}, \underline{n} \in \mathbb{N}^2\}$ be an iid random field with the stable law $(G_{\alpha',\beta',\gamma',c}())$ i.e. with the characteristic function

$$Ee^{itY_{\underline{1}}} = \exp\{it\gamma' - c|t|^{\alpha'}\{1 + i\beta' sign(t)\omega(t, \alpha')\}\}, t \in \mathbb{R},$$

where

$$\omega(t, \alpha') = \begin{cases} \frac{2}{\pi}\log t, & \text{if } \alpha' = 1, \\ \tan(\frac{\pi}{2}\alpha'), & \text{otherwise.} \end{cases}$$

Furthermore, let $\{\underline{N}_{\underline{n}}, \underline{n} \in \mathbb{N}^2\}$ and $\{\underline{U}_{\underline{n}}, \underline{n} \in \mathbb{N}^2\}$ be random fields of 2-dimensional vectors with the laws

$$P[\underline{N}_{\underline{n}} = \underline{k}] = e^{-\lambda_{n_1} - \lambda_{n_2}} \frac{\lambda_{n_1}^{k_1} \lambda_{n_2}^{k_2}}{k_1! k_2!}, \quad \underline{k} \in \mathbb{N}^2,$$

and

$$P[\underline{U}_{\underline{n}} = \underline{k}] = \begin{cases} \frac{1}{|\underline{k}|}, & \text{if } \underline{k} \le \underline{n}, \\ 0, & \text{otherwise,} \end{cases}$$

respectively. We assume that $\{\underline{U}_{\underline{n}}, \underline{n} \in \mathbb{N}^2\}$ is independent of $\{Y_{\underline{n}}, \underline{n} \in \mathbb{N}^2\}$. Furthermore, let $\{X_{\underline{n}} = Y_{\underline{U}_{\underline{n}}}, \underline{n} \in V\}$, then for every $x > 0$ we have

$$P[|X_{\underline{n}}| > x] = P[|Y_{\underline{1}}| > x] = 1 - G_{\alpha',\beta',\gamma',c}(x),$$

and for all $s < \alpha'$ we have $E|X_{\underline{n}}|^s = E|Y_{\underline{n}}|^s < \infty$, $\underline{n} \in V$. Let $0 < \beta < \frac{\alpha'}{p} - t - 1$, and

$$\lambda_n = \begin{cases} O(n^\beta), & \text{if } V = V_c, p(t+\beta+1) \neq 1, \\ O(\frac{n^\beta}{\log_+^{\tau+\delta} n}), & \text{if } V = V_c, p(t+\beta+1) = 1, \\ O(n^{\frac{\beta}{1+\delta}}), & \text{if } V = \mathbb{N}^2, \end{cases}$$

for some $\delta > 0$. If $0 < p(t+\beta+1) < 1$ then

$$\forall_{\epsilon>0} \quad \sum_{\underline{n}\in V} |\underline{n}|^t P\Big[\Big|\sum_{\underline{1}\le\underline{k}\le\underline{N}_{\underline{n}}} X_{\underline{k}}\Big| > \epsilon|\underline{n}|^{1/p}\Big] < \infty,$$

whereas if $1 \le p(t+\beta+1)$ then

$$\forall_{\epsilon>0} \quad \sum_{\underline{n}\in V} |\underline{n}|^t P\Big[\Big|\sum_{\underline{1}\le\underline{k}\le\underline{N}_{\underline{n}}} Y_{\underline{k}}\Big| > \epsilon|\underline{n}|^{1/p}\Big] < \infty.$$

3. Auxiliary results and proofs

Let us introduce the ordering in V by the "diagonal method":

$$(\underline{k} \prec \underline{n}) \iff (|\underline{k}| < |\underline{n}|) \vee (((|\underline{k}| = |\underline{n}|) \wedge (k_{i(\underline{k},\underline{n})} < n_{i(\underline{k},\underline{n})})),$$

where we put $i(\underline{k},\underline{n}) = \min\{1 \le i \le d : k_i \neq n_i\}$ for $\underline{k} \neq \underline{n}$ and $i(\underline{k},\underline{k}) = d$. This order stands the linear order of d-dimensional vectors. Let us denote $ord_V(\underline{n}) = card\{\underline{k} \in V : \underline{k} \prec \underline{n} \vee \underline{k} = \underline{n}\}$ (for eg. $ord_{\mathbb{N}^2}((3,2)) = 13, ord_{V_2}((3,2)) = 4$).

For proof, we need the following auxiliary results:

Lemma 3.1. *Let $\{X_{\underline{i}}, \underline{i} \in V\}$ denotes the field of independent random variables with $EX_{\underline{i}} = 0, \underline{i} \in V$. Then for arbitrary subset $S \subseteq V$ there exists a positive constant C_p depending only on $p > 2$ such that:*

$$E\Big|\sum_{\underline{i}\in S} X_{\underline{i}}\Big|^p \le C_p\Big\{\sum_{\underline{i}\in S} E|X_{\underline{i}}|^p + \Big(\sum_{\underline{i}\in S} EX_{\underline{i}}^2\Big)^{p/2}\Big\}.$$

Lemma 3.2. *Let* $\{X_{\underline{i}}, \underline{i} \in V\}$ *denote the field of independent random variables with* $EX_{\underline{i}} = 0, \underline{i} \in V$*, then for arbitrary integer* $j \geq 1$*, and* $t > 0$*:*

$$(7)\quad P\Big[\Big|\sum_{\underline{i}\in V} X_{\underline{i}}\Big| > 6^j t\Big] \leq C_j P\big[\sup_{\underline{i}\in V} |X_{\underline{i}}| > t/4^{j-1}\big] + D_j \sup_{k\in\mathbb{N}}\Big[P\big[\big|\sum_{l=1}^{k} X_{ord_V^{-1}(l)}\big| > t/4^j\big]\Big]^{2^j},$$

where $ord_V^{-1}()$ *denotes the inverse function of* $ord_V()$ *and* C_j, D_j *are positive constants.*

Applying the described above renumeration, we obtain Lemma 3.1 from Rosenthal's inequality [12] and Lemma 3.2 from proof of equation (3.3), p. 164 in [6] and Hoffmann-Jørgensen's Proposition 6.7 in [11]. We only remark that

$$\sup_{i\geq 1}\Big|X_{ord_V^{-1}(i)}\Big| = \sup_{\underline{k}\in V} |X_{\underline{k}}|,$$

but in general the second term on the right hand side of (7) can't be similarly written. ■

Lemma 3.3. *Let* $\{X_{\underline{n}},\ \underline{n} \in V\}$ *be the field of independent random variables with* $EX_{\underline{n}} = 0$*,* $EX_{\underline{n}}^2 < \infty$*,* $\underline{n} \in V$*, stochastically dominated by the random variable* X*, and let* $\{\beta_{\underline{n}},\ \underline{n} \in V\}$ *be the field of numbers bounded by* M*, then for arbitrary* $J \geq 2$*,* η*,* ϵ*,* $p > 0$ *and* β*,* $t \in \mathbb{R}$ *such that* $p(t+\beta+1) > 1+\eta$ *we have*

$$(8)\quad \sum_{\underline{n}\in V} |\underline{n}|^t P\Big[\big|\sum_{\underline{i}\in V} \beta_{\underline{i}} X_{\underline{i}}\big| > \epsilon|\underline{n}|^{1/p}\Big]$$
$$\leq C\sum_{\underline{n}\in V} |\underline{n}|^{t-2J/p}\Big(\sum_{\underline{i}\in V}\beta_{\underline{i}}^2\Big)^J + C\sum_{\underline{n}\in V}|\underline{n}|^{-\beta-1-\eta/p}\sum_{\underline{i}\in V} E|\beta_{\underline{i}}X_{\underline{i}}I[|X_{\underline{i}}| \leq |\underline{n}|^{1/p}]|^{p(t+\beta+1)+\eta}$$
$$+ C\sum_{\underline{n}\in V}|\underline{n}|^{-\beta-1+\eta/p}\sum_{\underline{i}\in V} E|\beta_{\underline{i}}X_{\underline{i}}I[|X_{\underline{i}}| > |\underline{n}|^{1/p}]|^{p(t+\beta+1)-\eta}.$$

