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Sebastian P. Kuniewski, Jolanta K. Misiewicz

POSITIVE DEFINITE NORM DEPENDENT MATRICES IN STOCHASTIC MODELING

Abstract. Positive definite norm dependent matrices are of interest in stochastic modeling of distance/norm dependent phenomena in nature. An example is the application of geostatistics in geographic information systems or mathematical analysis of varied spatial data. Because the positive definiteness is a necessary condition for a matrix to be a valid correlation matrix, it is desirable to give a characterization of the family of the distance/norm dependent functions that form a valid (positive definite) correlation matrix. Thus, the main reason for writing this paper is to give an overview of characterizations of norm dependent real functions and consequently norm dependent matrices, since this information is somehow hidden in the theory of geometry of Banach spaces.

1. Introduction

Modeling natural phenomena using probability theory has increased over last years. From the mathematical point of view, the most convenient in such modeling turned out to be constructions based on independent random variables. However, real physical processes demand modeling dependent events and using dependent random variables in a variety of constructions. One of the examples would be modeling spatial data where the correlation between random variables depends on the distance between them. An extensive discussion on statistical theory for spatial data analysis can be found in Cressie [8], who mostly used the standard Euclidean norm to represent the distance measure. The interest in modeling non-Euclidean distance measure emerges for instance in geostatistics where travel costs, travel duration and/or other issues are involved in the description of proximity relationships among spatial locations. For more practical problems involving the use of non-Euclidean distance measures in geostatistics see Curriero [9].

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This paper is organized as follows. Section 2 presents definitions of a positive definite matrix and a positive definite function. Next, the connection between the positive definite matrices and positive definite functions is given. Section 3 highlights some of the main results regarding positive definite norm dependent real functions to which the history of proofs (some results where proven independently by different mathematicians) is given. Section 3 is divided into 2 subsections. The first one considers an exponential function depending on the ℓ_p norm to the power q. The next one describes the necessary conditions for the function to be positive definite given dependency on ℓ_1 or ℓ_2 norm. Section 4 shows some examples of random fields generated with a norm dependent positive definite real functions.

2. Positive definiteness condition

One of the possible dependency structures for a random vector is based on the covariance (or correlation) matrix

$$\Sigma = \left(\mathrm{Cov}(X_i, X_j)\right)_{i,j}, \qquad \Re = \left(\rho(X_i, X_j)\right)_{i,j}.$$

The problem is creating such matrices, or checking whether a given matrix can be considered as a covariance matrix of a random vector because of the complexity of the positive definiteness condition. Next, we show some methods for creating positive definite norm (or quasi-norm)-dependent matrices based on a characterization of norm dependent characteristic functions.

Definition 1. An $n \times n$ matrix $\Sigma = (\sigma_{ij})_{ij=1}^{n}$, $\sigma_{ij} \in \mathbb{R}$, is positive definite if for every $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$, we have $\mathbf{c}\Sigma\mathbf{c}^T \geqslant 0$.

Definition 2. A function f defined on $\mathbb{R}^d$ taking values in the complex plane is positive definite if for every $n \in \mathbb{N}$ every choice of complex numbers $c_1, \ldots, c_n$ and every choice of $x_1, \ldots, x_n \in \mathbb{R}^d$, we have

$$\sum_{i,j=1}^{n} c_i \overline{c_j} f(x_i - x_j) \geqslant 0.$$

The following, very well known, fact describes the connection between positive definite real matrices and positive definite real functions. It is crucial for the whole paper. It shows that building norm dependent positive definite matrices one has to know the theory of norm dependent positive definite functions (i.e. characteristic functions after proper normalization) since some norms do not allow existence of such functions. Consequently, not for every norm the required positive definite norm dependent matrix can exist.

Note that if the function f is real and $f(-x) = f(x)$ then it is enough to consider real constants c_j, $j = 1, \ldots, n$.

Fact. *Let f be a function on $\mathbb{R}^d$ taking values in $\mathbb{R}$ such that $f(-x) = f(x)$. The function f is positive definite if and only if, for every $n \in \mathbb{N}$ and every choice of $x_1, \ldots, x_n \in \mathbb{R}^d$, the matrix*

$$\Sigma = \big(f(x_i - x_j)\big)_{i,j=1}^n = \begin{bmatrix} f(0) & f(x_1 - x_2) & \ldots & f(x_1 - x_n) \\ f(x_2 - x_1) & f(0) & \ldots & f(x_2 - x_n) \\ \ldots & \ldots & \ldots & \ldots \\ f(x_n - x_1) & f(x_n - x_2) & \ldots & f(0) \end{bmatrix}$$

is positive definite.

Proof. It is enough to notice that positive definiteness of the matrix Σ means that for every vector $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$, we have

$$\mathbf{c}\Sigma\mathbf{c}^T = \sum_{i,j=1}^n c_i c_j f(x_i - x_j),$$

which explains the equivalence. ∎

This Fact can be of great help for constructing positive definite matrices (for example a covariance matrix) or for verifying whether a given matrix of the form defined in the Fact is positive definite.

3. What is known about positive definite norm dependent functions

In this section, we are describing what is known about positive definite norm dependent functions together with the history of main results. Note that some proofs are the results of more than 50 years of hard work of many mathematicians. And, we still know very little in this area.

We give these results in their simplest version, since it is evident that if a real-valued function f on $[0, \infty)$ is such that $f(\|\mathbf{x}\|)$ is positive definite on $\mathbb{R}^d$ and A is a linear operator on $\mathbb{R}^d$ then also $f(\|A(\mathbf{x})\|)$ is positive definite on $\mathbb{R}^d$. This statement holds true for every norm (or even a quasi-norm) $\|\cdot\|$ on $\mathbb{R}^d$.

3.1. Exponential function and ℓ_p norm to the power q. First, consider the following function defined on $\mathbb{R}^d$:

$$f_{p,q}(\mathbf{x}) = \exp\Big\{-\Big(\sum_{k=1}^d |x_k|^p\Big)^{q/p}\Big\} = \exp\Big\{-\|\mathbf{x}\|_p^q\Big\}.$$

Theorem 3. *Assume that for each $d \in \mathbb{N}$, the function $f(\|\mathbf{x}\|_p)$ is positive definite on $\mathbb{R}^d$. Then $p \in (0, 2]$ and there exists a probability measure λ on*

$[0, \infty)$ *such that*

$$f(\|\mathbf{x}\|_p) = \int_0^\infty f_{p,p}(\mathbf{x}s)\lambda(ds).$$

Moreover, for every $d \in \mathbb{N}$ *and every finite symmetric measure* ν *on the unit sphere* S_{d-1} *in* $\mathbb{R}^d$*, the function* $f(\|\mathbf{x}\|_{p,\nu})$ *is positive definite on* $\mathbb{R}^d$*, where*

$$\|\mathbf{x}\|_{p,\nu}^p = \int_{S_{d-1}} |\langle \mathbf{x}, \mathbf{u}\rangle|^p \, \nu(d\mathbf{u}).$$

Proof. In this formulation, the theorem was proven by Bretagnolle, Dacunha-Castelle and Krivine in 1967 (see [5]). For the weakest possible assumptions and dependence on ℓ_p-norms see also the paper of Christensen and Ressel [7] written in 1976 and the paper of Misiewicz and Scheffer [22, Section 5], written in 1991. ∎

Theorem 4. *The function* $f_{p,q}(\mathbf{x})$ *is positive definite on* $\mathbb{R}^d$ *if and only if one of the following conditions hold*

1) $d = 2, \quad 0 < q \leqslant p \leqslant 2$ *or* $p \in (2, \infty]$ *and* $q \leqslant 1$;
2) $d \geqslant 3$, $0 < q \leqslant p \leqslant 2$.

