DE GRUYTER
OPEN

DEMONSTRATIO MATHEMATICA
Vol. XLVII No 1 2014

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ON TOTALLY CONTACT UMBILICAL CONTACT CR-LIGHTLIKE SUBMANIFOLDS OF INDEFINITE SASAKIAN MANIFOLDS

Abstract. After brief introduction, we prove that a totally contact umbilical CR-lightlike submanifold is totally contact geodesic. We obtain a necessary and sufficient condition for a CR-lightlike submanifold to be an anti-invariant submanifold. Finally, we characterize a contact CR-lightlike submanifold of indefinite Sasakian manifold to be a contact CR-lightlike product.

1. Introduction

Cauchy–Riemann (CR)-submanifolds of Kaehlerian manifolds with Riemannian metric were introduced by Bejancu [2] and further studied by [3]–[6] and many more. Then contact CR-submanifolds of Sasakian manifolds with definite metric were introduced and studied by Yano and Kon [13, 14]. Recently, Duggal and Sahin [8] introduced the theory of contact CR-lightlike submanifolds of indefinite Sasakian manifolds. Since significant applications of the contact geometry (Maclane [11], Nazaikinskii et al. [12], Arnol'd [1]) and very limited information available on its lightlike case, motivated us to extend this theory.

2. Lightlike submanifolds

We recall notations and fundamental equations for lightlike submanifolds, which are due to the book [7] by Duggal and Bejancu.

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ and g be the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M then M is called a lightlike

2010 *Mathematics Subject Classification*: 53B30, 53C25, 53D15.
Key words and phrases: indefinite Sasakian manifolds, lightlike submanifolds, contact CR-lightlike submanifolds, totally contact umbilical submanifolds.

DOI: 10.2478/dema-2014-0014

submanifold of $\bar{M}$. For a degenerate metric g on M,

$$TM^{\perp} = \cup\{u \in T_x\bar{M} : \bar{g}(u,v) = 0,\ \forall v \in T_xM,\ x \in M\} \tag{2.1}$$

is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $\mathrm{Rad}\, T_xM = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping

$$\mathrm{Rad}\, TM : x \in M \longrightarrow \mathrm{Rad}\, T_xM, \tag{2.2}$$

defines a smooth distribution on M of rank $r > 0$ then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold, and $RadTM$ is called the radical distribution on M.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM, that is

$$TM = \mathrm{Rad}\, TM \perp S(TM), \tag{2.3}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $RadTM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and to $\mathrm{Rad}\, TM$ in $S(TM^{\perp})^{\perp}$, respectively. Then, we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{2.4}$$

$$\begin{aligned} T\bar{M}|_M &= TM \oplus tr(TM) \\ &= (\mathrm{Rad}\, TM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \end{aligned} \tag{2.5}$$

Let u be a local coordinate neighborhood of M and consider the local quasi-orthonormal fields of frames of $\bar{M}$ along M, on u as $\{\xi_1, \dots, \xi_r, W_{r+1}, \dots, W_n, N_1, \dots, N_r, X_{r+1}, \dots, X_m\}$, where $\{\xi_1, \dots, \xi_r\}, \{N_1, \dots, N_r\}$ are local lightlike bases of $\Gamma(\mathrm{Rad}\, TM|_u)$, $\Gamma(ltr(TM)|_u)$ and $\{W_{r+1}, \dots, W_n\}$, $\{X_{r+1}, \dots, X_m\}$ are local orthonormal bases of $\Gamma(S(TM^{\perp})|_u)$ and $\Gamma(S(TM)|_u)$, respectively. For this quasi-orthonormal fields of frames, we have

Theorem 2.1. [7] *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of $\mathrm{Rad}\, TM$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)|_{\mathrm{u}})$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}|_{\mathrm{u}}$, where* u *is a coordinate neighborhood of M, such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \tag{2.6}$$

where $\{\xi_1, \dots, \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be the Levi–Civita connection on $\bar{M}$. Then according to the decomposition (2.5), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \forall X, Y \in \Gamma(TM), \tag{2.7}$$

$$\bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \forall X \in \Gamma(TM), U \in \Gamma(tr(TM)), \tag{2.8}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla^{\perp}_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$, called as the second fundamental form. A_V is a linear operator on M, called as shape operator.