Proof. Let us define

$$X_{\underline{i}}' = X_{\underline{i}} I\Big(|\beta_{\underline{i}}X_{\underline{i}}| \leq \frac{\epsilon|\underline{n}|^{1/p}}{24^j\cdot 36}\Big),$$
$$X_{\underline{i}}'' = X_{\underline{i}} I\big(|\beta_{\underline{i}}X_{\underline{i}}| > \delta|\underline{n}|^{1/p}\big),$$
$$X_{\underline{i}}''' = X_{\underline{i}} I\Big(\frac{\epsilon|\underline{n}|^{1/p}}{24^j\cdot 36} < |\beta_{\underline{i}}X_{\underline{i}}| \leq \delta|\underline{n}|^{1/p}\Big),$$

where j is integer such that $2^j \geq J$ and ϵ, without loss of generality considerations, is $\epsilon \leq \delta$. Then

$$
\begin{aligned}
P\Big[\big|\sum_{\underline{i}\in V}\beta_{\underline{i}}X_{\underline{i}}\big| > \epsilon|\underline{n}|^{1/p}\Big] &\le P\Big[\big|\sum_{\underline{i}\in V}\beta_{\underline{i}}(X'_{\underline{i}} - EX'_{\underline{i}})\big| > \frac{\epsilon|\underline{n}|^{1/p}}{4}\Big] \\
&\quad + P\Big[\big|\sum_{\underline{i}\in V}\beta_{\underline{i}}(EX'_{\underline{i}} + EX'''_{\underline{i}})\big| > \frac{\epsilon|\underline{n}|^{1/p}}{4}\Big] \\
&\quad + P\Big[\big|\sum_{\underline{i}\in V}\beta_{\underline{i}}X''_{\underline{i}}\big| > \frac{\epsilon|\underline{n}|^{1/p}}{4}\Big] \\
&\quad + P\Big[\big|\sum_{\underline{i}\in V}\beta_{\underline{i}}(X'''_{\underline{i}} - EX'''_{\underline{i}})\big| > \frac{\epsilon|\underline{n}|^{1/p}}{4}\Big] \\
&= I_1 + I_2 + I_3 + I_4, \quad \text{say.}
\end{aligned}
$$

To evaluate I_1 we use Lemma 3.2 to the field of random variables $|\underline{n}|^{-1/p}\beta_{\underline{i}}(X'_{\underline{i}} - EX'_{\underline{i}})$, i.e.

$$
\begin{aligned}
I_1 &\le C_j P\Big[\sup_{\underline{i}\in V}|\beta_{\underline{i}}(X'_{\underline{i}} - EX'_{\underline{i}})| > \frac{\epsilon|\underline{n}|^{1/p}}{24^j}\Big] \\
&\quad + D_j \sup_{k\in\mathbb{N}}\Big[P\big[\big|\sum_{l=1}^{k}\beta_{ord_V^{-1}(l)}(X'_{ord_V^{-1}(l)} - EX'_{ord_V^{-1}(l)})\big| > \frac{\epsilon|\underline{n}|^{1/p}}{24^j\cdot 4}\big]\Big]^{2^j}.
\end{aligned}
$$

Furthermore, because

$$
|\beta_{\underline{i}}X'_{\underline{i}}| \le \frac{\epsilon|\underline{n}|^{1/p}}{24^j\cdot 36}, \qquad \text{thus} \qquad |\beta_{\underline{i}}EX'_{\underline{i}}| \le \frac{\epsilon|\underline{n}|^{1/p}}{24^j\cdot 36},\ |\beta^2_{\underline{i}}X'^2_{\underline{i}}| \le \frac{\epsilon^2|\underline{n}|^{2/p}}{24^{2j}\cdot 36^2},
$$

and because $Var(X) \le EX^2$ for arbitrary random variable X, therefore:

$$
\begin{aligned}
&P\Big[\sup_{\underline{i}\in V}|\beta_{\underline{i}}(X'_{\underline{i}} - EX'_{\underline{i}})| > \frac{\epsilon|\underline{n}|^{1/p}}{24^j}\Big] \\
&\le P\Big[\sup_{\underline{i}\in V}|\beta_{\underline{i}}X'_{\underline{i}}| + \sup_{\underline{i}\in V}|\beta_{\underline{i}}EX'_{\underline{i}}| > \frac{\epsilon|\underline{n}|^{1/p}}{24^j}\Big] \\
&\le P\Big[\sup_{\underline{i}\in V}|\beta_{\underline{i}}X'_{\underline{i}}| > \frac{\epsilon|\underline{n}|^{1/p}}{24^j\cdot 2}\Big] + P\Big[\sup_{\underline{i}\in V}|\beta_{\underline{i}}EX'_{\underline{i}}| > \frac{\epsilon|\underline{n}|^{1/p}}{24^j\cdot 2}\Big] = 0
\end{aligned}
$$

and by Markov's inequality, we have

$$
\begin{aligned}
(9)\quad I_1 &\le D_j\Big(\frac{24^j\cdot 4}{\epsilon}\Big)^{2J}\sup_{k\in\mathbb{N}}\Big[\sum_{l=1}^{k}\beta^2_{ord_V^{-1}(l)}Var(X'_{ord_V^{-1}(l)})\Big]^J|\underline{n}|^{-2J/p} \\
&\le D_j\Big(\frac{24^j\cdot 4}{\epsilon}\Big)^{2J}\Big[\sum_{\underline{i}\in V}\beta^2_{\underline{i}}EX^2_{\underline{i}}\Big]^J|\underline{n}|^{-2J/p} \le C|\underline{n}|^{-2J/p}\Big(\sum_{\underline{i}\in V}\beta^2_{\underline{i}}\Big)^J.
\end{aligned}
$$

Using the facts:

$$EX_{\underline{i}}I[\beta_{\underline{i}}X_{\underline{i}}| \leq \delta|\underline{n}|^{1/p}] = -EX_{\underline{i}}I[|\beta_{\underline{i}}X_{\underline{i}}| > \delta|\underline{n}|^{1/p}],$$

$$X_{\underline{i}} = X_{\underline{i}}I[|X_{\underline{i}}| > |\underline{n}|^{1/p}] + X_{\underline{i}}I[|X_{\underline{i}}| \leq |\underline{n}|^{1/p}],$$

$$I[X+Y>\gamma] \leq I[X > \frac{\gamma}{2}] + I[Y > \frac{\gamma}{2}],$$

$$I[|X| > |Y|] \leq \Big|\frac{X}{Y}\Big|^{\gamma}, \text{ for all } \gamma > 0,$$

we get

$$(10) \quad I_2 = P\Big[|\sum_{\underline{i}\in V} EX_{\underline{i}}\beta_{\underline{i}}I[|\beta_{\underline{i}}X_{\underline{i}}| \leq \delta|\underline{n}|^{1/p}]| > \frac{\epsilon|\underline{n}|^{1/p}}{4}\Big]$$

$$= P\Big[\Big|-\sum_{\underline{i}\in V} EX_{\underline{i}}\beta_{\underline{i}}I[|\beta_{\underline{i}}X_{\underline{i}}| > \delta|\underline{n}|^{1/p}]\Big(I[|X_{\underline{i}}| > |\underline{n}|^{1/p}] + I[|X_{\underline{i}}| \leq |\underline{n}|^{1/p}]\Big)\Big| > \frac{\epsilon|\underline{n}|^{1/p}}{4}\Big]$$

$$\leq P\Big[\sum_{\underline{i}\in V} E\big|X_{\underline{i}}\beta_{\underline{i}}I[|X_{\underline{i}}| > |\underline{n}|^{1/p}]\big|I[|\beta_{\underline{i}}X_{\underline{i}}I[|X_{\underline{i}}| > |\underline{n}|^{1/p}]| > \delta|\underline{n}|^{1/p}] > \frac{\epsilon|\underline{n}|^{1/p}}{8}\Big]$$

$$+P\Big[\sum_{\underline{i}\in V} E\big|X_{\underline{i}}\beta_{\underline{i}}I[|X_{\underline{i}}| \leq |\underline{n}|^{1/p}]\big|I[|\beta_{\underline{i}}X_{\underline{i}}I[|X_{\underline{i}}| \leq |\underline{n}|^{1/p}]| > \delta|\underline{n}|^{1/p}] > \frac{\epsilon|\underline{n}|^{1/p}}{8}\Big]$$

