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DEMONSTRATIO MATHEMATICA
Vol. XLVII No 1 2014

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# SOME FIXED POINT RESULTS FOR MAPPINGS IN $G$-METRIC SPACES

**Abstract.** We prove a common fixed point theorem for a pair of self mappings satisfying a generalized contractive type condition in a complete $G$-metric space. We also deal with other fixed point results for a self mapping in the setting of generalized metric space. Our results generalize some recent results in the literature.

## 1. Introduction

Metric fixed point theory is playing an increasing role in mathematics and applied sciences. Over the past two decades a considerable amount of research work for the development of fixed point theory have executed by several authors. There have been a number of generalizations of metric spaces, such as Gähler [4, 5] (called 2-metric spaces) and Dhage [2, 3] (called $D$-metric spaces). Different authors proved that the results obtained by Gähler in 2-metric spaces are independent rather than generalizations of the corresponding results in metric spaces. However, Mustafa and Sims in [13] have pointed out that most of the results claimed by Dhage and others in $D$-metric spaces are incorrect. They also introduced an appropriate concept of generalized metric space, called $G$-metric space [9] and developed a new fixed point theory for various mappings in this new structure. Our aim in this study is to obtain some fixed point results in complete $G$-metric spaces. These results generalize some results of [11] and [14].

## 2. Preliminaries

We begin by briefly recalling some basic definitions and important results for $G$-metric spaces which will be needed in the sequel. Throughout this paper, we denote by $N$ the set of positive integers.

2010 *Mathematics Subject Classification*: 54H25, 47H10.
*Key words and phrases*: $G$-metric space, $G$-Cauchy sequence, $G$-continuity, fixed point.

DOI: 10.2478/dema-2014-0015

**Definition 2.1.** (see [9]) Let $X$ be a nonempty set, and let $G : X \times X \times X \to R^+$ be a function satisfying the following axioms:

($G_1$) $G(x, y, z) = 0$ if $x = y = z$,

($G_2$) $0 < G(x, x, y)$, for all $x, y \in X$, with $x \neq y$,

($G_3$) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$,

($G_4$) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),

($G_5$) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality).

Then the function $G$ is called a generalized metric, or, more specifically, a $G$-metric on $X$, and the pair $(X, G)$ is called a $G$-metric space.

**Proposition 2.1.** (see [9]) *Let $(X, G)$ be a $G$-metric space. Then for any $x, y, z$, and $a \in X$, it follows that*

(1) *if $G(x, y, z) = 0$, then $x = y = z$,*
(2) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
(3) $G(x, y, y) \leq 2G(y, x, x)$,
(4) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
(5) $G(x, y, z) \leq \frac{2}{3}\left(G(x, y, a) + G(x, a, z) + G(a, y, z)\right)$,
(6) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.

**Definition 2.2.** (see [9]) Let $(X, G)$ be a $G$-metric space, let $(x_n)$ be a sequence of points of $X$, we say that $(x_n)$ is $G$-convergent to $x$ if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$; that is , for any $\epsilon > 0$, there exists $n_0 \in N$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq n_0$. We call $x$ as the limit of the sequence $(x_n)$ and write $x_n \to x$.

**Definition 2.3.** (see [9]) Let $(X, G)$ be a $G$-metric space, a sequence $(x_n)$ is called $G$-Cauchy if given $\epsilon > 0$, there is $n_0 \in N$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq n_0$; that is, if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

**Definition 2.4.** (see [9]) A $G$-metric space $(X, G)$ is said to be $G$-complete (or a complete $G$-metric space) if every $G$-Cauchy sequence in $(X, G)$ is $G$-convergent in $(X, G)$.

**Proposition 2.2.** *In a $G$-metric space $(X, G)$, the following are equivalent.*

(1) *The sequence $(x_n)$ is $G$-Cauchy.*
(2) *For every $\epsilon > 0$, there exists $n_0 \in N$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq n_0$.*

**Definition 2.5.** (see [9]) Let $(X, G)$ and $(X', G')$ be $G$-metric spaces and let $f : (X, G) \to (X', G')$ be a function, then $f$ is said to be $G$-continuous at a point $a \in X$ if given $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$; $G(a, x, y) < \delta$ implies $G^{'}(f(a), f(x), f(y)) < \epsilon$. A function $f$ is $G$-continuous on $X$ if and only if it is $G$-continuous at all $a \in X$.