According to (2.4), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, (2.7) and (2.8) become

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{2.9}$$

$$\bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{2.10}$$

where, we put $h^l(X,Y) = L(h(X,Y)), h^s(X,Y) = S(h(X,Y)), D^l_X U = L(\nabla^{\perp}_X U)$, $D^s_X U = S(\nabla^{\perp}_X U)$.

As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued, respectively, therefore they are called the lightlike second fundamental form and the screen second fundamental form on M. In particular,

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \tag{2.11}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \tag{2.12}$$

where $X \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$.

Using (2.4)–(2.5) and (2.9)–(2.12), we obtain

$$\bar{g}(h^s(X,Y), W) + \bar{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.13}$$

$$\bar{g}(h^l(X,Y), \xi) + \bar{g}(Y, h^l(X,\xi)) + g(Y, \nabla_X \xi) = 0, \tag{2.14}$$

$$\bar{g}(A_N X, N') + \bar{g}(N, A_{N'} X) = 0, \tag{2.15}$$

$$\bar{g}(A_N X, \bar{P}Y) = \bar{g}(N, \bar{\nabla}_X \bar{P}Y), \tag{2.16}$$

for any $\xi \in \Gamma(RadTM)$, $W \in \Gamma(S(TM^{\perp}))$ and $N, N' \in \Gamma(ltr(TM))$. $\bar{P}$ is a projection of TM on $S(TM)$.

Now, we consider the decomposition (2.3), we can write

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*(X, \bar{P}Y), \tag{2.17}$$

$$\nabla_X \xi = -A^*_{\xi} X + \nabla^{*\perp}_X \xi, \tag{2.18}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}\, TM)$, where $\{\nabla^*_X \bar{P}Y, A^*_{\xi} X\}$ and $\{h^*(X, \bar{P}Y), \nabla^{*\perp}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(\mathrm{Rad}\, TM)$, respectively. Here ∇^* and $\nabla^{*\perp}_X$ are linear connections on $S(TM)$ and $\mathrm{Rad}\, TM$, respectively. By using (2.9)–(2.10) and (2.17)–(2.18), we obtain

$$\bar{g}(h^l(X, \bar{P}Y), \xi) = g(A^*_\xi X, \bar{P}Y), \tag{2.19}$$

$$\bar{g}(h^*(X, \bar{P}Y), N) = \bar{g}(A_N X, \bar{P}Y). \tag{2.20}$$

An odd-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called a contact metric manifold [10], if there are a $(1,1)$ tensor field ϕ, a vector field V and a one form η satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \qquad \bar{g}(V, V) = \epsilon, \tag{2.21}$$

$$\phi^2 X = -X + \eta(X)V, \tag{2.22}$$

$$\bar{g}(X, V) = \epsilon\eta(X), \quad \epsilon = \pm 1. \tag{2.23}$$

It follows that $\phi V = 0$, $\eta \circ \phi = 0$, $\eta(V) = \epsilon$, where V is called characteristic vector field and $(\phi, V, \eta, \bar{g})$ is called contact metric structure of $\bar{M}$ and $\bar{M}$ is called contact manifold.

If $d\eta(X, Y) = g(\phi X, Y)$ then M is said to have contact metric structure $(\phi, V, \eta, \bar{g})$. If $N_\phi + d\eta \otimes V = 0$, where N_ϕ is the Nijenhuis tensor field then $\bar{M}$ is called an indefinite Sasakian manifold, for which we have

$$\bar{\nabla}_X V = \phi X, \tag{2.24}$$

$$(\bar{\nabla}_X \phi)Y = -\bar{g}(X, Y)V + \epsilon\eta(Y)X. \tag{2.25}$$

3. Contact CR-lightlike submanifolds

Definition 3.1. [8] Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold, tangent to the structure vector field V, immersed in an indefinite Sasakian manifold $(\bar{M}, \bar{g})$. Then M is said to be a contact CR-lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(A) $\operatorname{Rad} TM$ is a distribution on M such that $\operatorname{Rad} TM \cap \phi(\operatorname{Rad} TM) = \{0\}$.
(B) There exist vector bundles D_0 and D' over M such that