History of the proof. Notice first that if the function $f_{p,q}(\mathbf{x})$ is positive definite then it is a characteristic function of a symmetric q-stable random vector $\mathbf{X}$ in $\mathbb{R}^d$ since the condition defining symmetric stable characteristic function

$$f_{p,q}(a\mathbf{x})f_{p,q}(b\mathbf{x}) = f_{p,q}((|a|^q + |b|^q)^{1/q}\mathbf{x})$$

holds for all $a, b \in \mathbb{R}$. The condition $q \leqslant 2$ is a student exercise based on the connection between moments of the variable $< \mathbf{x}, \mathbf{X} >$ and the derivatives of the corresponding characteristic function. It is enough to notice that in this case the second derivative of $f_{p,q}$ treated as a characteristic function of the first coordinate X_1 of $\mathbf{X}$ is zero at the origin, thus X_1 and $\|\mathbf{X}\|_p$ shall be constant in contradiction to our assumptions.

Observe also that if $f_{p,q}$ is positive definite, then every its scale mixture is positive definite and consequently $f_{p,s}$ is positive definite for every $0 < sq$ since

$$f_{p,s}(\mathbf{x}) = \int_0^\infty f_{p,q}(\mathbf{x}t)\gamma_{s/q}(dt),$$

where $\gamma_{s/q}$ is s/q-stable distribution on the positive half-line with the Laplace transform $e^{-t^{s/q}}$.

This shows that the sufficiency of the condition 1) holds in $\mathbb{R}^d$ for every $d \geqslant 2$. This fact was known already in 1937 to P. Lévy [17].

For further considerations we need to know that $f_{p,q}$ is positive definite on $\mathbb{R}^d$ if and only if the space $(\mathbb{R}^d, \| \, \|_p)$ embeds isometrically into a space

$(\mathbb{R}^d, \| \|_{L_q(\nu)})$ in the sense that there exists a finite measure ν on $\mathbb{R}^d$ (for the uniqueness we shall assume that ν has support in $S_{d-1} \subset \mathbb{R}^d$) such that

$$\|\mathbf{x}\|_p^q = \int_{\mathbb{R}^d} | < \mathbf{x}, \mathbf{y} > |^q \nu(d\mathbf{y}).$$

This is a simple implication from the general representation of the characteristic function for symmetric stable random vector. One can find it e.g. in [16], Th. 2.4.3 stating that φ is a characteristic function of a symmetric α-stable random vector in $\mathbb{R}^d$ with $\alpha \in (0, 2]$ if and only if there exists a unique symmetric finite measure ν on $S_{d-1} \subset \mathbb{R}^d$ such that

$$\varphi(\mathbf{x}) = \exp\left\{ - \int_{S_{d-1}} | < \mathbf{x}, \mathbf{y} > |^\alpha \nu(d\mathbf{y}) \right\}.$$

This result can be expressed also in the following form: $\exp\{-\| \cdot \|^q\}$ is a characteristic function on $\mathbb{R}^d$ if and only if there exists a finite measure ν on S_{d-1} such that $\| \cdot \| = \| \cdot \|_{q,\nu}$, or equivalently, if and only if $(\mathbb{R}^d, \| \cdot \|)$ embeds isometrically into some L_q-space (for example, into $L_q(S_{d-1}, \nu)$).

Now we can describe the proof of necessity of the condition 1). It has a long history going back to first investigations of symmetric stable random vectors [17], and the first Schoenberg problem [25] (see also Introduction). In 1963, Herz [14] proved that if $1 < q < 2$ and ℓ_p^n embeds isometrically into some L_q-space then $q \leq p \leq q(q-1)^{-1}$. In 1973, Witsenhausen [26] proved that if $p > 2.7$, $n \geq 3$, then ℓ_p^n does not embeds isometrically into L_1-space. In 1976, Dor [10] (see also [5]) proved that if $p, q \in [1, \infty)$ and ℓ_p^n embeds isometrically into some L_q-space then $1 \leq q \leq p \leq 2$. In 1991, Koldobsky [15] proved that if $p > 2$, and if $n \geq 3$ then ℓ_p^n does not embed isometrically into any L_q-space, $q \leq 2$. Note that the result of Koldobsky solves finally, after 53 years, the first Schoenberg question. And in 1995, Grząślewicz and Misiewicz [12] noticed that all the previous considerations does not include all the cases when $p < 1$ or $q < 1$. They proved that if $0 < p < q \leq 2$ then ℓ_p^2 does not embed isometrically into any L_q-space.

We continue with the case $p > 2$. In Theorem 2.1 of [10], Dor proved that the function $f_{p,q}$ is not positive definite if $1 < q < 2 < p$. In the case when $p > 1$ and $q = 1$ (and consequently for each $q < 1$), positive definiteness of the function $f_{p,q}(\mathbf{x})$ on $\mathbb{R}^2$ follows from the well known theorem stating that every two-dimensional Banach space (e.g. $(\mathbb{R}^2, \| \cdot \|_p)$) embeds isometrically into some L_1-space. This theorem has been proven by several authors in different ways in different areas of mathematics; see e.g. Ferguson 1962 [11], Herz 1963 [14], Lindenstrauss 1964 [18], Assouad 1979–1980 [2–4] or Misiewicz and Ryll-Nardzewski 1989 [21]. ∎

Important remark. We have seen that for each $p \leqslant 2$, every $q \leqslant p$ and every symmetric finite measure on S_{d-1} the function $\exp\{-\| \, \|^q_{L_p(\nu)}\}$ is positive definite on $\mathbf{R}^d$. However, not every norm can be written in this form and for some norms none norm-dependent function can be positive definite. In particular, we know that if the function $f(\|\mathbf{x}\|_p)$ for some $p \in (2, \infty]$ is positive definite on $\mathbb{R}^d$, $d \geqslant 3$, then $f \equiv 1$. This was shown in the following papers: In 1989, Misiewicz [20] proved that if the function $f(\|\mathbf{x}\|_\infty)$ is positive definite on $\mathbb{R}^d$, $d \geqslant 3$ then $f \equiv 1$. The similar result for the norm $\| \cdot \|_p$, $p \in (2, \infty)$, was proven independently by two authors: Lisitsky in 1991 [19] and Zastawny in 1991 [28].

3.2. ℓ_2 and ℓ_1-norm dependent positive definite functions. Of course sometimes we want to consider functions other than exponentials. Unfortunately, we know very little about the characterization of the norm dependent positive definite continuous functions, and we know even less when these functions are not continuous. The only exceptions are ℓ_2 and ℓ_1 norms in $\mathbb{R}^d$, and the full characterization of the functions f such that $f(\| \cdot \|_2)$ is positive definite on $\mathbb{R}^d$, and the functions f such that $f(\| \cdot \|_1)$ is positive definite on $\mathbb{R}^d$, respectively, are known. These exceptions are described below.