**Proposition 2.3.** (see [9]) *Let $(X, G)$ and $(X^{'}, G^{'})$ be G-metric spaces, then a function $f : X \to X^{'}$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at $x$; that is, whenever $(x_n)$ is G-convergent to $x$, $(f(x_n))$ is G-convergent to $f(x)$.*

**Proposition 2.4.** (see [9]) *Let $(X, G)$ be a G-metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

## 3. Main results

**Theorem 3.1.** *Let $(X, G)$ be a complete G-metric space. Suppose the mappings $S, T : X \to X$ satisfy*

$$\max \begin{Bmatrix} G(S(x), T(y), T(y)), \\ G(T(x), S(y), S(y)) \end{Bmatrix} \leq a_1 G(x, y, y) + a_2 \min \begin{Bmatrix} G(x, T(y), T(y)) + G(y, S(x), S(x)), \\ G(x, S(y), S(y)) + G(y, T(x), T(x)) \end{Bmatrix} + a_3 \min \begin{Bmatrix} G(x, S(x), S(x)) + G(y, T(y), T(y)), \\ G(x, T(x), T(x)) + G(y, S(y), S(y)) \end{Bmatrix}, \tag{3.1}$$

*for all $x, y \in X$, where $a_1, a_2, a_3 \geq 0$ with $a_1 + 2a_2 + 2a_3 < 1$. Then $S$ and $T$ have a unique common fixed point in $X$.*

**Proof.** Let $x_0 \in X$ be arbitrary and define a sequence $(x_n)$ by

$$x_n = \begin{cases} S(x_{n-1}), & \text{if } n \text{ is odd}, \\ T(x_{n-1}), & \text{if } n \text{ is even}. \end{cases}$$

For any odd positive integer $n \in N$, we have by (3.1)

$$G(x_n, x_{n+1}, x_{n+1}) = G(S(x_{n-1}), T(x_n), T(x_n)) \leq \max \begin{Bmatrix} G(S(x_{n-1}), T(x_n), T(x_n)), \\ G(T(x_{n-1}), S(x_n), S(x_n)) \end{Bmatrix}$$

$$\leq a_1\, G(x_{n-1}, x_n, x_n) + a_2 \min\left\{\begin{array}{l} G(x_{n-1}, T(x_n), T(x_n)) + G(x_n, S(x_{n-1}), S(x_{n-1})), \\ G(x_{n-1}, S(x_n), S(x_n)) + G(x_n, T(x_{n-1}), T(x_{n-1})) \end{array}\right\} + a_3 \min\left\{\begin{array}{l} G(x_{n-1}, S(x_{n-1}), S(x_{n-1})) + G(x_n, T(x_n), T(x_n)), \\ G(x_{n-1}, T(x_{n-1}), T(x_{n-1})) + G(x_n, S(x_n), S(x_n)) \end{array}\right\}.$$

Thus,

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq a_1\, G(x_{n-1}, x_n, x_n) \\ &\quad + a_2\{G(x_{n-1}, T(x_n), T(x_n)) + G(x_n, S(x_{n-1}), S(x_{n-1}))\} \\ &\quad + a_3\{G(x_{n-1}, S(x_{n-1}), S(x_{n-1})) + G(x_n, T(x_n), T(x_n))\} \\ &= a_1\, G(x_{n-1}, x_n, x_n) \\ &\quad + a_2\{G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_n)\} \\ &\quad + a_3\{G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})\} \\ &\leq a_1\, G(x_{n-1}, x_n, x_n) \\ &\quad + a_2\{G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})\} \\ &\quad + a_3\{G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})\}, \end{aligned}$$

which gives that

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3}\, G(x_{n-1}, x_n, x_n).$$