$$S(TM) = \{\phi(RadTM) \oplus D'\} \perp D_0 \perp \{V\}, \tag{3.1}$$

$$\phi D_0 = D_0, \quad \phi(D') = L_1 \perp ltr(TM), \tag{3.2}$$

where D_0 is non degenerate and L_1 is a vector subbundle of $S(TM^\perp)$.

Therefore

$$TM = D \oplus \{V\} \oplus D', \quad D = \operatorname{Rad} TM \perp \phi(\operatorname{Rad} TM) \perp D_0. \tag{3.3}$$

A contact CR-lightlike submanifold is proper if $D_0 \neq \{0\}$ and $L_1 \neq \{0\}$. If $D_0 = \{0\}$ then M is said to be anti-invariant lightlike submanifold.

Example 3.1. [8] Let M be a lightlike hypersurface of $\bar{M}$. For $\xi \in \Gamma(\operatorname{Rad} TM)$, we have $\bar{g}(\phi\xi, \xi) = 0$, this implies $\phi\xi \in \Gamma(TM)$ and we have a rank-1 distribution $\phi(TM^\perp)$ on M such that $\phi(TM^\perp) \cap TM^\perp = \{0\}$. This implies that $\phi(TM^\perp)$ is a vector subbundle of $S(TM)$. Since for any

$N \in \Gamma(ltr(TM))$, $\bar{g}(\phi N, \xi) = -\bar{g}(N, \phi\xi) = 0$ and $\bar{g}(\phi N, N) = 0$, therefore $\phi N \in \Gamma(S(TM))$. Taking $D' = \phi(tr(TM))$, we obtain $S(TM) = \{\phi(TM^{\perp}) \oplus D'\} \perp D_0$, where D_0 is a non degenerate distribution and $\phi(D') = tr(TM)$. Hence, M is a contact CR-lightlike hypersurface.

Denote the orthogonal complement subbundle to L_1 in $S(TM^{\perp})$ by $L_1^{\perp}$, therefore

$$\phi X = fX + wX, \quad \forall X \in \Gamma(TM), \tag{3.4}$$

where $fX \in \Gamma(D)$, $wX \in \Gamma(L_1 \perp ltr(TM))$ and

$$\phi W = BW + CW, \quad \forall W \in \Gamma(S(TM^{\perp}), \tag{3.5}$$

where $BW \in \Gamma(\phi L_1) \subset \Gamma(D')$, $CW \in \Gamma(L_1^{\perp})$.

Using (2.9), (2.10), (3.4) and (3.5) in (2.25) and then comparing tangential and transversal components, we have

$$(\nabla_X f)Y = A_{wY}X + B(h^s(X,Y)) + \phi(h^l(X,Y)) - g(X,Y)V + \epsilon\eta(Y)X, \tag{3.6}$$

and

$$(\nabla^t_X w)Y = C(h^s(X,Y)) - h(X, fY), \tag{3.7}$$

where

$$(\nabla_X f)Y = \nabla_X fY - f\nabla_X Y. \tag{3.8}$$

$$(\nabla^t_X w)Y = \nabla^t_X wY - w\nabla_X Y. \tag{3.9}$$

Definition 3.2. If the second fundamental form h of a submanifold, tangent to structure vector field V, of an indefinite Sasakian manifold $\bar{M}$ is of the form

$$h(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha + \eta(X)h(Y,V) + \eta(Y)h(X,V), \tag{3.10}$$

for any $X, Y \in \Gamma(TM)$, where α is a vector field transversal to M, then M is called totally contact umbilical and totally contact geodesic if $\alpha = 0$.