3.2.1. ℓ_2-norm dependent functions. Consider the function Ω_d defined in the following way:

$$\begin{aligned}\Omega_d(r) &= \frac{2\Gamma(d/2)}{\Gamma((d-1)/2)\Gamma(1/2)} \int_0^{\pi/2} \cos\Big(r \sin\varphi\Big) \cos^{d-2}(\varphi)\, d\varphi \\ &= \Gamma\Big(\frac{d}{2}\Big)\Big(\frac{2}{r}\Big)^{\frac{d}{2}-1} J_{\frac{d-2}{2}}(r),\end{aligned}$$

where $J_\nu(r)$ is a Bessel function of the first kind. In 1938, Schoenberg [23] gave the full characterization of ℓ_2-norm dependent characteristic functions in the language of the function Ω_d. The function Ω_d is the characteristic function of one-dimensional projection of the uniform distribution ω_d on the unit sphere in $\mathbb{R}^d$. If $\mathbf{U}^{(d)} = (U_1, \ldots, U_d)$ has distribution ω_d then its characteristic function is given by

$$\mathbf{E} \exp\{< \mathbf{x}, \mathbf{U}^{(d)} >\} = \Omega_d\left(\|\mathbf{x}\|_2\right).$$

Theorem 5. *The function $\varphi(\mathbf{x}) = f(\|\mathbf{x}\|_2)$ is a characteristic function on $\mathbb{R}^d$ if and only if there exists a probability measure λ on $[0, \infty)$ such that*

$$f(r) = \int_0^\infty \Omega_d(rs)\, \lambda(dr).$$

This result can be also formulated in the following form: a symmetric random vector $\mathbf{X}$ in $\mathbb{R}^d$ is spherically generated (rotationally invariant), i.e. has the characteristic function of the form $\varphi(\mathbf{x}) = f(\|\mathbf{x}\|_2)$ if and only if

$\mathbf{X} \stackrel{d}{=} \mathbf{U}^{(d)}Q$, for a nonnegative variable Q with distribution λ independent of $\mathbf{U}^{(d)}$. Here $\stackrel{d}{=}$ denotes equality of distributions.

With the exponential functions $f_{p,q}$, we can only obtain correlation matrices with all elements positive and sometimes it is necessary to consider negative correlations between some variables. This can be done by using the function $\Omega_d(\|\cdot\|_2)$, but, as we can see on Figure 1 below presenting graphs of functions Ω_d, $d = 2, 3, 4$, not all negative values are available.

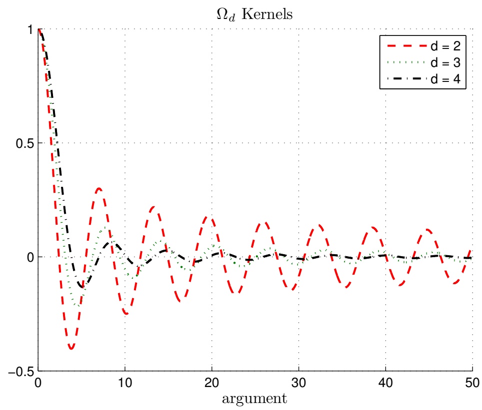

Fig. 1. Ω_d for $d = 2, 3, 4$.

The characteristic functions of the form $f(\|\mathbf{x}\|_2)$ are the characteristic functions of spherically invariant random vectors. We can see that not every symmetric positive definite function φ on $\mathbb{R}$ with $\varphi(0) = 1$ has the property that $\varphi(\|\cdot\|_2)$ is a characteristic function of an elliptically contoured random vector. They all have to be scale mixtures of the function Ω_d. This property is not easy to verify. One of the verifying methods is using the following Askey result (for details see [1]):

Theorem 6. *Let $d \geqslant 2$ and let $\varphi : [0, \infty) \mapsto \mathbb{R}$ be continuous and such that*

1) $\varphi(0) = 1$, $\lim_{t\to\infty} \varphi(t) = 0$;
2) $(-1)^k \varphi^{(k)}(t) \geqslant 0$ *is convex for* $k = [\frac{d}{2}]$.

Then, for every positive definite $n \times n$-matrix $\Re$, $\varphi\big((\langle \mathbf{x}, \Re\mathbf{x}\rangle)^{1/2}\big)$ is the characteristic function of some elliptically contoured random vector.

3.2.2. ℓ_1-norm dependent functions. In 1983, S. Cambanis, R. Keener and G. Simons [6] found all the extreme points for ℓ_1-dependent distributions on $\mathbb{R}^d$, i.e. distributions having the characteristic functions of the form $f(\|\mathbf{x}\|_1)$. This result was based on the following, surprisingly general, definite integral identity:

$$\int_0^{\pi/2} f\left(\frac{s^2}{\sin^2\theta} + \frac{t^2}{\cos^2\theta}\right) d\theta = \int_0^{\pi/2} f\left(\frac{(|s|+|t|)^2}{\sin^2\theta}\right) d\theta,$$

which holds for each $s, t \in \mathbb{R}$ and every function f for which the integrals make sense. The result of Cambanis, Keener and Simons can be formulated in the following way:

Theorem 7. *The function $f(\|\mathbf{x}\|_1)$ is positive definite on $\mathbb{R}^d$ if and only if there exists a probability measure λ on $[0,\infty)$ such that*

$$f(r) = \int_0^\infty \varphi_d(rx)\lambda(dx),$$

where

$$\begin{aligned}\varphi_d(r) &= \frac{2\Gamma(\frac{d}{2})}{\sqrt{\pi}\Gamma(\frac{d-1}{2})} \int_1^\infty \Omega_d(ur)u^{-d+1}(u^2-1)^{(d-3)/2}du \\ &= \frac{2\Gamma(\frac{d}{2})}{\sqrt{\pi}\Gamma(\frac{d-1}{2})} \int_0^1 \Omega_d(r/u)(1-u^2)^{(d-3)/2}du,\end{aligned}$$

for $\Omega_d(\|\mathbf{x}\|_2)$ the characteristic function of the vector $\mathbf{U}^{(d)}$.

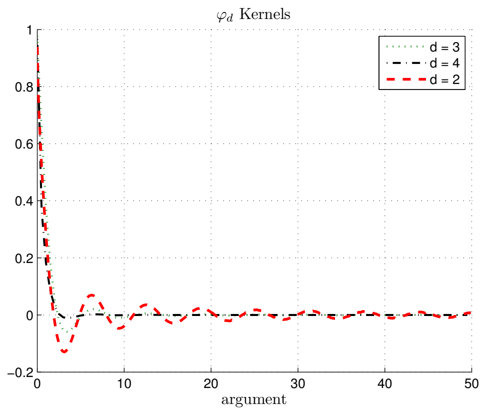

Fig. 2. φ_d for $d = 2, 3, 4$.

Figure 2 shows the graphs of the function φ_d, for $d = 2, 3, 4$. As we can see all the possible positive values are admissible for the function φ_d, but when d grows the set of admissible negative values is getting smaller. It is known, that $\varphi_d(\mathbb{R}) \to [0,1]$ if $d \to \infty$.

4. Norm dependent standard normal random fields with computer realizations

For the purpose of this paper, we only consider zero mean Gaussian random fields. We want to show, at least in a graphical way, the differences between norm dependent random fields with different norms or quasi-norms and different correlation functions. We simulate Gaussian random fields on $\mathbb{R}^2$ with correlation functions dependent on the norm of the distance between the points indexing random variables i.e. $\rho(X_{\mathbf{x}}, X_{\mathbf{y}}) = f(\|\mathbf{x}-\mathbf{y}\|)$. We present the exponential correlation function $f = f_{p,q}$ for different parameters p and q and a norm $\|\mathbf{x}\|_{\nu,p}^{p}$ as defined in Theorem 4, the correlation function $f = \Omega_2$ and finally, we present an example using $\|\mathbf{x}\|_1$ norm and corresponding correlation function $f = \varphi_2$.