If $n$ is even then by (3.1), we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(T(x_{n-1}), S(x_n), S(x_n)) \\ &\leq \max\left\{\begin{array}{l} G(S(x_{n-1}), T(x_n), T(x_n)), \\ G(T(x_{n-1}), S(x_n), S(x_n)) \end{array}\right\} \\ &\leq a_1\, G(x_{n-1}, x_n, x_n) \\ &\quad + a_2 \min\left\{\begin{array}{l} G(x_{n-1}, T(x_n), T(x_n)) + G(x_n, S(x_{n-1}), S(x_{n-1})), \\ G(x_{n-1}, S(x_n), S(x_n)) + G(x_n, T(x_{n-1}), T(x_{n-1})) \end{array}\right\} \\ &\quad + a_3 \min\left\{\begin{array}{l} G(x_{n-1}, S(x_{n-1}), S(x_{n-1})) + G(x_n, T(x_n), T(x_n)), \\ G(x_{n-1}, T(x_{n-1}), T(x_{n-1})) + G(x_n, S(x_n), S(x_n)) \end{array}\right\} \\ &\leq a_1\, G(x_{n-1}, x_n, x_n) \\ &\quad + a_2\{G(x_{n-1}, S(x_n), S(x_n)) + G(x_n, T(x_{n-1}), T(x_{n-1}))\} \\ &\quad + a_3\{G(x_{n-1}, T(x_{n-1}), T(x_{n-1})) + G(x_n, S(x_n), S(x_n))\}. \end{aligned}$$

Thus,

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) \le\; & a_1\, G(x_{n-1}, x_n, x_n) \\ & + a_2 \{G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_n)\} \\ & + a_3 \{G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})\} \\ \le\; & a_1\, G(x_{n-1}, x_n, x_n) \\ & + a_2 \{G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})\} \\ & + a_3 \{G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})\}, \end{aligned}$$

which implies that

$$G(x_n, x_{n+1}, x_{n+1}) \le \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3}\, G(x_{n-1}, x_n, x_n).$$

Thus, for any positive integer $n$,

$$G(x_n, x_{n+1}, x_{n+1}) \le \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3}\, G(x_{n-1}, x_n, x_n). \tag{3.2}$$

Let $r = \frac{a_1+a_2+a_3}{1-a_2-a_3}$, then $0 \le r < 1$ since $a_1, a_2, a_3 \ge 0$ with $a_1 + 2a_2 + 2a_3 < 1$. Thus, (3.2) becomes

$$G(x_n, x_{n+1}, x_{n+1}) \le r\, G(x_{n-1}, x_n, x_n). \tag{3.3}$$

By repeated application of (3.3), we obtain

$$G(x_n, x_{n+1}, x_{n+1}) \le r^n\, G(x_0, x_1, x_1). \tag{3.4}$$

Then, by repeated use of the rectangle inequality and (3.4), we have that, for all $n, m \in N$, $n < m$,

$$\begin{aligned} G(x_n, x_m, x_m) &\le G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_m) \\ &\le \left(r^n + r^{n+1} + \cdots + r^{m-1}\right) G(x_0, x_1, x_1) \\ &\le \frac{r^n}{1 - r}\, G(x_0, x_1, x_1). \end{aligned}$$

Then, $\lim G(x_n, x_m, x_m) = 0$, as $n, m \to \infty$, since $\lim \frac{r^n}{1-r}\, G(x_0, x_1, x_1) = 0$, as $n, m \to \infty$. For $n, m, l \in N$, $(G_5)$ implies that

$$G(x_n, x_m, x_l) \le G(x_n, x_m, x_m) + G(x_l, x_m, x_m),$$

taking limit as $n, m, l \to \infty$, we get $G(x_n, x_m, x_l) \to 0$. So $(x_n)$ is a $G$-Cauchy sequence. By completeness of $(X, G)$, there exists $u \in X$ such that $(x_n)$ is $G$-convergent to $u$.