$$\leq P\Big[\sum_{\underline{i}\in V}(\delta^p|\underline{n}|)^{-t-\beta-1+1/p+\eta/p}E\big|\beta_{\underline{i}}X_{\underline{i}}I[|X_{\underline{i}}| > |\underline{n}|^{1/p}]\big|^{p(t+\beta+1)-\eta} > \frac{\epsilon|\underline{n}|^{1/p}}{8}\Big]$$

$$+P\Big[\sum_{\underline{i}\in V}(\delta^p|\underline{n}|)^{-t-\beta-1+1/p-\eta/p}E\big|\beta_{\underline{i}}X_{\underline{i}}I[|X_{\underline{i}}| \leq |\underline{n}|^{1/p}]\big|^{p(t+\beta+1)+\eta} > \frac{\epsilon|\underline{n}|^{1/p}}{8}\Big]$$

$$\leq C\sum_{\underline{i}\in V}|\underline{n}|^{-t-\beta-1-\eta/p}E|\beta_{\underline{i}}X_{\underline{i}}I[|X_{\underline{i}}| \leq |\underline{n}|^{1/p}]|^{p(t+\beta+1)+\eta}$$

$$+C\sum_{\underline{i}\in V}|\underline{n}|^{-t-\beta-1+\eta/p}E|\beta_{\underline{i}}X_{\underline{i}}I[|X_{\underline{i}}| > |\underline{n}|^{1/p}]|^{p(t+\beta+1)-\eta} = K, \text{ say,}$$

for arbitrary $0 < \eta < p(t+\beta+1)-1$.

Let us remark that

$$\sum_{\underline{i}\in V} P[|\beta_{\underline{i}}X_{\underline{i}}| \geq C|\underline{n}|^{1/p}]$$

$$\leq \sum_{\underline{i}\in V} P[|\beta_{\underline{i}}X_{\underline{i}}|I[|X_{\underline{i}}| \leq |\underline{n}|^{1/p}] \geq C|\underline{n}|^{1/p}/2]$$

$$+\sum_{\underline{i}\in V} P[|\beta_{\underline{i}}X_{\underline{i}}|I[|X_{\underline{i}}| > |\underline{n}|^{1/p}] \geq C|\underline{n}|^{1/p}/2]$$

$$\leq C \sum_{\underline{i} \in V} |\underline{n}|^{-t-\beta-1-\eta/p} E|\beta_{\underline{i}} X_{\underline{i}} I[|X_{\underline{i}}| \leq |\underline{n}|^{1/p}]|^{p(t+\beta+1)+\eta}$$
$$+ C \sum_{\underline{i} \in V} |\underline{n}|^{-t-\beta-1+\eta/p} E|\beta_{\underline{i}} X_{\underline{i}} I[|X_{\underline{i}}| > |\underline{n}|^{1/p}]|^{p(t+\beta+1)-\eta} = K.$$

On the other hand

$$I_3 \leq \sum_{\underline{i} \in V} P[|\beta_{\underline{i}} X_{\underline{i}}| \geq \delta |\underline{n}|^{1/p}] = K. \tag{11}$$

Moreover, from Lemma 3.1, Markov's inequality and evaluation I_1 and I_3 we have

$$I_4 \leq \Big\{ C \sum_{\underline{i} \in V} P[|\beta_{\underline{i}} X_{\underline{i}}| > \frac{\epsilon |\underline{n}|^{1/p}}{36 \cdot 24^j}] + C|\underline{n}|^{-2J/p} \Big(\sum_{\underline{i} \in V} \beta_{\underline{i}}^2 E X_{\underline{i}}^2 I[|\beta_{\underline{i}} X_{\underline{i}}| \leq \delta |\underline{n}|^{1/p}] \Big)^J \Big\}$$
$$\leq K + C|\underline{n}|^{-2J/p} \Big(\sum_{\underline{i} \in V} \beta_{\underline{i}}^2 \Big)^J, \tag{12}$$

such that (9)–(12) ends the proof of Lemma 3.3. ■

The following Lemma is well known and its proof is standard:

Lemma 3.4. *Let $\{X_{\underline{n}}, \underline{n} \in V\}$ be a field of random variables which are stochastically dominated by a random variable X. For any $\alpha > 0$ and $b > 0$, the following statements hold:*

(i) $E|X_{\underline{n}}|^\alpha I[|X_{\underline{n}}| \leq b] \leq C\{E|X|^\alpha I[|X| \leq b] + b^\alpha P(|X| > b)\}$,
(ii) $E|X_{\underline{n}}|^\alpha I[|X_{\underline{n}}| > b] \leq C E|X|^\alpha I[|X| > b]$.

The following easy lemma may be found in [13] (cf. Examples 2.3 and 2.5):

Lemma 3.5. *We have*

$$M_V(x) = O(x(\log_+ x)^{\tau-1}), \ \textit{where } \tau \textit{ is defined in } (3).$$
$$d_V(x) = o(x^\delta), \ \textit{for each } \delta > 0, \ \textit{as } x \to \infty.$$

Lemma 3.6. *For some $\delta > 0$, the following evaluations hold:*

$$\sum_{|\underline{n}| \geq i, \underline{n} \in V} \frac{1}{|\underline{n}|^\beta} = \sum_{j \geq i} \frac{d_V(j)}{j^\beta} \leq \frac{C\beta}{\beta - 1} i^{1-\beta} (\log_+ i)^{\tau-1}, \ \textit{for } \beta > 1,$$
$$\sum_{|\underline{n}| \geq i, \underline{n} \in V} \frac{1}{|\underline{n}| \log_+^{\tau+\delta} |\underline{n}|} = \sum_{j \geq i} \frac{d_V(j)}{j \log_+^{\tau+\delta} j} \leq \frac{C}{\delta} (\log_+ i)^{-\delta},$$
$$\sum_{|\underline{n}| \leq i, \underline{n} \in V} \frac{1}{|\underline{n}|^\beta} = \sum_{j \leq i} \frac{d_V(j)}{j^\beta} \leq \begin{cases} \frac{C}{|1-\beta|} \big(2 \vee i^{1-\beta} (\log_+ i)^{\tau-1} \big), & \textit{for } \beta \neq 1, \\ C(\log_+ i)^\tau, & \textit{for } \beta = 1. \end{cases}$$

Sketch of the proof of Lemma 3.6. We apply Lemma 3.5 and use Cauchy-Maclaurin's Theorem ([17], p. 71) and Lagrange's Theorem ([17], §5.42, p 96). For example, for some $\theta \in [0,1]$:

$$\begin{aligned}\sum_{j\geq i}\frac{d_{\mathbb{N}^d}(j)}{j\log_+^{d+\delta} j} &\leq \sum_{j\geq i}\frac{M_{\mathbb{N}^d}(j)-M_{\mathbb{N}^d}(j-1)}{j\log_+^{d+\delta} j}\\ &\leq \sum_{j\geq i} M_{\mathbb{N}^d}(j)\Big(\frac{1}{j\log^{d+\delta} j}-\frac{1}{(j+1)\log^{d+\delta}(j+1)}\Big)-\frac{M_{\mathbb{N}^d}(i-1)}{i\log^{d+\delta}(i)}\\ &\leq \sum_{j\geq i} M_{\mathbb{N}^d}(j)\Big(\frac{\log_+^{d+\delta}(j+1)+\frac{(d+\delta)j}{j+\theta}\log_+^{d+\delta-1}(j+\theta)}{j(j+1)\log_+^{d+\delta}(j)\log_+^{d+\delta}(j+1)}\Big)\\ &\leq C\sum_{j\geq i}\Big(\frac{j\log_+^{d-1}(j)}{j(j+1)\log_+^{d+\delta}(j)}\Big)\leq C\sum_{j\geq i}\frac{1}{j\log_+^{1+\delta}(j)}\leq C\frac{1}{\log_+^{\delta}(i)}.\end{aligned}$$

The proof in other cases runs similarly. ∎

In the case of multidimensional indexes, the Lemma 2 ([14]) may be expressed as

Lemma 3.7. *Let X be a random variable and let $r,\delta,p>0$ be arbitrary. Then the following statements hold:*

(i) $\sum_{\underline{n}\in V}\frac{1}{|\underline{n}|^{1+\delta/p}}E|X|^{r+\delta}I[|X|\leq|\underline{n}|^{1/p}]\leq C(\frac{p}{\delta}+1)p^{\tau-1}E|X|^r(\log_+|X|)^{\tau-1}$,

(ii) $\sum_{\underline{n}\in V}\frac{1}{|\underline{n}|^{1-\delta/p}}E|X|^{r-\delta}I[|X|>|\underline{n}|^{1/p}]$
$\leq C\frac{p}{\delta}E|X|^r(\log_+|X|)^{\tau-1}(p^{\tau-1}\vee 2)$,

(iii) $\sum_{\underline{n}\in V}\frac{1}{|\underline{n}|^{1-r/p}}P[|X|>|\underline{n}|^{1/p}]$
$\leq \frac{Cp}{r}(E|X|^r(\log_+|X|)^{\tau-1}p^{\tau-1}\vee 2P(|X|>1))$.