For a totaly contact umbilical M, we have

$$h^l(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha_l + \eta(X)h^l(Y,V) + \eta(Y)h^l(X,V), \tag{3.11}$$

$$h^s(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha_s + \eta(X)h^s(Y,V) + \eta(Y)h^s(X,V). \tag{3.12}$$

Theorem 3.1. *Let M be a totally contact umbilical proper contact CR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ and screen distribution be totally geodesic in M. Then $\nabla_X \phi X = \phi\nabla_X X$, for any $X \in \Gamma(D_0)$.*

Proof. From (3.7) and (3.9), we have $\nabla^t_X wY - w\nabla_X Y = C(h^s(X,Y)) - h(X, fY)$. Let $X \in \Gamma(D_0)$ then $\phi X = fX$ and $wX = 0$ therefore $\nabla^t_X wX - w\nabla_X X = C(h^s(X,X)) - h(X, fX)$ becomes $-w\nabla_X X = C(h^s(X,X)) -$

$h(X, \phi X)$. Since M is totally contact umbilical then using (3.10), we have $-w\nabla_X X = C(h^s(X,X)) - \{g(X,\phi X)\}\alpha$, or $w\nabla_X X + C(h^s(X,X)) = 0$. Hence

$$\nabla_X X \in \Gamma(D), \qquad h^s(X,X) \in \Gamma(L_1). \tag{3.13}$$

Since $D = \mathrm{Rad}\, TM \perp \phi(\mathrm{Rad}\, TM) \perp D_0$ therefore $\nabla_X X \in D_0$, $\nabla_X X \in \mathrm{Rad}(TM)$ or $\nabla_X X \in \phi(\mathrm{Rad}(TM))$. Let $N \in \mathrm{Rad}(TM)$ therefore using (2.11) and (2.20), we have $g(\nabla_X X, N) = -\bar{g}(X, \bar{\nabla}_X N) = g(h^*(X,X), N)$. Also using (2.21) and (2.25), we have $g(\nabla_X X, \phi N) = -\bar{g}(X, \phi\bar{\nabla}_X N) = \bar{g}(h^*(\phi X, X), N)$. Since screen distribution is totally geodesic in M therefore $\nabla_X X \in D_0$.

Let $X, Y \in \Gamma(D_0)$ then $g(\nabla_X \phi X, Y) = \bar{g}(\bar{\nabla}_X \phi X, Y) = \bar{g}(\phi\bar{\nabla}_X X - g(X,X)V + \epsilon\eta(X)X, Y) = \bar{g}(\phi\bar{\nabla}_X X, Y)$, by using (2.25) and $\eta(X) = 0$. Using (2.21) and (2.22), we have

$$g(\nabla_X \phi X, Y) = -\bar{g}(\bar{\nabla}_X X, \phi Y) = -g(\nabla_X X, \phi Y) = g(\phi\nabla_X X, Y)$$

then non degeneracy of D_0, gives the result. ■

Lemma 3.1. [8] *Let M be a totally contact umbilical proper contact CR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then $\alpha_l = 0$.*

Theorem 3.2. *Let M be a totally contact umbilical proper contact CR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ and screen distribution be totally geodesic in M. Then M is a totally contact geodesic.*

Proof. For $W \in \Gamma(S(TM^\perp))$ and $X \in \Gamma(D_0)$, using (2.25), (3.13) and Theorem 3.1, we have

$$\begin{aligned} \bar{g}(\phi\bar{\nabla}_X X, \phi W) &= \bar{g}(\bar{\nabla}_X \phi X + g(X,X)V, \phi W) = \bar{g}(\bar{\nabla}_X \phi X, \phi W) \\ &= \bar{g}(\nabla_X \phi X + h^s(X,\phi X), BW + CW) \\ &= \bar{g}(\nabla_X \phi X, \phi W) + \bar{g}(h^s(X,\phi X), CW) \\ &= \bar{g}(\phi\nabla_X X, \phi W) + g(X,\phi X)\bar{g}(\alpha_s, CW) \\ &= \bar{g}(\nabla_X X, W) = 0. \end{aligned} \tag{3.14}$$