Further, by rectangle inequality and (3.1), we have

$$\begin{aligned} G(u,T(u),T(u)) &\le G(u,x_{2n+1},x_{2n+1}) + G(x_{2n+1},T(u),T(u)) \\ &= G(u,x_{2n+1},x_{2n+1}) + G(S(x_{2n}),T(u),T(u)) \\ &\le G(u,x_{2n+1},x_{2n+1}) + \max\left\{\begin{matrix} G(S(x_{2n}),T(u),T(u)), \\ G(T(x_{2n}),S(u),S(u)) \end{matrix}\right\} \\ &\le G(u,x_{2n+1},x_{2n+1}) + a_1\, G(x_{2n},u,u) \\ &\quad + a_2 \min\left\{\begin{matrix} G(x_{2n},T(u),T(u)) + G(u,S(x_{2n}),S(x_{2n})), \\ G(x_{2n},S(u),S(u)) + G(u,T(x_{2n}),T(x_{2n})) \end{matrix}\right\} \\ &\quad + a_3 \min\left\{\begin{matrix} G(x_{2n},S(x_{2n}),S(x_{2n})) + G(u,T(u),T(u)), \\ G(x_{2n},T(x_{2n}),T(x_{2n})) + G(u,S(u),S(u)) \end{matrix}\right\} \\ &\le G(u,x_{2n+1},x_{2n+1}) + a_1\, G(x_{2n},u,u) \\ &\quad + a_2 \left\{G(x_{2n},T(u),T(u)) + G(u,S(x_{2n}),S(x_{2n}))\right\} \\ &\quad + a_3 \left\{G(x_{2n},S(x_{2n}),S(x_{2n})) + G(u,T(u),T(u))\right\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} G(u,T(u),T(u)) \le\ & G(u,x_{2n+1},x_{2n+1}) + a_1\, G(x_{2n},u,u) \\ & + a_2 \left\{G(x_{2n},T(u),T(u)) + G(u,x_{2n+1},x_{2n+1})\right\} \\ & + a_3 \left\{G(x_{2n},x_{2n+1},x_{2n+1}) + G(u,T(u),T(u))\right\}, \end{aligned}$$

taking the limit as $n \to \infty$, and using the fact that the function $G$ is continuous on its variables, we have

$$G(u,T(u),T(u)) \le (a_2 + a_3)\, G(u,T(u),T(u)).$$

Since $0 \le (a_2 + a_3) < 1$,

$$G(u,T(u),T(u)) = 0,$$

which implies that, $u = T(u)$.

Similarly, we can show that $S(u) = u$. Thus, $u$ is a common fixed point of $S$ and $T$.

To prove uniqueness, suppose that there exists another point $v$ in $X$ such that $v = S(v) = T(v)$. Then,

$$\begin{aligned} G(u,v,v) &= G(S(u),T(v),T(v)) \\ &\le \max\left\{\begin{matrix} G(S(u),T(v),T(v)), \\ G(T(u),S(v),S(v)) \end{matrix}\right\} \end{aligned}$$

$$\begin{aligned} &\le a_1\, G(u,v,v) \\ &\quad + a_2 \min \left\{ \begin{array}{l} G(u,T(v),T(v)) + G(v,S(u),S(u)), \\ G(u,S(v),S(v)) + G(v,T(u),T(u)) \end{array} \right\} \\ &\quad + a_3 \min \left\{ \begin{array}{l} G(u,S(u),S(u)) + G(v,T(v),T(v)), \\ G(u,T(u),T(u)) + G(v,S(v),S(v)) \end{array} \right\} \\ &= a_1\, G(u,v,v) \\ &\quad + a_2 \left\{ G(u,v,v) + G(v,u,u) \right\} \\ &\quad + a_3 \left\{ G(u,u,u) + G(v,v,v) \right\}, \end{aligned}$$

which gives that,

$$G(u,v,v) \le \frac{a_2}{1-a_1-a_2}\, G(v,u,u).$$

Again by the same argument, we will find that

$$G(v,u,u) \le \frac{a_2}{1-a_1-a_2}\, G(u,v,v).$$

Hence,

$$G(u,v,v) \le \left( \frac{a_2}{1-a_1-a_2} \right)^2 G(u,v,v),$$

which implies that $u = v$, since $0 \le \frac{a_2}{1-a_1-a_2} < 1$. ■

**Theorem 3.2.** *Let $(X,G)$ be a complete G-metric space. Suppose the mappings $S,T : X \to X$ satisfy*

$$\begin{aligned} \max \left\{ \begin{array}{l} G(S(x),T(y),T(y)), \\ G(T(x),S(y),S(y)) \end{array} \right\} &\le a_1\, G(x,y,y) \\ &\quad + a_2 \min \left\{ \begin{array}{l} G(x,x,T(y)) + G(y,y,S(x)), \\ G(x,x,S(y)) + G(y,y,T(x)) \end{array} \right\} \\ &\quad + a_3 \min \left\{ \begin{array}{l} G(x,x,S(x)) + G(y,y,T(y)), \\ G(x,x,T(x)) + G(y,y,S(y)) \end{array} \right\}, \end{aligned}$$