Proof of Lemma 3.7. We have

$$\begin{aligned}(13)\qquad \sum_{\underline{n}\in V}\frac{1}{|\underline{n}|^{1+\delta/p}}E|X|^{r+\delta}I[|X|\leq|\underline{n}|^{1/p}]\\ &=\sum_{j=1}^{\infty}\frac{1}{j^{1+\delta/p}}d_V(j)\,E|X|^{r+\delta}I[|X|^p\leq j]\\ &=\sum_{j=1}^{\infty}\frac{1}{j^{1+\delta/p}}d_V(j)\sum_{i=1}^{j}E|X|^{r+\delta}I[i-1<|X|^p\leq i]\\ &=\sum_{i=1}^{\infty}E|X|^{r+\delta}I[i-1<|X|^p\leq i]\sum_{j=i}^{\infty}\frac{d_V(j)}{j^{1+\delta/p}}\end{aligned}$$

in case (i),

$$(14)\qquad \sum_{\underline{n}\in V} \frac{1}{|\underline{n}|^{1-\delta/p}}\, E|X|^{r-\delta}\, I[|X| > |\underline{n}|^{1/p}]$$

$$= \sum_{\underline{n}\in V} \frac{1}{|\underline{n}|^{1-\delta/p}} \sum_{i=|\underline{n}|}^{\infty} E|X|^{r-\delta}\, I[i^{1/p} < |X| \le (i+1)^{1/p}]$$

$$= \sum_{i=1}^{\infty} E|X|^{r-\delta}\, I[i^{1/p} < |X| \le (i+1)^{1/p}] \sum_{\substack{\underline{n}\in V\\ |\underline{n}|\le i}} \frac{1}{|\underline{n}|^{1-\delta/p}},$$

in case (ii) and

$$(15)\quad \sum_{\underline{n}\in V} \frac{1}{|\underline{n}|^{1-r/p}}\, P[|X| > |\underline{n}|^{1/p}] = \sum_{\underline{n}\in V} \frac{1}{|\underline{n}|^{1-r/p}} \sum_{i=|\underline{n}|}^{\infty} P[i < |X|^p \le i+1]$$

$$= \sum_{i=1}^{\infty} P[i < |X|^p \le i+1] \sum_{\substack{\underline{n}\in V\\ |\underline{n}|\le i}} \frac{1}{|\underline{n}|^{1-r/p}}$$

in the last case.

Now, taking into account Lemma 3.6 and (13–15), we get

$$\sum_{\underline{n}\in \mathbb{N}^d} \frac{1}{|\underline{n}|^{1+\delta/p}}\, E|X|^{r+\delta}\, I[|X| \le |\underline{n}|^{1/p}]$$

$$\le C\left(\frac{p}{\delta}+1\right) \sum_{i=1}^{\infty} E|X|^{r+\delta} I[i-1 < |X|^p \le i] i^{-\delta/p} (\log_+ i)^{d-1}$$

$$\le C\left(\frac{p}{\delta}+1\right) p^{d-1} E|X|^r (\log_+ |X|)^{d-1},$$

and

$$\sum_{\underline{n}\in V_c} \frac{1}{|\underline{n}|^{1+\delta/p}}\, E|X|^{r+\delta}\, I[|X| \le |\underline{n}|^{1/p}]$$

$$\le C\left(\frac{p}{\delta}+1\right) \sum_{i=1}^{\infty} E|X|^{r+\delta} I[i-1 < |X|^p \le i] i^{-\delta/p} \le C\left(\frac{p}{\delta}+1\right) E|X|^r.$$

Similarly, we get

$$\sum_{\underline{n}\in \mathbb{N}^d} \frac{1}{|\underline{n}|^{1-\delta/p}}\, E|X|^{r-\delta}\, I[|X| > |\underline{n}|^{1/p}] \le C\frac{p}{\delta} E|X|^r (\log_+ |X|)^{d-1} (p^{d-1} \vee 2),$$

$$\sum_{\underline{n}\in V_c} \frac{1}{|\underline{n}|^{1-\delta/p}}\, E|X|^{r-\delta}\, I[|X| > |\underline{n}|^{1/p}] \le 2C\frac{p}{\delta} E|X|^r,$$

$$\sum_{\underline{n}\in\mathbb{N}^d} \frac{1}{|\underline{n}|^{1-r/p}} P[|X| > |\underline{n}|^{1/p}]| \le C\frac{p}{r}(E|X|^r(\log_+|X|)^{d-1}p^{d-1} \vee 2P(|X|>1)),$$

$$\sum_{\underline{n}\in V_c} \frac{1}{|\underline{n}|^{1-r/p}} P[|X| > |\underline{n}|^{1/p}] \le C\frac{p}{r}(E|X|^r \vee 2P(|X| > 1)),$$

which ends the proof. ∎

Lemma 3.8. *Let $\{X_{\underline{n}}, \underline{n} \in V\}$ be a field of random variables which are stochastically dominated by a random variable X satisfying*

$$E|X|^{p(t+\beta+1)}(\log_+|X|)^{\tau-1} < \infty,$$

where $p(t+\beta+1) > 0$, β, $t \in \mathbb{R}$ and $p > 0$. Let $\{a_{\underline{n},\underline{i}}, \underline{i}, \underline{n} \in V\}$ be a field of random variables independent of field $\{X_{\underline{n}},\ \underline{n} \in V\}$ and such that

$$|a_{\underline{n},\underline{i}}| = O(1),\ \underline{i}, \underline{n} \in V,\ a.s.,$$

$$\sum_{\underline{i}\in V} E|a_{\underline{n},\underline{i}}|^q = O(|\underline{n}|^\beta),$$

for some $q < p(t+\beta+1)$. Then for any $\delta > 0$

$$\sum_{\underline{n}\in V} |\underline{n}|^t \sum_{\underline{i}\in V} E||\underline{n}|^{-1/p} a_{\underline{n},\underline{i}} X_{\underline{i}} I[|X_{\underline{i}}| \le |\underline{n}|^{1/p}]|^{p(t+\beta+1)+\delta} \le CE|X|^{p(t+\beta+1)}(\log_+|X|)^{\tau-1},$$

and for any positive δ such that $p(t+\beta+1)-\delta \ge q$ and $p(t+\beta+1)-\delta > 0$, we have

$$\sum_{\underline{n}\in V} |\underline{n}|^t \sum_{\underline{i}\in V} E||\underline{n}|^{-1/p} a_{\underline{n},\underline{i}} X_{\underline{i}} I[|X_{\underline{i}}| > |\underline{n}|^{1/p}]|^{p(t+\beta+1)-\delta} \le CE|X|^{p(t+\beta+1)}(\log_+|X|)^{\tau-1}.$$