Consider a closed convex, bounded set $E \subset \mathbb{R}^2$ and any fixed norm (or quasi-norm) q on $\mathbb{R}^2$. For each admissible function $f\colon [0,\infty) \to [0,\infty)$ (i.e. such a function that $f(q(\cdot))$ is positive definite), $f(0) = 1$, by a (q, f)-dependent random field $\mathcal{F}(E, q, f)$ we consider a collection of second order random variables

$$\mathcal{F}(E, q, f) = \{X_{\mathbf{x}} \colon \mathbf{x} \in E\}$$

such that

$$\rho\,(X_{\mathbf{x}}, X_{\mathbf{y}}) = \frac{\mathrm{Cov}(X_{\mathbf{x}}, X_{\mathbf{y}})}{\sqrt{\mathrm{Var}X_{\mathbf{x}}\mathrm{Var}X_{\mathbf{y}}}} = f\big(q(\mathbf{x}-\mathbf{y})\big).$$

For convenience, we assume that the expected value of the generated field is zero, but in general we can have

$$m(\mathbf{x}) := \mathbf{E}\,(X_{\mathbf{x}})$$

as an arbitrary function $m\colon E \to \mathbb{R}$.

Each random field presented in the examples below is generated on a matrix domain (discrete version of the set E) using Matlab programming language and the script in Listing 1. The parameters of the `StandNormal-RandField` function are explained in the script.

ℓ_2 norm dependent Ω_2 correlation function:
First we consider the ℓ_2 norm and Ω_2 function which for dimension 2 (random field on $\mathbb{R}^2$) is:

$$\Omega_2(r) = J_0(r),$$

where J_ν is a Bessel function of the first kind. The `StandNormalRandField` function parameter `call` determines which example is being calculated and in this case it is `'Omega'`. We consider two examples for the Omega correlation function. First one with coefficient of correlation $d = 1$ (no distance scaling) and second one with coefficient of correlation $d = 0.6$ (scaled distance; original distance of 1 is now 0.6). Figure 3 shows the two examples where each is presented in two graphs; left is the 3D plot using `meash` function in Matlab of the random field and right is the colormap (colors represent different values of random variables) using `pcolor` function in Matlab. The resulting pattern observed on the random fields is due to the Omega correlation function changing in the first 10 units of its argument correlation from positive to negative twice, see Fig. 3. To generate a random field with Omega correlation run `StandNormalRandField('Omega',50,50,[],[],1,[],[],[],'File')`. Additionally, one may consider a scale mixture of the function $f = \Omega_2(\|\cdot\|_2)$ because Theorem 5 guaranties that it is positive definite. An example is the φ function defined in Theorem 7.

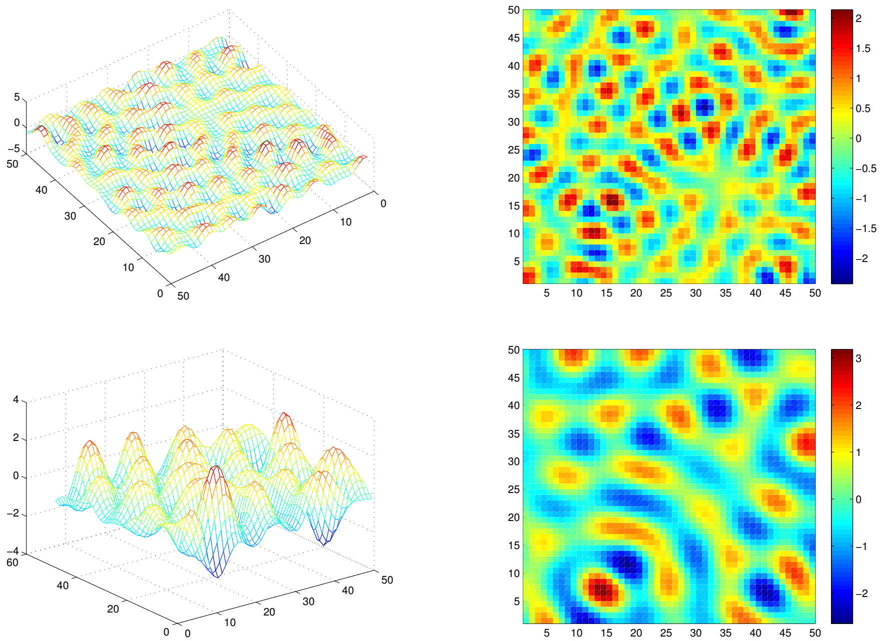

Fig. 3. $f = \Omega_2$ and $\|\mathbf{x}\|_2$-dependent random field.

ℓ_1 norm dependent φ_2 correlation function:
$\varphi_2(\|\cdot\|_1)$ or every scale mixture of the function $\varphi_2(\|\cdot\|_1)$ is positive definite, Theorem 7. In `StandNormalRandField` we define the φ_2 function as an integral from 0 to 1 which can be found to be:

$$\varphi_2(r) = \frac{2}{\pi}\int_{u=0}^{1} \Omega_2(r/u)\frac{\mathrm{d}u}{\sqrt{1-u^2}}, \tag{1}$$

and represent it as `Int_phi` function. The resulting random field generated by running

`StandNormalRandField('phi',50,50,[],[],.1,[],[],[],'file')`
with coefficient of correlation $d = 0.1$ is presented in Figure 4. Because of the ℓ_1 norm we can see stronger dependence along vertices.

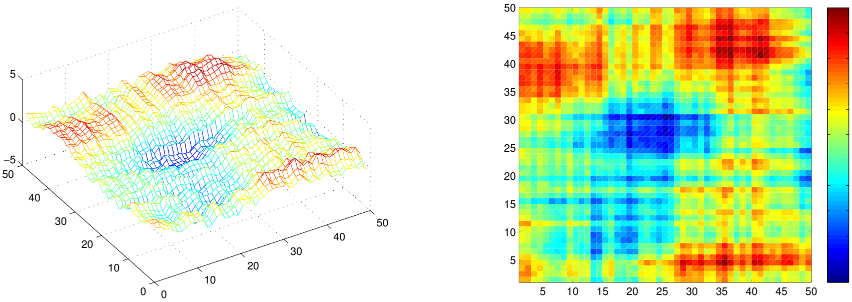

Fig. 4. $f = \varphi_2$ and $\|\mathbf{x}\|_1$-dependent random field.

ℓ_p^q norm dependent exp correlation function $f_{p,q}$:
In Figure 5, we present simulation of $\|\cdot\|_p$-dependent Gaussian random field $\mathcal{F}(E, \|\cdot\|_p, f_{p,p})$ for $p = 0.3$. Note that $\|\cdot\|_p$ for $p < 1$ is only a quasi-norm for which the triangular inequality takes the form:

$$\|\mathbf{x} + \mathbf{y}\|_p \leq 2^{\frac{1}{p}-1} \left(\|\mathbf{x}\|_p + \|\mathbf{y}\|_p\right),$$

where the constant $2^{\frac{1}{p}-1} = 2^{7/3} \approx 5.04$ is the smallest possible for $p = 0.3$. The corresponding unit sphere is presented on the left graph in Figure 5 from where we can see that the strong correlation is almost only preserved on the axis x and y. The corresponding random field with the `mesh` plot (middle graph) and the `pcolor` plot (right graph) is also shown. To generate a similar random filed with the same specifications use
`StandNormalRandField('Exp',50,50,.3,.3,.01,[],[],[],'file')`.