*for all $x,y \in X$, where $a_1,a_2,a_3 \ge 0$ with $a_1 + 2a_2 + 2a_3 < 1$. Then $S$ and $T$ have a unique common fixed point in $X$.*

**Proof.** Let $x_0 \in X$ be arbitrary and define a sequence $(x_n)$ by

$$x_n = \begin{cases} S(x_{n-1}), & \text{if } n \text{ is odd,} \\ T(x_{n-1}), & \text{if } n \text{ is even.} \end{cases}$$

Then by the argument similar to that used in Theorem 3.1, we have for any positive integer $n$,

$$G(x_n, x_n, x_{n+1}) \leq r^n \, G(x_0, x_0, x_1). \tag{3.5}$$

Then, by repeated use of the rectangle inequality and (3.5), we have that, for all $n, m \in N$, $n < m$

$$\begin{aligned} G(x_m, x_n, x_n) &\leq G(x_m, x_{m-1}, x_{m-1}) + G(x_{m-1}, x_{m-2}, x_{m-2}) \\ &\quad + G(x_{m-2}, x_{m-3}, x_{m-3}) + \cdots + G(x_{n+1}, x_n, x_n) \\ &\leq \left(r^n + r^{n+1} + \cdots + r^{m-1}\right) G(x_0, x_0, x_1) \\ &\leq \frac{r^n}{1-r} \, G(x_0, x_0, x_1). \end{aligned}$$

So, $(x_n)$ becomes a $G$-Cauchy sequence. By completeness of $(X, G)$, there exists $u \in X$ such that $(x_n)$ is $G$-convergent to $u$.

As in the proof of Theorem 3.1, we can show that

$$G(u, u, T(u)) \leq (a_2 + a_3) \, G(u, u, T(u)).$$

Thus, the desired conclusion follows from the same argument used in Theorem 3.1. ∎

Combining Theorems 3.1 and 3.2, we state the following theorem:

**Theorem 3.3.** *Let $(X, G)$ be a complete $G$-metric space. Suppose the mappings $S, T : X \to X$ satisfying one of the following conditions:*

$$\begin{aligned} \max \left\{ \begin{matrix} G(S(x), T(y), T(y)), \\ G(T(x), S(y), S(y)) \end{matrix} \right\} &\leq a_1 \, G(x, y, y) \\ &+ a_2 \min \left\{ \begin{matrix} G(x, T(y), T(y)) + G(y, S(x), S(x)), \\ G(x, S(y), S(y)) + G(y, T(x), T(x)) \end{matrix} \right\} \\ &+ a_3 \min \left\{ \begin{matrix} G(x, S(x), S(x)) + G(y, T(y), T(y)), \\ G(x, T(x), T(x)) + G(y, S(y), S(y)) \end{matrix} \right\} \end{aligned} \tag{3.6}$$

*or*

$$(3.7)\qquad \max\left\{\begin{array}{l} G(S(x),T(y),T(y)), \\ G(T(x),S(y),S(y)) \end{array}\right\}$$
$$\leq a_1\, G(x,y,y) + a_2 \min\left\{\begin{array}{l} G(x,x,T(y)) + G(y,y,S(x)), \\ G(x,x,S(y)) + G(y,y,T(x)) \end{array}\right\}$$
$$+ a_3 \min\left\{\begin{array}{l} G(x,x,S(x)) + G(y,y,T(y)), \\ G(x,x,T(x)) + G(y,y,S(y)) \end{array}\right\},$$

*for all $x, y \in X$, where $a_1, a_2, a_3 \geq 0$ with $a_1 + 2a_2 + 2a_3 < 1$. Then $S$ and $T$ have a unique common fixed point in $X$.*

As an application of Theorem 3.3, we have the following corollary.