Proof. Applying Lemma 3.4, we obtain

$$\begin{aligned}&\sum_{\underline{n}\in V} |\underline{n}|^t \sum_{\underline{i}\in V} E||\underline{n}|^{-1/p} a_{\underline{n},\underline{i}} X_{\underline{i}} I[|X_{\underline{i}}| \le |\underline{n}|^{1/p}]|^{p(t+\beta+1)+\delta}\\ &\le C\sum_{\underline{n}\in V} |\underline{n}|^{-\beta-1-\delta/p}\Big(\sum_{\underline{i}\in V} E|a_{\underline{n},\underline{i}}|^q\Big)\\ &\quad\times \big(E|X|^{p(t+\beta+1)+\delta} I[|X| \le |\underline{n}|^{1/p}] + |\underline{n}|^{(p(t+\beta+1)+\delta)/p} P[|X| > |\underline{n}|^{1/p}]\big)\end{aligned}$$

and

$$\begin{aligned}&\sum_{\underline{n}\in V} |\underline{n}|^t \sum_{\underline{i}\in V} E||\underline{n}|^{-1/p} a_{\underline{n},\underline{i}} X_{\underline{i}} I[|X_{\underline{i}}| > |\underline{n}|^{1/p}]|^{p(t+\beta+1)-\delta}\\ &\le C\sum_{\underline{n}\in V} |\underline{n}|^{-\beta-1+\delta/p}\big(\sum_{\underline{i}\in V} E|a_{\underline{n},\underline{i}}|^q\big)\big(E|X|^{p(t+\beta+1)-\delta} I[|X| > |\underline{n}|^{1/p}]\big),\end{aligned}$$

Now Lemma 3.7 finishes the proof. ∎

Lemma 3.9. *Let $\mathbb{X} = \{X_{\underline{k}}, \underline{k} \in V\}$ be the random field independent of the random field $\mathbb{Y} = \{Y_{\underline{k}}, \underline{k} \in V\}$. Let $\mu_{\mathbb{X}}, \mu_{\mathbb{Y}}, \mu_{\mathbb{X},\mathbb{Y}}$ be probability measures of fields $\mathbb{X}, \mathbb{Y}$ and $(\mathbb{X}, \mathbb{Y})$, respectively. Let $\{A_{\underline{i}}, \underline{i} \in V\}$ be the family of compact sets in $\Re$. Let $g(\mathbf{x}, \mathbf{y})$ and $h(\mathbf{x}, \mathbf{y})$ be two measurable Borel functions such that for every field of numbers $\mathbf{y} \in \{A_{\underline{i}}, \underline{i} \in V\}$:*

$$Eg(\mathbb{X}, \mathbf{y}) \leq Eh(\mathbb{X}, \mathbf{y}) < C,$$
$$Y_{\underline{i}} \in A_{\underline{i}}, \ a.s., \ \underline{i} \in V,$$

then

$$Eg(\mathbb{X}, \mathbb{Y}) \leq Eh(\mathbb{X}, \mathbb{Y}). \tag{16}$$

Proof. From the independency of $\mathbb{X}$ and $\mathbb{Y}$ and the Fubini's Theorem, we have

$$\begin{aligned} Eg(\mathbb{X}, \mathbb{Y}) &= \int g(\mathbf{x}, \mathbf{y}) d\mu_{\mathbb{X},\mathbb{Y}} = \iint g(\mathbf{x}, \mathbf{y}) d\mu_{\mathbb{X}} d\mu_{\mathbb{Y}} \\ &= \int\limits_{\mathbf{y} \in \{A_{\underline{i}}, \underline{i} \in V\}} Eg(\mathbb{X}, \mathbf{y}) d\mu_{\mathbb{Y}} \leq \int\limits_{\mathbf{y} \in \{A_{\underline{i}}, \underline{i} \in V\}} Eh(\mathbb{X}, \mathbf{y}) d\mu_{\mathbb{Y}} \\ &= Eh(\mathbb{X}, \mathbb{Y}). \ \blacksquare \end{aligned}$$

Proof of Theorem 2.1.

(I) For any $\delta > 0$ such that $p(t+\beta+1)+\delta \leq 1$, using Markov's inequality and Lemma 3.8, we get

$$\begin{aligned} &\sum_{\underline{n} \in V} |\underline{n}|^t P\Big(\Big|\sum_{\underline{i} \in V} |\underline{n}|^{-1/p} a_{\underline{n},\underline{i}} X_{\underline{i}} I[|X_{\underline{i}}| \leq |\underline{n}|^{1/p}]\Big| > \epsilon \Big) \\ &\leq \frac{1}{\epsilon^{p(t+\beta+1)+\delta}} \sum_{\underline{n} \in V} |\underline{n}|^t \sum_{\underline{i} \in V} E\Big| |\underline{n}|^{-1/p} a_{\underline{n},\underline{i}} X_{\underline{i}} I[|X_{\underline{i}}| \leq |\underline{n}|^{1/p}]\Big|^{p(t+\beta+1)+\delta} < \infty. \end{aligned} \tag{17}$$

Similarly, taking $\delta > 0$ such that $p(t + \beta + 1) - \delta \geq q$ and $p(t + \beta + 1) - \delta > 0$, we have

$$\begin{aligned} &\sum_{\underline{n} \in V} |\underline{n}|^t P\Big(\Big|\sum_{\underline{i} \in V} |\underline{n}|^{-1/p} a_{\underline{n},\underline{i}} X_{\underline{i}} I[|X_{\underline{i}}| > |\underline{n}|^{1/p}]\Big| > \epsilon \Big) \\ &\leq \frac{1}{\epsilon^{p(t+\beta+1)-\delta}} \sum_{\underline{n} \in V} |\underline{n}|^t \sum_{\underline{i} \in V} E\Big| |\underline{n}|^{-1/p} a_{\underline{n},\underline{i}} X_{\underline{i}} I[|X_{\underline{i}}| > |\underline{n}|^{1/p}]\Big|^{p(t+\beta+1)-\delta} < \infty. \end{aligned} \tag{18}$$

From (17) and (18), we obtain (2) in the case $0 < p(t + \beta + 1) < 1$.

(II) Consider the case $p(t + \beta + 1) < 2$. Taking $\delta > 0$ such that $p(t + \beta + 1) + \delta \leq 2$, we get by Markov's inequality, Marcinkiewicz–Zygmund's inequality, the c_r-inequality and Jensen's inequality for arbitrary family of real numbers $\{\beta_{\underline{n}}, \underline{n} \in V\}$ such that $0 \leq \beta_{\underline{n}} \leq C$, arbitrary $m \in \mathbb{R}$, similarly as in [14]:

$$
\begin{aligned}
(19)\quad & P\Big[\Big|\sum_{\underline{i}\in V}\beta_{\underline{i}}\Big(X_{\underline{i}}I[|X_{\underline{i}}|\le m]-EX_{\underline{i}}I[|X_{\underline{i}}|\le m]\Big)\Big|>\epsilon m\Big]\\
&\le \frac{1}{(m\epsilon)^{p(t+\beta+1)+\delta}}E\Big|\sum_{\underline{i}\in V}\beta_{\underline{i}}\big(X_{\underline{i}}I[|X_{\underline{i}}|\le m]-EX_{\underline{i}}I[|X_{\underline{i}}|\le m]\big)\Big|^{p(t+\beta+1)+\delta}\\
&\le C\frac{1}{m^{p(t+\beta+1)+\delta}}\sum_{\underline{i}\in V}E\big|\beta_{\underline{i}}X_{\underline{i}}I[|X_{\underline{i}}|\le m]\big|^{p(t+\beta+1)+\delta}.
\end{aligned}
$$

Now putting

$$
\begin{aligned}
g(\{\alpha_{\underline{i}}\},\{\beta_{\underline{i}}\}) &= I\Big[\Big|\sum_{\underline{i}\in V}\beta_{\underline{i}}\big(\alpha_{\underline{i}}I[|\alpha_{\underline{i}}|\le m]-EX_{\underline{i}}I[|X_{\underline{i}}|\le m]\big)\Big|>\epsilon m\Big],\\
h(\{\alpha_{\underline{i}}\},\{\beta_{\underline{i}}\}) &= C\frac{1}{m^{p(t+\beta+1)+\delta}}\sum_{\underline{i}\in V}\big|\beta_{\underline{i}}\alpha_{\underline{i}}I[|\alpha_{\underline{i}}|\le m]\big|^{p(t+\beta+1)+\delta}
\end{aligned}
$$

from (19), we see that $Eg(\{X_{\underline{i}}\},\{\beta_{\underline{i}}\})\le Eh(\{X_{\underline{i}}\},\{\beta_{\underline{i}}\})$ such that Lemma 3.9 implies for arbitrary $\underline{n}$

$$
\begin{aligned}
(20)\quad & P\Big[\Big|\sum_{\underline{i}\in V}a_{\underline{n},\underline{i}}(X_{\underline{i}}I[|X_{\underline{i}}|\le|\underline{n}|^{1/p}]-EX_{\underline{i}}I[|X_{\underline{i}}|\le|\underline{n}|^{1/p}])\Big|>\epsilon|\underline{n}|^{1/p}\Big]\\
&\qquad\le C|\underline{n}|^{-t-\beta-1-\delta/p}\sum_{\underline{i}\in V}E|a_{\underline{n},\underline{i}}X_{\underline{i}}|^{p(t+\beta+1)+\delta}I[|X_{\underline{i}}|\le|\underline{n}|^{1/p}].
\end{aligned}
$$