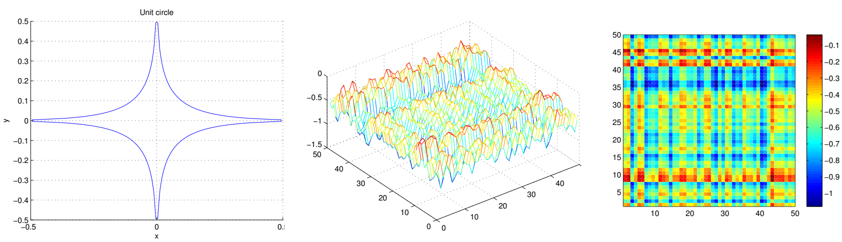

Fig. 5. The exponential correlation function $f_{p,q}$ with $p = 0.3$ and $q = 0.3$.

Next, Figure 6 presents an example of the computer simulation of $\|\cdot\|_1$ dependent random field $\mathcal{F}(E, \|\cdot\|_1, f_{1,q})$. The ℓ_1 norm, for which the unit circle is shown on the left of Figure 6 is used. To generate a similar example use `StandNormalRandField('Exp',50,50,1,1,.01,[],[],[],'file')`.

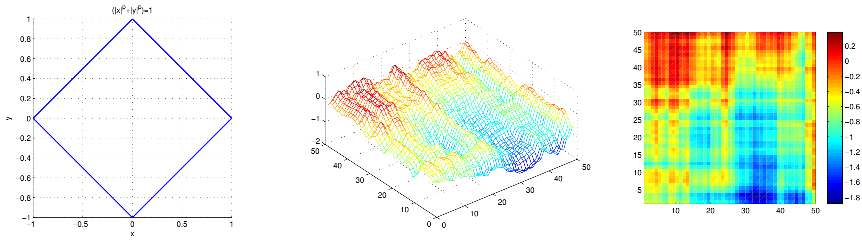

Fig. 6. The exponential correlation function $f_{p,q}$ with $p = 1$ and $q = 1$.

Figure 7 shows two examples of the computer simulation of a $\|\cdot\|_2$-dependent Gaussian random field $\mathcal{F}(E, \|\cdot\|_2, f_{2,q})$. In the first case, the power of the ℓ_2 norm $q = 1$ is used. This means that all the distances are original Euclidian distances. In the second example, $q = 2$ is considered in which case all the distances within the unit circle are decreased and all the distances outside the unit circle are increased ultimately affecting correlations and the resulting random field. To generate the example use `StandNormalRandField('Exp',50,50,2,q,.05,[],[],[],'file')`.

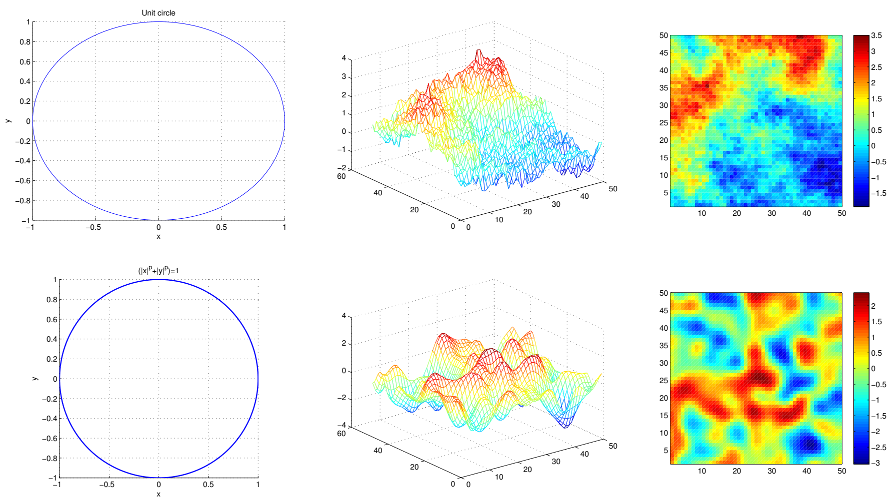

Fig. 7. The exponential correlation function $f_{p,q}$ with $p = 2$ and $q = 1$ (three upper pictures), $p = 2$ and $q = 2$ (three lower pictures).

Finally, Figure 8 presents the simulation of $\|\cdot\|_p$-dependent random field $\mathcal{F}(E, \|\cdot\|_p, f_{p,q})$ for $p > 2$. Note that this is possible only in the case of E being a subset of $\mathbb{R}^2$. In the first figure we have $p = 10$, $q = 1$, and in the

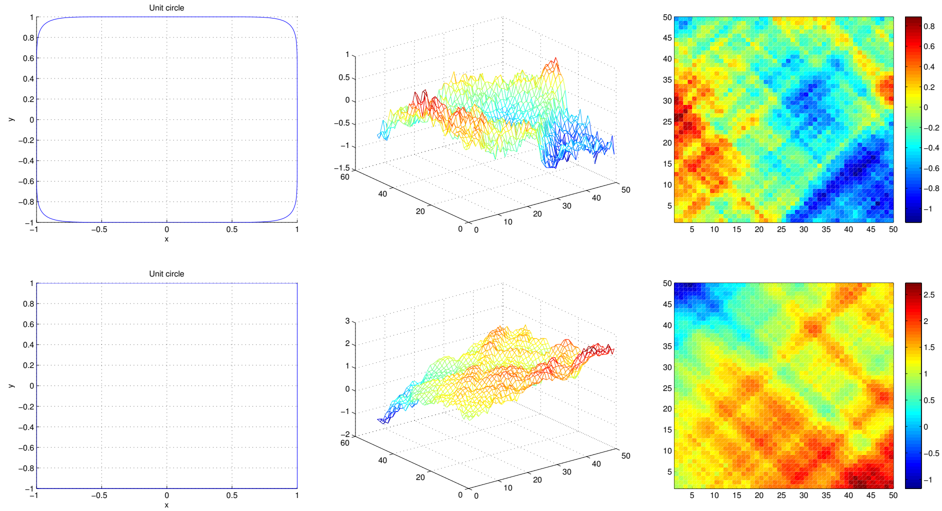

Fig. 8. The exponential correlation function $f_{p,q}$ with $p = 10$ and $q = 1$ (three upper pictures), $p = \infty$ and $q = 1$ (three lower pictures).

second $p = \infty$ and $q = 1$. Both unit spheres for $\|\cdot\|_p$-norms (first pictures on the left) are very close to the unit square causing stronger diagonal dependency in both random fields. To generate the example use `StandNormalRandField('Exp',50,50,p,1,.01,[],[],[],'file')`, for $p =$ `inf` or $p = 10$.

***Mixed* norm dependent exp correlation function $f_{p,p}$:**

From Theorem 4, we know that if a function $f_{p,q}$ is positive definite on $\mathbb{R}^2$ then it has to be a mixture of the exponential function $f_{p,p}$ where the norm $\|\mathbf{x}\|_p$ for $p \in (0, 2]$ can be redefined by:

$$\|\mathbf{x}\|^p_{\nu,p} \stackrel{def}{=} \int_{S_1} | < \mathbf{x}, \mathbf{u} > |^p \nu(d\mathbf{u}),$$

an integral with respect to a finite symmetric measure ν defined on the unit circle S_1 in $\mathbb{R}^2$. The restriction that ν has a unit circle ($S_1 \subset \mathbb{R}^2$) as a support guaranties only the uniqueness of this measure given all the values of the norm $\|\mathbf{x}\|^p_{\nu,p} < \infty$. For our purpose the uniqueness of ν is irrelevant, thus we can take any ν symmetric with compact support.