**Corollary 3.1.** *Let $(X, G)$ be a complete $G$-metric space, and let $T : X \to X$ be a mapping satisfying one of the following conditions:*

$$G(T(x),T(y),T(y)) \leq a_1 G(x,y,y) + a_2\{G(x,T(y),T(y)) + G(y,T(x),T(x))\}$$
$$+ a_3\{G(x,T(x),T(x)) + G(y,T(y),T(y))\}$$

*or*

$$G(T(x),T(y),T(y)) \leq a_1\, G(x,y,y) + a_2\, \{G(x,x,T(y)) + G(y,y,T(x))\}$$
$$+ a_3\, \{G(x,x,T(x)) + G(y,y,T(y))\},$$

*for all $x, y \in X$, where $a_1, a_2, a_3 \geq 0$ with $a_1 + 2a_2 + 2a_3 < 1$. Then $T$ has a unique fixed point in $X$.*

**Proof.** Put $S = T$ in Theorem 3.3. ■

**Remark 3.1.** Putting $a_1 = a_3 = 0$ in Corollary 3.1, we obtain Theorem 2.9 from [11].

**Theorem 3.4.** *Let $(X, G)$ be a complete $G$-metric space, and let $T : X \to X$ be such that for each positive integer $n$,*

$$(3.8)\qquad G(T^n(x), T^n(y), T^n(y)) \leq a_n\, G(x,y,y),$$

*for all $x, y \in X$, where $a_n > 0$ is independent of $x, y$. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $T$ has a unique fixed point in $X$.*

**Proof.** Let $x_0 \in X$ be arbitrary and define a sequence $(x_n)$ by

$$x_n = T(x_{n-1}) = T^n(x_0), \qquad \text{for } n = 1, 2, 3, \ldots.$$

Then, by repeated use of the rectangle inequality and (3.8), we have that, for all $n, m \in N$, $n < m$

$$(3.9)\quad \begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_m) \\ &= \sum_{r=n}^{m-1} G(x_r, x_{r+1}, x_{r+1}) \\ &= \sum_{r=n}^{m-1} G(T^r(x_0), T^r(x_1), T^r(x_1)) \\ &\leq \Big(\sum_{r=n}^{m-1} a_r\Big) G(x_0, x_1, x_1). \end{aligned}$$

If $x_1 = x_0$, then a fixed point is obtained. Therefore, we assume that $x_1 \neq x_0$. Let $k$ be a positive integer such that $k > G(x_0, x_1, x_1)$. Since the series $\sum_{n=1}^{\infty} a_n$ is convergent, for $\epsilon > 0$ arbitrary, there exists a positive integer $n_0$ such that

$$\sum_{r=n}^{m-1} a_r < \frac{\epsilon}{k}, \qquad \text{for } m > n \geq n_0.$$

Then, for $m > n \geq n_0$, we have from (3.9)

$$G(x_n, x_m, x_m) \leq \frac{\epsilon}{k} G(x_0, x_1, x_1) < \epsilon.$$

By Proposition 2.2, the sequence $(x_n)$ becomes a $G$-Cauchy sequence. Using the completeness of $(X, G)$, there exists $u \in X$ such that $(x_n)$ is $G$-convergent to $u$.

But by $(G_5)$ and (3.8), we have

$$\begin{aligned} G(u, T(u), T(u)) &\leq G(u, x_{n+1}, x_{n+1}) + G(x_{n+1}, T(u), T(u)) \\ &= G(u, x_{n+1}, x_{n+1}) + G(T(x_n), T(u), T(u)) \\ &\leq G(u, x_{n+1}, x_{n+1}) + a_1 G(x_n, u, u), \end{aligned}$$

taking the limit as $n \to \infty$, and using the fact that the function $G$ is continuous on its variables, we have

$$G(u, T(u), T(u)) = 0,$$

which implies that $u = T(u)$ and $u$ becomes a fixed point of $T$.

For uniqueness, suppose that $v \neq u$ is such that $T(v) = v$. Then for any positive integer $n$, we have

$$G(u, v, v) = G(T^n(u), T^n(v), T^n(v)) \leq a_n\, G(u, v, v).$$

Since by $G_2$, $G(u, v, v) > 0$, it must be the case that $a_n \geq 1$ for all $n$. So, $a_n$ can not tend to zero and this contradiction shows that $u = v$. ■

As an application of Theorem 3.4, we have the following Corollary.