Now multiplying both two sides (20) by $|\underline{n}|^t$, summing on $\underline{n}\in V$ and applying Lemma 3.8, we get

$$
\begin{aligned}
(21)\quad & \sum_{\underline{n}\in V}|\underline{n}|^tP\Big[\Big|\sum_{\underline{i}\in V}a_{\underline{n},\underline{i}}\big(X_{\underline{i}}I[|X_{\underline{i}}|\le|\underline{n}|^{1/p}]-EX_{\underline{i}}I[|X_{\underline{i}}|\le|\underline{n}|^{1/p}]\big)\Big|>\epsilon|\underline{n}|^{1/p}\Big]\\
&\qquad\le C\sum_{\underline{n}\in V}|\underline{n}|^{-\beta-1-\delta/p}\sum_{\underline{i}\in V}E|a_{\underline{n},\underline{i}}X_{\underline{i}}|^{p(t+\beta+1)+\delta}I[|X_{\underline{i}}|\le|\underline{n}|^{1/p}]\\
&\qquad\le CE|X|^{p(t+\beta+1)}(\log_+|X|)^{\tau-1}<\infty.
\end{aligned}
$$

(III) Now, in the case $1<p(t+\beta+1)\le 2$, taking $\delta>0$ such that $p(t+\beta+1)-\delta\ge\max\{q,1\}$ and using Markov's inequality, Marcinkiewicz–Zygmund's inequality, the c_r-inequality and Jensen's inequality for arbitrary family of real numbers $\{\beta_{\underline{n}},\underline{n}\in V\}$ like in [14], we get

$$\begin{aligned}(22)\quad & P\Big[\Big|\sum_{\underline{i}\in V}\beta_{\underline{i}}\Big(X_{\underline{i}}I[|X_{\underline{i}}|>m]-EX_{\underline{i}}I[|X_{\underline{i}}|>m]\Big)\Big|>\epsilon m\Big]\\ &\le \frac{1}{(m\epsilon)^{p(t+\beta+1)-\delta}}E\Big|\sum_{\underline{i}\in V}\beta_{\underline{i}}\big(X_{\underline{i}}I[|X_{\underline{i}}|>m]-EX_{\underline{i}}I[|X_{\underline{i}}|>m]\big)\Big|^{p(t+\beta+1)-\delta}\\ &\le C\frac{1}{m^{p(t+\beta+1)-\delta}}\sum_{\underline{i}\in V}E\big|\beta_{\underline{i}}X_{\underline{i}}I[|X_{\underline{i}}|>m]\big|^{p(t+\beta+1)-\delta}.\end{aligned}$$

In this case, we put

$$g(\{\alpha_{\underline{i}}\},\{\beta_{\underline{i}}\}) = I\Big[\Big|\sum_{\underline{i}\in V}\beta_{\underline{i}}\big(\alpha_{\underline{i}}I[|\alpha_{\underline{i}}|>m]-EX_{\underline{i}}I[|X_{\underline{i}}|>m]\big)\Big|>\epsilon m\Big],$$

$$h(\{\alpha_{\underline{i}}\},\{\beta_{\underline{i}}\}) = C\frac{1}{m^{p(t+\beta+1)-\delta}}\sum_{\underline{i}\in V}|\beta_{\underline{i}}\alpha_{\underline{i}}I[|\alpha_{\underline{i}}|>m]|^{p(t+\beta+1)-\delta}$$

and use Lemma 3.9 to obtain that

$$\begin{aligned}(23)\quad & P\Big[\Big|\sum_{\underline{i}\in V}a_{\underline{n},\underline{i}}\big(X_{\underline{i}}I[|X_{\underline{i}}|>|\underline{n}|^{1/p}]-EX_{\underline{i}}I[|X_{\underline{i}}|>|\underline{n}|^{1/p}]\big)\Big|>\epsilon|\underline{n}|^{1/p}\Big]\\ &\qquad\le C|\underline{n}|^{-t-\beta-1+\delta/p}\sum_{\underline{i}\in V}E|a_{\underline{n},\underline{i}}X_{\underline{i}}|^{p(t+\beta+1)-\delta}I[|X_{\underline{i}}|>|\underline{n}|^{1/p}].\end{aligned}$$

Finally, we have

(24)

$$\begin{aligned}&\sum_{\underline{n}\in V}|\underline{n}|^{t}P\Big[\Big|\sum_{\underline{i}\in V}a_{\underline{n},\underline{i}}\big(X_{\underline{i}}I[|X_{\underline{i}}|>|\underline{n}|^{1/p}]-EX_{\underline{i}}I[|X_{\underline{i}}|>|\underline{n}|^{1/p}]\big)\Big|\le\epsilon|\underline{n}|^{1/p}\Big]\\ &\qquad\le C\sum_{\underline{n}\in V}|\underline{n}|^{-\beta-1+\delta/p}\sum_{\underline{i}\in V}E|a_{\underline{n},\underline{i}}X_{\underline{i}}|^{p(t+\beta+1)-\delta}I[|X_{\underline{i}}|>|\underline{n}|^{1/p}]\\ &\qquad\le CE|X|^{p(t+\beta+1)}(log_{+}|X|)^{r-1}<\infty.\end{aligned}$$

Since $EX_{\underline{n}}=0$, (2) holds by (21) and (24) when $1<p(t+\beta+1)<2$.

(IV) Let us consider the case $p(t+\beta+1)=1$ and $t\ge -1$ (otherwise the proof is obvious). Since, from Lemma 3.4(ii), (4), (5), and the fact that $E|X|<\infty$ in both cases of definition V, we have:

$$(25)\qquad E|X_{\underline{n}}|I[|X_{\underline{n}}|>b]\le D_1E|X|I[|X|>b],$$

$$(26)\qquad \sum_{\underline{i}\in V}E|a_{\underline{n},\underline{i}}|\le D_2|\underline{n}|^{\beta},\ \text{ for } \underline{n}\in V,$$

$$(27)\qquad E|X|I[|X|>K]\le\frac{\epsilon}{16D_1D_2}\ \text{for sufficiently large } K,$$

then for $\underline{n}$ such that $|\underline{n}| > K^p$, we have

$$\Big|\sum_{\underline{i}\in V} |\underline{n}|^{-1/p} E a_{\underline{n},\underline{i}} X_{\underline{i}} I[|X_{\underline{i}}| > |\underline{n}|^{1/p}]\Big| \le \frac{\epsilon}{16}. \tag{28}$$

Furthermore, because $EX_{\underline{n}} = 0, \underline{n} \in V$,

$$\begin{aligned} &P\Big[\Big|\sum_{\underline{i}\in V} a_{\underline{n},\underline{i}} X_{\underline{i}}\Big| > \epsilon |\underline{n}|^{1/p}\Big] \\ &\le P\Big[\Big|\sum_{\underline{i}\in V} a_{\underline{n},\underline{i}}\big(X_{\underline{i}} I[|X_{\underline{i}}| \le |\underline{n}|^{1/p}] - EX_{\underline{i}} I[|X_{\underline{i}}| \le |\underline{n}|^{1/p}]\big)\Big| > \frac{\epsilon}{2}|\underline{n}|^{1/p}\Big] \\ &\quad + P\Big[\Big|\sum_{\underline{i}\in V} a_{\underline{n},\underline{i}} X_{\underline{i}} I[|X_{\underline{i}}| > |\underline{n}|^{1/p}]\Big| > \frac{\epsilon}{4}|\underline{n}|^{1/p}\Big] \\ &\quad + P\Big[\Big|\sum_{\underline{i}\in V} a_{\underline{n},\underline{i}} EX_{\underline{i}} I[|X_{\underline{i}}| > |\underline{n}|^{1/p}]\Big| > \frac{\epsilon}{4}|\underline{n}|^{1/p}\Big] \\ &= H_{\underline{n},1} + H_{\underline{n},2} + H_{\underline{n},3}, \text{ say.} \end{aligned}$$