Next example is generated using the exponential correlation function and the norm $\|\mathbf{x}\|_p$ redefined using a discrete symmetric measure ν defined on $\mathbb{R}^2$ by:

$$\nu \simeq \sum_{i,j \in \{-1,0,1\}} \delta_{i,j},$$

where δ_x is the Dirac delta measure. Note that not all chosen points in the delta Dirac measure above belong to the unit sphere. Essentially we would get the same norm given by:

$$\|\mathbf{x}\|^q_{\nu,p} \stackrel{def}{=} (|x|^p + |y|^p + |x-y|^p + |x+y|^p)^{q/p},$$

if we were to use $\delta_{\{+-\sqrt{2},+-\sqrt{2}\}}$ instead of $\delta_{\{+-1,+-1\}}$. Using `StandNormalRandField('ExpDiscreteNorm',50,50,.5,.5,.01,[],[], [],'file')` we can generate random fields like shown in Figure 9.

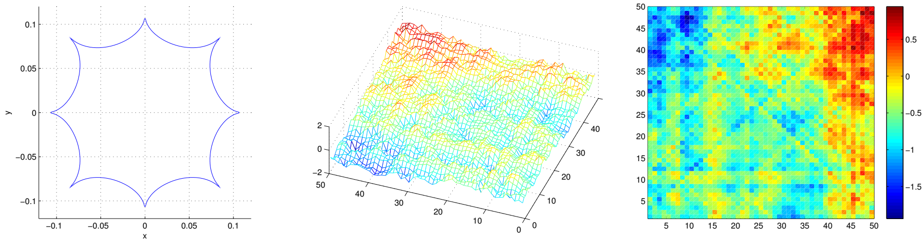

Fig. 9. The exponential correlation function $f_{p,q}$ with $p = 0.5$ and $q = 0.5$ and the norm constructed as described above.

In the last example, we define a continuous, finite and symmetric measure ν proportional to $\max\{0, \sin(c\theta)\}$ for a constant $c = 4 * n$ and integer $n \geq 1$. The resulting norm is given by:

$$(2) \qquad \|\mathbf{x}\|^p_{\nu,p} \stackrel{def}{=} \int_0^{2\pi} \left|x\cos\theta + y\sin\theta\right|^p \ \max\{0, \sin(c\theta)\}\, d\theta.$$

We generate two random fields using the exponential correlation function $f_{p,p}$ for $p = 1.3$ and the norm defined in equation 1. In the first one, the norm $\|\mathbf{x}\|^p_{\nu,p}$ is used and in the second one, the following additional linear operator $A = \left(\begin{smallmatrix} 4 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is applied. This is because if $A\colon \mathbb{R}^d \mapsto \mathbb{R}^d$ is a linear operator and $f(\|\mathbf{x}\|)$ is a positive definite function on $\mathbb{R}^d$ then $f(\|A(\mathbf{x})\|)$ is also positive definite. The resulting computer simulation of the random fields using `StandNormalRandField('ExpIntNorm',50,50,1.3,[],.1,a_{1,1} ,a_{2,2},4,'file')` function are presented in Figure 10 where is the first case $a_{1,1} = 1$ and $a_{2,2} = 1$ and in the second case $a_{1,1} = 4$ and $a_{2,2} = 1$.

As we can see applying linear transformation rascals the original distance. The unit circle of defined norm is plotted for both cases on the left side in Figure 10.

In general, regarding the exponential correlation function on $\mathbb{R}^2$ if $\|\cdot\|$ is a norm (but not quasi-norm) then $\exp\{-\|\cdot\|\}$ is positive definite.

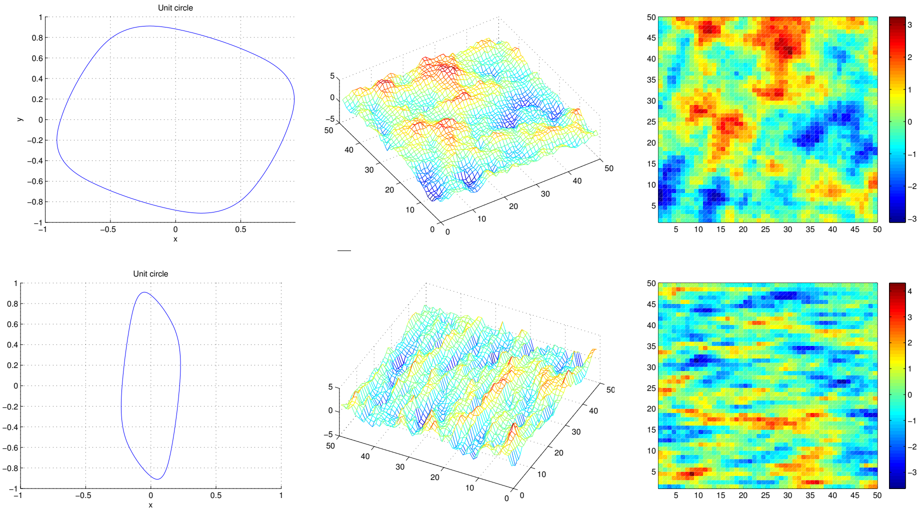

Fig. 10. The exponential correlation function $f_{p,p}$ with $p = 1.3$ and the norm $\|\mathbf{x}\|_{\nu,p}^{p}$ (above) and $\|A(\mathbf{x})\|_{\nu,p}^{p}$.

5. Conclusions

Positive definiteness condition is a necessary condition for a random vector or random field to be generated. In this paper, we have shown what kind of functions one can use as a correlation function in order to produce a valid positive definite correlation matrix. List of theorems with some additional facts on the history of proves is given to highlight the possibilities and the graphical presentation is considered to visualize the impact of using different norms and correlation functions on generated random field. For higher flexibility, we also add a Matlab program to generate other examples of norm dependent random fields using different norms and correlation functions. The Matlab script can be found in Appendix.

6. Appendix

A script written in Matlab programming language to generate Standard Normal random fields with exponential, Ω_2 and φ_2 functions, is presented in Listing 1 below. For exponential function the script supports ℓ_p norm to the power q for p and q parameters as described in Theorem 4, integral norm defined in equation 2 for measure μ defined in MeasureNU function at the end of the script where any finite symmetric measure on a unit circle may be used instead of $\max(0, \sin(c\theta))$ for $c = 4n$ and integer $n \geq 1$. For other measure, replace the $\max(0, \sin(c\theta))$ in MeasureNU. Last norm defined by discrete points on a unit circle and presented in example in Figure 9 also uses exponential function. Other correlation functions are Ω_2 correlation function defined for ℓ_2 and φ_2 defined for ℓ_1 norms.

Please note that when coping the script to Matlab correct error messages on lines 53, 126, 129 and 153.