But, using (2.21), (3.12) and (3.13), we have

$$\begin{aligned} \bar{g}(\phi\bar{\nabla}_X X, \phi W) &= \bar{g}(\bar{\nabla}_X X, W) = g(\nabla_X X, W) + g(h^s(X,X), W) \\ &= g(h^s(X,X), W) = g(X,X)g(\alpha_s, W), \end{aligned} \tag{3.15}$$

therefore from (3.14) and (3.15), we have $g(X,X)g(\alpha_s, W) = 0$ then non degeneracy of D_0 and $S(TM^\perp)$ implies that $\alpha_s = 0$. Using this with Lemma 3.1, we get the result. ■

Theorem 3.3. [8] *Let M be a contact CR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then $D \oplus \{V\}$ is integrable if and only if $h(X, \phi Y) = h(\phi X, Y)$, for any $X, Y \in \Gamma(D \oplus \{V\})$.*

Theorem 3.4. *Let M be a contact CR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then D' is integrable if and only if $A_{\phi Z}W = A_{\phi W}Z$, for any $W, Z \in \Gamma(D')$.*

Proof. From (3.6) and (3.8), for $W, Z \in \Gamma(D')$, we have $-f\nabla_Z W = A_{wW}Z + Bh^s(Z,W) + \phi h^l(Z,W) - g(Z,W)V$. Then we obtain $-f[Z,W] = A_{wW}Z - A_{wZ}W$, which completes the proof. ∎

Lemma 3.2. *For $Y \in \Gamma(D')$ and $Z \in \Gamma(D)$, we have*

$$g(\nabla_X Y, Z) = g(fA_{wY}X, Z).$$

Proof. Using (3.6), we have $\nabla_X fY - f\nabla_X Y = A_{wY}X + B(h^s(X,Y)) + \phi(h^l(X,Y)) - g(X,Y)V + \epsilon\eta(Y)X$. Let $Y \in \Gamma(D')$ then $\phi Y = wY, fY = 0$, therefore we have $f\nabla_X Y = -A_{wY}X - B(h^s(X,Y)) - \phi(h^l(X,Y)) + g(X,Y)V$. Let $Z \in \Gamma(D)$ then $\phi Z \in D$, therefore $g(f\nabla_X Y, \phi Z) = -g(A_{wY}X, \phi Z) - g(B(h^s(X,Y)), \phi Z) - g(\phi(h^l(X,Y), \phi Z) + g(X,Y)g(V,\phi Z) = -g(A_{wY}X, \phi Z)$. Using (2.21), we get $g(\nabla_X Y, Z) - \epsilon\eta(\nabla_X Y)\eta(Z) = -g(A_{wY}X, \phi Z)$, this implies $g(\nabla_X Y, Z) = g(fA_{wY}X, Z)$. Hence the result follows. ∎

Next, for any $\lambda \in \Gamma(tr(TM))$, we put

$$\phi\lambda = P\lambda + F\lambda, \tag{3.16}$$

where $P\lambda$ and $F\lambda$ are tangential and transversal components of $\phi\lambda$, respectively. Using (2.22), we get

$$f^2 = -I - Pw + \eta \otimes V, \tag{3.17}$$

$$wf + Fw = 0, \tag{3.18}$$

$$F^2 = -I - wP, \tag{3.19}$$

$$fP + PF = 0. \tag{3.20}$$

Clearly, if M is tangent to structure vector field V, then it is a CR-submanifold if and only if one of the following conditions is satisfied [9]:

$$\text{(i) } wF = 0, \quad \text{(ii) } Fw = 0, \quad \text{(iii) } fP = 0, \quad \text{(iv) } PF = 0. \tag{3.21}$$

Theorem 3.5. *Let M be a contact CR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then f is parallel if and only if M is an anti-invariant submanifold.*