On the other hand, the term $\sum_{\underline{n}\in V} |\underline{n}|^t H_{\underline{n},1}$ on the right hand side is evaluated in (21), whereas the term $\sum_{\underline{n}\in V} |\underline{n}|^t H_{\underline{n},2}$ may be bounded by

$$\begin{aligned} &C \sum_{\underline{n}\in V} |\underline{n}|^t E\Big|\sum_{\underline{i}\in V} |\underline{n}|^{-1/p} a_{\underline{n},\underline{i}} X_{\underline{i}} I[|X_{\underline{i}}| > |\underline{n}|^{1/p}]\Big|^{p(t+\beta+1)-\delta} \\ &\qquad\qquad \le CE|X|^{p(t+\beta+1)} (log_+|X|)^{\tau-1} < \infty. \end{aligned}$$

In the case of (a),

$$\Big|\sum_{\underline{i}\in V} a_{\underline{n},\underline{i}} EX_{\underline{i}} I[|X_{\underline{i}}| > |\underline{n}|^{1/p}]\Big| \le \sum_{\underline{i}\in V} |a_{\underline{n},\underline{i}}| D_1 E|X| I[|X| > |\underline{n}|^{1/p}] \le \frac{\epsilon}{4}|\underline{n}|^{1/p},$$

such that $H_{\underline{n},3} = 0$ for $|\underline{n}| > K^p$.

For the proof of Theorem 2.1(ii)(b) we consider first the case $|\underline{n}| > K^p$, too. Because from (28),

$$P\Big[\Big|\sum_{\underline{i}\in V} E a_{\underline{n},\underline{i}} EX_{\underline{i}} I[|X_{\underline{i}}| > |\underline{n}|^{1/p}]\Big| > \frac{\epsilon}{16}|\underline{n}|^{1/p}\Big] = 0,$$

we have

$$H_{\underline{n},3} \le P\Big[\Big|\sum_{\underline{i}\in V} (a_{\underline{n},\underline{i}} - E a_{\underline{n},\underline{i}}) EX_{\underline{i}} I[|X_{\underline{i}}| > |\underline{n}|^{1/p}]\Big| > \frac{\epsilon}{16}|\underline{n}|^{1/p}\Big].$$

From (b) there exists constant D_3 such that $\sum_{\underline{i}\in V} E|a_{\underline{n},\underline{i}} - E a_{\underline{n},\underline{i}}| \le D_3 \frac{|\underline{n}|^\beta}{\log_+^{\tau+\gamma}|\underline{n}|}$, such that using Markov's inequality, (25) and Lemma 3.6, we get

$$\sum_{\underline{n}\in V:|\underline{n}|>K^p} |\underline{n}|^t H_{\underline{n},3}$$
$$\leq \sum_{\underline{n}\in V:|\underline{n}|>K^p} \frac{16D_1|\underline{n}|^{t-1/p}}{\epsilon} \sum_{\underline{i}\in V} E|a_{\underline{n},\underline{i}} - Ea_{\underline{n},\underline{i}}|E|X|I[|X| > |\underline{n}|^{1/p}]$$
$$\leq \sum_{\underline{n}\in V:|\underline{n}|>K^p} \frac{16D_1D_3}{\epsilon} \frac{|\underline{n}|^{t+\beta-1/p}}{\log_+^{\tau+\gamma}|\underline{n}|} E|X|$$
$$\leq \frac{16D_1D_3}{\epsilon} E|X| \sum_{\underline{n}\in V:|\underline{n}|>K^p} \frac{1}{|\underline{n}|\log_+^{\tau+\gamma}|\underline{n}|}$$
$$\leq \frac{16CD_1D_3}{\gamma\epsilon} E|X|(\log_+ K)^{-\gamma} < \infty.$$

In both cases (a) and (b), from Lemma 3.6, we have additionally

$$\sum_{\underline{n}\in V:|\underline{n}|\leq K^p} |\underline{n}|^t P\Big[\Big|\sum_{\underline{i}\in V} a_{\underline{n},\underline{i}}X_{\underline{i}}\Big| > \epsilon|\underline{n}|^{1/p}\Big] \leq \begin{cases} \frac{C}{|1+t|}K^{(1+t)p}\log_+^{\tau-1}K, & \text{if } t > -1, \\ C\log_+^{\tau}K, & \text{if } t = -1, \end{cases} < \infty.$$

If the condition (c) holds, we obtain

$$\sum_{\underline{n}\in V} |\underline{n}|^t H_{\underline{n},3} \leq D_1\frac{4}{\epsilon}\sum_{\underline{n}\in V}|\underline{n}|^{t-1/p}\sum_{\underline{i}\in V}E|a_{\underline{n},\underline{i}}|E|X|I[|X| > |\underline{n}|^{1/p}]$$
$$\leq D_1D_2\frac{4}{\epsilon}\sum_{\underline{n}\in V}|\underline{n}|^{t+\beta-1/p}E|X|I[|X| > |\underline{n}|^{1/p}]$$
$$\leq D_1D_2\frac{4}{\epsilon}\sum_{\underline{n}\in V}|\underline{n}|^{-1}E|X|I[|X| > |\underline{n}|^{1/p}]$$
$$\leq D_1D_2\frac{4}{\epsilon}\sum_{\underline{n}\in V}|\underline{n}|^{-1}\frac{E|X|\log_+^{\tau+\gamma}|X|}{\left(\frac{1}{p}\right)^{\tau+\gamma}\log_+^{\tau+\gamma}|\underline{n}|} < \infty,$$

which ends the proof of Theorem 2.1 in the case $p(t+\beta+1) = 1$.

(V) Let $\{\beta_{\underline{n},\underline{i}},\ \underline{n},\underline{i}\in V\}$ be a field of real numbers bounded by M. From Lemma 3.3, we obtain

$$\sum_{\underline{n}\in V}|\underline{n}|^t P[|\sum_{\underline{i}\in V}\beta_{\underline{n},\underline{i}}X_{\underline{i}}| > \epsilon|\underline{n}|^{1/p}] \leq C\sum_{\underline{n}\in V}|\underline{n}|^{t-2J/p}\big(\sum_{\underline{i}\in V}\beta_{\underline{n},\underline{i}}^2\big)^J$$
$$+ C\sum_{\underline{n}\in V}|\underline{n}|^{-\beta-1-\eta/p}\sum_{\underline{i}\in V}E|\beta_{\underline{n},\underline{i}}X_{\underline{i}}I[|X_{\underline{i}}| \leq |\underline{n}|^{1/p}]|^{p(t+\beta+1)+\eta}$$
$$+ C\sum_{\underline{n}\in V}|\underline{n}|^{-\beta-1+\eta/p}\sum_{\underline{i}\in V}E|\beta_{\underline{n},\underline{i}}X_{\underline{i}}I[|X_{\underline{i}}| > |\underline{n}|^{1/p}]|^{p(t+\beta+1)-\eta}.$$