Listing 1. Normal norm dependent random fields

```
function StandNormalRandField(call,k1,k2,p,q,d,a,b,c,filename)

% Generates Standard Normal random field on a rectangular
% domain matrix (k1,k2) with correlation function chosen
% by input call and various norms for instance
% l_{p,q}(x)=((x_1^p+x_2^p)^(1/p))^q norm.
%
% INPUTS
% call - string with the correlation function and norm to use:
%        'Exp'   - Exponential correlation function with l_p to
%                   the power q norm;
%        'Omega' - Omega correlation function with norm l_2
%        'phi'   - phi correlation function with norm l_1
%        'ExpIntNorm'    - Exponential correlation function
%                          with norm defined by IntNorm
%                          function and measure @measure_nu
%        'ExpDiscreteNorm'   - Exponential correlation function
%                              with norm defined by
% k1 - rows of the rectangular domain matrix (k1,k2)
% k2 - columns of the rectangular domain matrix (k1,k2)
% norm parameters p q bounds:  0<q<=p<=2 or p>2 and q <=1
% p  - norm parameter: p=inf - max norm
% q  - power of the norm
% d  - coefficient of correlation
% a  - scalar for x distance (IntNorm)
% b  - scalar for y distance (IntNorm)
% c  - scalar in the MeasureNU
% filename - name of file storing the random field
%% input check
if ~k1==floor(k1)
    error('k1␣must␣be␣integer␣(dimension␣of␣the␣random␣field)');
elseif ~k2==floor(k2)
    error('k2␣must␣be␣integer␣(dimension␣of␣the␣random␣field)');
elseif d<=0
    error('The␣scalar␣parameter␣d␣must␣be␣positive:␣d>0');
end
%% initializing random field
% the surface of random field
MgC=zeros(k1,k2);
% vector size/ Correlation matrix
n=k1*k2;
% indexing the rows (ii) and columns (jj) of MgC
[ii,jj]=find(MgC==MgC);
% defining the correlation matrix sigma
msig1=zeros(n,n);
msig2=msig1;
% indexing the rows (k_ms1) and columns (w_ms2) of msig1
[k_ms1,w_ms2]=find(msig1==msig1);
%% Exponential correlation function
if strcmpi(call,'Exp')
    if ~((q>0 && q <=2 && p>=q && p<=2) || (q>0 && q <=1 && p>2))
        error('The␣norm␣and␣power␣parameters␣(p,q)␣must␣satisfy:
␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣0<q<=p<=2␣or␣p>2␣and␣q␣<=1');
    end
    if p==inf % max norm
        % defining the absolute distances d1 and  d2 in the MgC
        % matrix
        msig1(1:n^2)=(abs(ii(k_ms1)-ii(w_ms2))); % d1 distance:
        %                                          ii(k_ms1)-ii(w_ms2)
```

```
        msig2(1:n^2)=(abs(jj(k_ms1)-jj(w_ms2))); % d2 distance:
        %                                          jj(k_ms1)-jj(w_ms2)
        % correlation matrix sigma
        sigma=exp(-d*(max(msig1,msig2).^q));
    else
        % defining the absolute distances d1 and  d2 in the MgC
        % matrix
        msig1(1:n^2)=abs(ii(k_ms1)-ii(w_ms2)).^p; % d1 distance:
        %                                          ii(k_ms1)-ii(w_ms2)
        msig2(1:n^2)=abs(jj(k_ms1)-jj(w_ms2)).^p; % d2 distance:
        %                                          jj(k_ms1)-jj(w_ms2)
        % correlation matrix sigma
        sigma=exp(-d*(msig1+msig2).^(q/p));
    end
%% Omega correlation function
elseif strcmpi(call,'Omega')
    % defining the absolute distances d1 and  d2 in the MgC
    % matrix
    msig1(1:n^2)=abs(ii(k_ms1)-ii(w_ms2)).^2; % d1 distance:
    %                                          ii(k_ms1)-ii(w_ms2)
    msig2(1:n^2)=abs(jj(k_ms1)-jj(w_ms2)).^2; % d2 distance:
    %                                          jj(k_ms1)-jj(w_ms2)
    % l_2 norm
    x=d*(msig1+msig2).^(1/2);
    sigma=ones(size(x));
        % looking for distinct x's
    [y] = sort(x(1:end));
    t = (diff(y) == 0);
    if any(t)
        y(t) = []; % containg all distinct x's
    end
    % correlation matrix omega for d=2 (dimentions)
    sigma(x==0)=1;
    for i=2:length(y)
     sigma(x==y(i))=besselj(0,y(i));
     % filling in the sigma matrix at all x=y(i)
    end
%% phi correlation function
elseif strcmpi(call,'phi')
    % defining the absolute distances d1 and d2 in the MgC
    % matrix
    msig1(1:n^2)=abs(ii(k_ms1)-ii(w_ms2)); % d1 distance:
    %                                      ii(k_ms1)-ii(w_ms2)
    msig2(1:n^2)=abs(jj(k_ms1)-jj(w_ms2)); % d2 distance:
    %                                      jj(k_ms1)-jj(w_ms2)
    % l_1 norm
    x=d*(msig1+msig2);
    sigma=ones(size(x));
    % looking for distinct x's
    [y] = sort(x(1:end));
    t = (diff(y) == 0);
    if any(t)
        y(t) = []; % containg all distinct x's
    end
    % correlation matrix
    sigma(x==0)=1;
    for i=2:length(y)
     sigma(x==y(i))=Int_phi(y(i));
     % filling in the sigma matrix at all x=y(i)
    end
%% ExpIntNorm correlation function
elseif strcmpi(call,'ExpIntNorm')
    if p>2 || p<0
        error('The norm parameter p must satisfy:  0<p<=2');
    elseif isempty(a) || isempty(b)
        error('Scaling parameters of x and y distance
                                    (a,b) must be specified');
```

```
    elseif isempty(c)
        error('Scaling parameters c of the measure
                                MeasureNU must be specified');
    end
    % defining the distances d1 and d2 in the MgC
    % matrix
    msig1(1:n^2)=(ii(k_ms1)-ii(w_ms2)); % d1 distance:
    %                                     ii(k_ms1)-ii(w_ms2)
    msig2(1:n^2)=(jj(k_ms1)-jj(w_ms2)); % d2 distance:
    %                                     jj(k_ms1)-jj(w_ms2)
    % norm
    norm=zeros(n,n);
    e1=-(k1-1):1:(k1-1);
    e2=-(k2-1):1:(k2-1);
    for i=1:length(e1)
        for j=1:length(e2)
            norm(msig1==e1(i) & msig2==e2(j)) = ...
                IntNorm(e1(i),e2(j),a,b,p,500,@MeasureNU,c);
        end
    end
    % correlation matrix sigma
    sigma=exp(-d*norm);  % norm is n x n matrix
%% ExpDiscreteNorm correlation function
elseif strcmpi(call,'ExpDiscreteNorm')
    if ~((q>0 && q <=2 && p>=q && p<=2) || (q>0 && q <=1 && p>2))
        error('The norm and power parameters (p,q) must satisfy:
                                  0<q<=p<=2 or p>2 and q <=1');
    end
    msig12=msig1;
    msig13=msig1;
    % defining the absolute distances d1 and d2 in the MgC
    % matrix
    msig1(1:n^2)=abs(ii(k_ms1)-ii(w_ms2)).^p; % d1 distance:
    %                                     ii(k_ms1)-ii(w_ms2)
    msig2(1:n^2)=abs(jj(k_ms1)-jj(w_ms2)).^p; % d2 distance:
    %                                     jj(k_ms1)-jj(w_ms2)
    msig12(1:n^2)=(abs(ii(k_ms1)-ii(w_ms2)+...
                         jj(k_ms1)-jj(w_ms2))).^p; % x+y
    msig13(1:n^2)=(abs(ii(k_ms1)-ii(w_ms2) -...
                         (jj(k_ms1)-jj(w_ms2))))).^p; % x-y
    % correlation matrix sigma
    sigma=exp(-d*(msig1+msig2+msig12+msig13).^(q/p));
else
    error('Call is not correct');
end
%% generating standard multi normal vector Z with correlation sigma
Z = mvnrnd(zeros(1,n), sigma);
% Putting Z into the matrix MgC
MgC(1:n)=Z;
% saving the random field in filename.mat
save(filename,'MgC')
% plot
figure % norm
xmin=-1; xmax=1; ymin=-1; ymax=1;
% [x from -1 to 1 y from -1 to 1]
    hold on
    ezplot(@(x,y)plotnorm(x,y,p,a,b,c,call),[xmin xmax ymin ymax])
    Title('Unit circle')
    grid on;
% saving the unit circle plot
saveas(gcf,filename,'fig')
figure % random field
    mesh(MgC)
figure % random field in a colormap
    pcolor(MgC)
    shading flat
    colorbar
```

```
%% Ploting norm
function z = plotnorm(x,y,p,a,b,c,call)
% Solution (x,y) of the equation below is the Unit Circle the norm
if strcmpi(call,'Exp')
    if p==inf
        z=max(abs(x),abs(y))-1;
    else
        z = (abs(x).^p + abs(y).^p).^(1/p) - 1;
    end
elseif strcmpi(call,'Omega')
    z = (abs(x).^2 + abs(y).^2).^(1/2) - 1;
elseif strcmpi(call,'phi')
    z = (abs(x) + abs(y)) - 1;
elseif strcmpi(call,'ExpIntNorm')
    z = IntNorm(x,y,a,b,p,1000,@MeasureNU,c)-1;
elseif strcmpi(call,'ExpDiscreteNorm')
    z = (abs(x).^p + abs(y).^p+abs(x-y).^p+abs(x+y).^p) - 1;
end
%% phi integral
function out=Int_phi(r)
% r - distance vector: r=abs(x)+abs(y)
lr=length(r);
out=zeros(1,lr);
    u=0.0001:0.0001:.99999;
    for i=1:lr
        Omegan=gamma(2/2)*(2./(r(i)./u)).^(2/2-1).*besselj(0,r(i)./u);
        uf=1./sqrt(1-u.^2);
        % integral approximation using trapezoidal rule
        out(i)=trapz(u,Omegan.*uf);
    end
    out=2/pi*out;
%% norm integral
function out=IntNorm(x,y,a,b,p,sizeu,MeasureNU,c)
% array over an angle of a unit circle (parametryzation)
u=linspace(0,2*pi,sizeu);
% integral approximation using trapezoidal rule
nu=MeasureNU(u,c);
out=trapz(u,abs(a*x.*cos(u)+b*y.*sin(u)).^p.*nu);
%% continuous symmetric measure on a unit circle
function out=MeasureNU(x,c)
% nu measure: continuous finite symmetric measure
%             on the unit circle
% c - real constant chosen such that the measure
%       nu is symetric on the unit circle (or R^2)!!!
%       that is: c=4*n (n-integer)
% measure nu: must be symmetric
out=max(0,sin(c*x));
```