Proof. Put $Y = V$ in (3.6). We get $(\nabla_X f)V = A_{wV}X + P(h(X,V)) - g(X,V)V + \epsilon\eta(V)X$, since $\phi V = 0$ hence $wV = 0 = fV$. Therefore $(\nabla_X f)V = X - \epsilon\eta(X)V + P(h(X,V))$. Let f be parallel then we have $X - \epsilon\eta(X)V + P(h(X,V)) = 0$, applying f to this equation then using (3.21), we get $fX = 0$, hence M is an anti-invariant submanifold. Converse is trivial from (3.8). ∎

Theorem 3.6. *Let M be a contact CR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then $D \oplus \{V\}$ defines totally geodesic foliation in M if and only if $h^l(X, \phi Y) = 0, Bh^s(X, \phi Y) = 0$, for any $X, Y \in D \oplus \{V\}$.*

Proof. Clearly $D \oplus \{V\}$ defines a totally geodesic foliation in M if and only if $\nabla_X Y \in \Gamma(D \oplus \{V\})$, for any $X, Y \in D \oplus \{V\}$. Since $D' = \phi(L_1 \perp ltr(TM))$, therefore $\nabla_X Y \in \Gamma(D \oplus \{V\})$ if and only if

$$\bar{g}(\nabla_X Y, \phi W_i) = 0, \quad i \in \{1, \dots, s\}, \tag{3.22}$$

and

$$\bar{g}(\nabla_X Y, \phi \xi_j) = 0, \quad j \in \{1, \dots, r\}, \tag{3.23}$$

where $\{N_1, N_2, \dots, N_r\}$ is a basis of $\Gamma(ltr(TM))$ with respect to the basis $\{\xi_1, \xi_2, \dots, \xi_r\}$ of $\Gamma(\mathrm{Rad}(TM))$ and $\{W_1, W_2, \dots, W_s\}$ is a basis of $\Gamma(L_1)$. Using (2.9), (2.21), (2.22) and (2.25), we have $\bar{g}(\nabla_X Y, \phi W_i) = -\bar{g}(\bar{\nabla}_X \phi Y, W_i)$ $= -\bar{g}(h^s(X, \phi Y), W_i)$ and similarly, $\bar{g}(\nabla_X Y, \phi \xi_j) = -\bar{g}(h^l(X, \phi Y), \xi_j)$. Hence the assertion follows. ∎

Theorem 3.7. [5] *A CR-submanifold of a Kaehler manifold $\bar{M}$ is a CR-product if and only if P is parallel, that is, $\bar{\nabla} P = 0$, where $\bar{J} X = PX + FX$.*

Now, we give the characterization of contact CR-lightlike product of an indefinite Sasakian manifold.

Definition 3.3. A contact CR-lightlike submanifold M of an indefinite Sasakian manifold $\bar{M}$ is called contact CR-lightlike product if both the distributions $D \oplus \{V\}$ and D' define totally geodesic foliation in M.

Theorem 3.8. *Let M be a contact CR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. If f is parallel, that is, $\nabla f = 0$ then M is a contact CR-lightlike product.*

Proof. Let $X, Y \in \Gamma(D')$ then $(\nabla_X f)Y = 0$ implies that $\nabla_X fY - f \nabla_X Y$ $= 0$. Since $Y \in \Gamma(D')$, therefore $fY = 0$, hence $f \nabla_X Y = 0$, $\forall X, Y \in \Gamma(D')$, this implies that the distribution D' defines a totally geodesic foliation in M.

Let $X, Y \in D \oplus \{V\}$, then $wY = 0$ and using (3.6), we obtain $(\nabla_X f)Y =$ $B(h^s(X, Y)) + \phi(h^l(X, Y)) - g(X, Y)V + \epsilon \eta(Y) X$. Taking into account that f is parallel, we get $B(h^s(X, Y)) + \phi(h^l(X, Y)) - g(X, Y)V + \epsilon \eta(Y) X$ $= 0$. Comparing the transversal components, we get $B(h^s(X, Y)) = 0$ and $h^l(X, Y) = 0$. Hence, the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M. Consequently, M is a contact CR-lightlike product of an indefinite Sasakian manifold.

Acknowledgment. The authors would like to thank the anonymous referee for his/her comments that helped us to improve this paper.

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Received March 28, 2012; revised version July 24, 2012.