**Corollary 3.2.** (see [14]) *Let $(X,G)$ be a complete G-metric space, and let $T : X \to X$ be a mapping satisfying the following condition for all $x,y \in X$,*

$$G(T(x),T(y),T(y)) \leq k\,G(x,y,y), \tag{3.10}$$

*where $0 \leq k < 1$. Then $T$ has a unique fixed point in $X$.*

**Proof.** For $x,y \in X$, we obtain from (3.10) that

$$G(T^2(x),T^2(y),T^2(y)) \leq k\,G(T(x),T(y),T(y)) \leq k^2\,G(x,y,z).$$

Similarly, for any positive integer $n$,

$$G(T^n(x),T^n(y),T^n(y)) \leq k^n\,G(x,y,y), \quad \text{for all } x,y \in X.$$

But, the series $\sum_{n=1}^{\infty} k^n$ is convergent. Now, Theorem 3.4 applies to obtain a unique fixed point of $T$. ∎

**Theorem 3.5.** *Let $(X,G)$ be a complete G-metric space, and let $T : X \to X$ be G-continuous. Suppose that there exists a mapping $Q : X \to [0,\infty)$ such that*

$$G(x,T(x),T(x)) \leq Q(x) - Q(T(x)), \tag{3.11}$$

*for all $x \in X$. Then $T$ has a fixed point in $X$.*

**Proof.** Let $x_0 \in X$ be arbitrary and define a sequence $(x_n)$ by

$$x_n = T(x_{n-1}), \quad \text{for } n = 1,2,3,\ldots.$$

Then, for any positive integer $r$, we have by using (3.11) that

$$\begin{aligned} G(x_r,x_{r+1},x_{r+1}) &= G(x_r,T(x_r),T(x_r)) \\ &\leq Q(x_r) - Q(T(x_r)) = Q(x_r) - Q(x_{r+1}). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{r=0}^{n-1} G(x_r,x_{r+1},x_{r+1}) &\leq \sum_{r=0}^{n-1} [Q(x_r) - Q(x_{r+1})] \\ &= Q(x_0) - Q(x_n) \leq Q(x_0). \end{aligned}$$

So, the series $\sum_{r=0}^{\infty} G(x_r,x_{r+1},x_{r+1})$ is convergent. Then, for all $n,m \in N$, $n < m$, we have by repeated use of the rectangle inequality that

$$\begin{aligned} G(x_n,x_m,x_m) &\leq G(x_n,x_{n+1},x_{n+1}) + G(x_{n+1},x_{n+2},x_{n+2}) \\ &\quad + G(x_{n+2},x_{n+3},x_{n+3}) + \cdots + G(x_{m-1},x_m,x_m) \\ &= \sum_{r=n}^{m-1} G(x_r,x_{r+1},x_{r+1}). \end{aligned} \tag{3.12}$$

The convergence of the series $\sum_{r=0}^{\infty} G(x_r, x_{r+1}, x_{r+1})$ gives, for arbitrary $\epsilon > 0$, there exists a positive integer $n_0$ such that

$$\sum_{r=n}^{m-1} G(x_r, x_{r+1}, x_{r+1}) < \epsilon \text{ } for \text{ } m > n \geq n_0.$$

Then, for $m > n \geq n_0$, we have from (3.12) that

$$G(x_n, x_m, x_m) < \epsilon.$$

By Proposition 2.2, the sequence $(x_n)$ becomes a $G$-Cauchy sequence. Using the completeness of $(X, G)$, there exists $u \in X$ such that $(x_n)$ is $G$-convergent to $u$. $G$-continuity of $T$ implies that

$$T(u) = \lim_n T(x_n) = \lim_n x_{n+1} = u.$$

Thus, $u$ is a fixed point of $T$. ■

**Remark 3.2.** A fixed point of $T$, in the above theorem, is not unique. The identity mapping $I$ satisfies the condition (3.11) but a fixed point of $I$ is not unique.

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*Received May 20, 2011; revised version February 2, 2012.*