References

[1] R. Askey, *Radial characteristic functions*, Tech. Report No 1262 Math. Research Center, University of Wisconsin-Maddison, 1973.

[2] P. Assouad, *Analyses fonctionelle - une espace hypermetrique non plongeable dans un eapace* L_1, C. R. Acad. Sci. Paris 285 (1977), 361–363.

[3] P. Assouad, *Plongements isometriques dans* L_1*; aspect analytique, initiation seminar on analysis*, G. Choquet - M. Rogalski - J. Saint-Raymond 19th Year, Exp. No. 14, Publ. Math. Univ. Pierre et Marie Curie, 41, Univ. Paris VI, Paris 1980, 1980.

[4] P. Assouad, *Caracterizations des suous-espaces normes de* L_1 *de dimension finie*, Seminaire d'Analyse Fonctionelle, 1979–1980, preprint.

[5] J. Bretagnolle, D. Dacunha-Castelle, J. L. Krivine, *Lois stables et espaces L_p*, in: Symp. on Probability Methods in Analysis, Lecture Notes in Math. 31, Springer, pp. 48–54, 1967.

[6] S. Cambanis, R. Keener, G. Simons, *On α-symmetric distributions*, J. Multivariate Anal. 13 (1983), 213–233.

[7] J. P. R. Christensen, P. Ressel, *Norm dependent positive definite functions on B-spaces*, Lecture Notes in Math. 990 (1983), 47–53.

[8] N. Cressie, *Statistics for Spatial Data*, Wiley, New York, 1991.

[9] F. Curriero, *On the use of non-Euclidean distance measures in geostatistics*, Math. Geol. 38(8) (2006), 907–926.

[10] L. E. Dor, *Potentials and isometric embeddings in L_1*, Israel J. Math. 24 (1976), 260–268.

[11] T. S. Ferguson, *A representation of the symmetric bivariate Cauchy distribution*, Ann. Math. Statist. 33 (1962), 1256–1266.

[12] R. Grząślewicz, J. K. Misiewicz, *Isometric embeddings of subspaces of L_α-spaces and maximal representation for symmetric stable processes, Functional Analysis*, Trier 1994, Eds.: Dierolf, Dineen, Domański, Walter de Gruyter & Co., Berlin-New York, pp. 179–182, 1966.

[13] I. Gradstein, I. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, 1980.

[14] C. S. Herz, *A class of negative definite functions*, Proc. Amer. Math. Soc. 14 (1963), 670–676.

[15] A. L. Koldobsky, *Schoeberg's problem on positive definite functions*, Algebra and Analysis (Leningrad Math. J.) 3 (1991), 78–85.

[16] J. Kuelbs, *A representation theorem for symmetric stable processes and stable measures on H*, Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 26 (1973), 259–271.

[17] P. Lévy, *Théorie de l'Addition des Variables Aléatoires*, Gauthier-Villars, Paris, 1937.

[18] J. Lindenstrauss, *On the extension of operators with finite dimensional range*, Illinois J. Math. 8 (1964), 488–499.

[19] A. D. Lisitsky, *One more solution of the Schoenberg problem*, preprint 1991.

[20] J. K. Misiewicz, *Positive definite functions on l^∞*, Statist. Probab. Lett. 8 (1989), 255–260.

[21] J. K. Misiewicz, Cz. Ryll-Nardzewski, *Norm dependent positive definite functions and measures on vector spaces*, In *Probability Theory on Vector Spaces IV, Lancut 1987*, pp. 284–292, Springer-Verlag LNM 1391, 1989.

[22] J. K. Misiewicz, C. L. Scheffer, *Pseudo-isotropic measures*, Nieuw Arch. Wisk. 8(2) (1990), 111–152.

[23] I. J. Schoenberg, *Metric spaces and completely monotone functions*, Ann. of Math. 39 (1938), 811–841.

[24] I. J. Schoenberg, *On certain metric spaces arising from Euclidean spaces by change of metric and their imbedding in Hilbert spaces*, Ann. of Math. 38 (1937), 787–793.

[25] I. J. Schoenberg, *Metric spaces and positive definite functions*, Trans. Amer. Math. Soc. 44 (1938), 522–536.

[26] H. S. Witsenhausen, *Metric inequalities and the zonoid problem*, Proc. Amer. Math. Soc. 40 (1973), 517–520.

[27] A. M. Yaglom, *Some classes of random fields in n-dimensional space, related to stationary random process*, Theory Probab. Appl. 2(3) (1957), 273–320.

[28] W. P. Zastawny, *Positive definite norm-dependent functions. Solution of the Schoenberg problem*, Dokl. Russian Akad. Nauk 325 (1991), 901–903.

Sebastian P. Kuniewski
FACULTY OF ELECTRICAL ENGINEERING
MATHEMATICS AND COMPUTER SCIENCE
DELFT UNIVERSITY OF TECHNOLOGY
THE NETHERLANDS
E-mail: s.k.kuniewski@tudelft.nl

Jolanta K. Misiewicz
DEPARTMENT OF MATHEMATICS AND INFORMATION SCIENCE
WARSAW TECHNICAL UNIVERSITY
WARSAW, POLAND
E-mail: j.misiewicz@mini.pw.edu.pl

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