Putting

$$g(\{X_{\underline{i}}\},\{\beta_{\underline{n},\underline{i}}\}) = I[|\sum_{\underline{i}\in V}\beta_{\underline{n},\underline{i}}X_{\underline{i}}| > \epsilon|\underline{n}|^{1/p}],$$

$$h(\{X_{\underline{i}}\},\{\beta_{\underline{n},\underline{i}}\}) = C|\underline{n}|^{-2J/p}\big(\sum_{\underline{i}\in V}\beta^2_{\underline{n},\underline{i}}\big)^J + C\frac{1}{|\underline{n}|^{t+1+\beta+\eta/p}}\sum_{\underline{i}\in V}|\beta_{\underline{n},\underline{i}}X_{\underline{i}}I[|X_{\underline{i}}| \le |\underline{n}|^{1/p}]|^{p(t+\beta+1)+\eta} + C\frac{1}{|\underline{n}|^{t+1+\beta-\eta/p}}\sum_{\underline{i}\in V}|\beta_{\underline{n},\underline{i}}X_{\underline{i}}I[|X_{\underline{i}}| > |\underline{n}|^{1/p}]|^{p(t+\beta+1)-\eta},$$

applying Lemma 3.9, multiplying both two sides by $|\underline{n}|^t$ and summing on $\underline{n}\in V$, we get

$$\sum_{\underline{n}\in V}|\underline{n}|^tP\Big(\Big|\sum_{\underline{i}\in V}a_{\underline{n},\underline{i}}X_{\underline{i}}\Big| > \epsilon|\underline{n}|^{1/p}\Big) \le C\sum_{\underline{n}\in V}|\underline{n}|^{t-2J/p}E\Big(\sum_{\underline{i}\in V}a^2_{\underline{n},\underline{i}}\Big)^J + C\sum_{\underline{n}\in V}\frac{1}{|\underline{n}|^{1+\beta+\eta/p}}E\Big(\sum_{\underline{i}\in V}|a_{\underline{n},\underline{i}}X_{\underline{i}}I[|X_{\underline{i}}| \le |\underline{n}|^{1/p}]|^{p(t+\beta+1)+\eta}\Big) + C\sum_{\underline{n}\in V}\frac{1}{|\underline{n}|^{1+\beta-\eta/p}}E\Big(\sum_{\underline{i}\in V}|a_{\underline{n},\underline{i}}X_{\underline{i}}I[|X_{\underline{i}}| > |\underline{n}|^{1/p}]|^{p(t+\beta+1)-\eta}\Big).$$

Finally, using Lemma 3.8 and (6), we have

$$\sum_{\underline{n}\in V}|\underline{n}|^tP\Big(\Big|\sum_{\underline{i}\in V}a_{\underline{n},\underline{i}}X_{\underline{i}}\Big| > \epsilon|\underline{n}|^{1/p}\Big) \le C\sum_{\underline{n}\in V}|\underline{n}|^{t-2J/p}E\big(\sum_{\underline{i}\in V}a^2_{\underline{n},\underline{i}}\big)^J + CE|X|^{p(t+\beta+1)}\big(\log_+|X|\big)^{\tau-1} + CE|X|^{p(t+\beta+1)}\big(\log_+|X|\big)^{\tau-1} < \infty,$$

which ends the proof of Theorem 2.1 in case $p(t+\beta+1)\ge 2$.

Thus Theorem 2.1(i) follows from **(I), (II)** and **(III)**, Theorem 2.1(ii) from **(II)** and **(IV)** and Theorem 2.1(iii) is the consequence of **(V)**. ■

Proofs of Corollaries 2.1 and 2.2. For proofs we put in Theorem 2.1 $a_{\underline{n},\underline{i}} = I[N_{\underline{n}} \ge |\underline{i}|]$ and $a_{\underline{n},\underline{i}} = I[\underline{N}_{\underline{n}} \ge \underline{i}]$ in items (a) and (b), respectively. We only remark, that

$$\sum_{\underline{i}\in V}P[N_{\underline{n}} \ge |\underline{i}|] = EM_V(N_{\underline{n}}),$$

$$E\big(\sum_{\underline{i}\in V}I[N_{\underline{n}} \ge |\underline{i}|]\big)^J = E\big(M_V(N_{\underline{n}})\big)^J,$$

$$\sum_{\underline{i}\in V}P[\underline{N}_{\underline{n}} \ge \underline{i}] \le E|\underline{N}_{\underline{n}}|,$$

$$E\Big(\sum_{\underline{i}\in V} I[\underline{N}_{\underline{n}} \geq \underline{i}]\Big)^J \leq E|\underline{N}_{\underline{n}}|^J.$$

Whereas for proof of Corollary 2.2, we put

$$a_{\underline{n},\underline{i}} = \begin{cases} \varepsilon_{\underline{i}},\ \underline{1} \leq \underline{i} \leq \underline{n}, \\ 0,\ \text{otherwise}, \end{cases} \text{ and } a_{\underline{n},\underline{i}} = \varepsilon_{\underline{n},\underline{i}}$$

in items (a) and (b), respectively. Furthermore, from Lemma 3.1 and Lemma 3.5:

$$\sum_{\underline{i}\in V,\underline{i}\leq\underline{n}} P[\varepsilon_{\underline{i}} = 1] = \hat{p}M_V(|\underline{n}|) = O\big(|\underline{n}|(\log_+|\underline{n}|)^{\tau-1}\big),$$

$$E\Big(\sum_{\underline{i}\in V,\underline{i}\leq\underline{n}} \varepsilon_{\underline{i}}^2\Big)^J \leq C\Big(\sum_{\underline{i}\in V,\underline{i}\leq\underline{n}} P[\varepsilon_{\underline{i}} = 1] + \Big(\sum_{\underline{i}\in V,\underline{i}\leq\underline{n}} P[\varepsilon_{\underline{i}} = 1]\Big)^{J/2}\Big)$$

$$\leq 2C\hat{p}^{J/2}M_V^{J/2}(|\underline{n}|) = O\big(|\underline{n}|^{J/2}\big(\log_+|\underline{n}|\big)^{J(\tau-1)/2}\big),$$

$$E\Big(\sum_{\underline{i}\in V} \varepsilon_{\underline{n},\underline{i}}^2\Big)^J \leq 2C\Big(\sum_{\underline{i}\in V} \hat{p}_{\underline{n},\underline{i}}\Big)^{J/2}. \ \blacksquare$$

Acknowledgement. The authors wish to thank the anonymous referee for his/her helpfull remarks.

References

[1] P. Erdős, *On a theorem of Hsu and Robbins*, Acta. Math. Statist. 20 (1949), 286–291.

[2] L. E. Baum, M. Katz, *Convergence rates in the law of large numbers*, Trans. Amer. Math. Soc. 120 (1965), 108–123.

[3] A. Gut, *Complete convergence. Asymptotics statistics. Proceedings of the Fifth Prague Symposium*, Physica Verlag held September 4–9, 1993, (1994), 237–247.

[4] A. Gut, *Complete convergence and convergence rates for randomly indexed partial sums with an application to some first passage times*, Acta Math. Acad. Sci. Hungar. 42 (1983), 225–232; Correction, ibid. 45 (1985), 235–236.

[5] A. Gut, *Marcinkiewicz laws and convergence rates in the law of large numbers for random variables with multidimensional indices*, The Annals of Probab. 6(3) (1978), 469–482.

[6] J. Hoffmann-Jørgensen, *Sums of independent Banach space valued random variables*, Studia Math. 52 (1974), 159–186.

[7] P. L. Hsu, H. Robbins, *Complete conevergence and the law of large numbers*, Proc. Nat. Acad. Sci. USA 33 (1947), 25–31.

[8] T.-C. Hu, M. O. Cabrera, S. H. Sung, A. Volodin, *Complete convergence for arrays of rowwise independent random variables*, Commun. Korean Math. Soc. 18 (2003), 375–383.

[9] T.-C. Hu, F. Móricz, R. L. Taylor, *Strong laws of large numbers for arrays of rowwise independent random variables*, Acta Math. Hung. 54(1-2) (1989), 153–162.

[10] T.-C. Hu, D. Szynal, A. I. Volodin, *A note on complete convergence for arrays*, Statist. Probab. Lett. 38 (1998), 27–31.

[11] M. Ledoux, M. Talagrand, *Probability in Banach Space*, Springer, 1991.
[12] H. P. Rosenthal, *On the subspaces of $L^p (p > 2)$ spanned by sequences of independent random variables*, Israel J. Math. 8 (1970), 273–303.
[13] R. T. Smythe, *Sums of independent random variables on partially ordered sets*, Ann. Probability 2 (1974), 906–917.
[14] S. H. Sung, *Complete convergence for weighted sum of random variables*, Statist. Probab. Lett. 77 (2007), 303–311.
[15] S. H. Sung, A. I. Volodin, T.-C. Hu, *More on complete convergence for arrays*, Statist. Probab. Lett. 71 (2005), 303–311.
[16] L. V. Thahn, G. Yin, *Almost sure and complete convergence of randomly weighted sums of independent random elements in Banach spaces*, Taiwanese Journal of Mathematics 15 (2011), 1759–1781.
[17] E. T. Whittaker, G. N. Watson, *A course of modern analysis. An introduction to the general theory of infinite processes and of analytic functions with an account of the principal trnscendental functions*, Cambridge University Press, Fourth Edition reprinted 2002.

Received December 2, 2011; revised version October 5, 2012.