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THE EXPLICIT DETERMINATIONS OF DUAL PLANE CURVES AND DUAL HELICES IN TERMS OF ITS DUAL CURVATURE AND DUAL TORSION

Abstract. In this paper, we give the explicit determinations of dual plane curves, general dual helices and dual slant helices in terms of its dual curvature and dual torsion as a fundamental theory of dual curves in a dual 3-space.

1. Introduction

Study of a theory of plane curves in differential geometry is the most elementary and classical topic. A theory of plane curves plays a fundamental role in a theory of the general curves. It is well-known that a plane curve in Euclidean space is completely determined by a function called by the *curvature*. Moreover, the position vector of a curve with a given curvature function is well-known and this is very useful.

In a theory of space curves, especially, a helix is the most elementary and interesting topic. A helix, moreover, pays attention to natural scientists as well as mathematicians because of its various applications, for example, DNA, carbon nanotube, α-helix, and so on.

A helix in a Euclidean 3-space is defined as a regular curve with constant curvature and constant torsion, and a helix has the properties that its tangent (principal normal, respectively) vector field makes constant angle with some fixed line. These properties of a helix have led its generalizations, called by a *general helix* and *slant helix*. A famous *Lancret theorem* states that general helices are characterized by the constant ratio of curvature and torsion. Also, slant helices are characterized by a differential equation of curvature and

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torsion ([4]) which is

$$\frac{\kappa^2}{\kappa^2+\tau^2}\left(\frac{\tau}{\kappa}\right)' \equiv \text{constant}.$$

Moreover, the position vectors of general helices and slant helices in a Euclidean 3-space are also studied (see, [1, 2]).

Recently, mathematicians studied a theory of curves in a 3-dimensional dual space motivated by E. Study mapping (see [3, 6–8]). E. Study mapping is the corresponding between a dual spherical curve and a ruled surface in a Euclidean 3-space.

On the other hand, a 3-dimensional dual space $\mathbb{D}^3$ can be considered as a 6-dimensional space containing a Euclidean 3-space $\mathbb{E}^3$. Thus, a curve in $\mathbb{D}^3$ is a natural extension of a curve in $\mathbb{E}^3$.

In this paper, we study a fundamental theory of dual plane curves, general dual helices and dual slant helices in $\mathbb{D}^3$. Also, we give characterizations of these dual curves in terms of its dual curvature and dual torsion. In §2, we give some basic facts of the dual number, the functions, the dual space and the dual space curves. Afterwards, we study the dual plane curves together with a explicit determination of a dual plane curve in terms of its dual curvature in §3. Lastly, we give the Lancret theorem of the general dual helices in $\mathbb{D}^3$ and classifications of the general dual helices and the dual slant helices in §4.

2. Preliminaries

Dual numbers were introduced by W. K. Clifford in 1873. A dual number has the form $\tilde{a} = a + \epsilon a^*$ where a and a^* are real numbers and ϵ stands for the *dual unit*, subjected to the ruleds:

$$\epsilon \neq 0, \quad 0\epsilon = \epsilon 0 = 0, \quad 1\epsilon = \epsilon 1 = \epsilon, \quad \epsilon^2 = 0.$$

As a complex number, the composition rules for dual numbers is defined as the definitions;

1. *Equality* : $\tilde{a} = \tilde{b}$ if and only if $a = b$ and $a^* = b^*$.
2. *Addition* : $(a + \epsilon a^*) + (b + \epsilon b^*) = (a + b) + \epsilon(a^* + b^*)$.
3. *Multiplication* : $(a + \epsilon a^*)(b + \epsilon b^*) = ab + \epsilon(ab^* + a^*b)$.

Let $\mathbb{D}$ denote the set of dual numbers;

$$\mathbb{D} = \{\tilde{a} = a + \epsilon a^* \mid a, a^* \in \mathbb{R}\}.$$

It is well-known that dual numbers form a ring over the real numbers, but not field. In fact, the pure dual numbers are zero divisors.

Dual function of dual number presents a mapping of a dual numbers space on itself. The analytic condition for dual function

$$\tilde{f}(x+\epsilon x^*) = f(x+\epsilon x^*) + \epsilon f^*(x+\epsilon x^*)$$

is

$$\frac{\partial f^*}{\partial x^*} = \frac{\partial f}{\partial x}$$

and the derivative of $\tilde{f}$ with respect to a dual variable is

$$\frac{d\tilde{f}(\tilde{x})}{d\tilde{x}} = \frac{\partial f^*}{\partial x^*} + \epsilon\frac{\partial f^*}{\partial x}.$$

A differentiable function $f(\tilde{x})$ can be expressed by

$$f(x+\epsilon x^*) = f(x) + \epsilon x^* f'(x),$$

from which

$$\frac{df(\tilde{x})}{d\tilde{x}} = f'(x) + \epsilon x^* f''(x),$$

where $f'(x)$ is the derivative of f.

An ordered triple of dual numbers $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ is called *a dual vector* and the set of dual vectors is denotedy by

$$\begin{aligned}\mathbb{D}^3 = \mathbb{D}\times\mathbb{D}\times\mathbb{D} = \{\tilde{\mathbf{x}}|\tilde{\mathbf{x}} &= (x_1+\epsilon x_1^*, x_2+\epsilon x_2^*, x_3+\epsilon x_3^*)\\ &= (x_1+x_2+x_3) + \epsilon(x_1^*, x_2^*, x_3^*)\\ &= \mathbf{x}+\epsilon\mathbf{x}^*,\ \mathbf{x},\mathbf{x}^*\in\mathbb{E}^3\}.\end{aligned}$$

For any dual number $\tilde{\lambda} = \lambda + \epsilon\lambda^*$ and any $\tilde{\mathbf{x}} = \mathbf{x}+\epsilon\mathbf{x}^*$, $\tilde{\mathbf{y}} = \mathbf{y}+\epsilon\mathbf{y}^* \in \mathbb{D}^3$, we define:

1. *Equality* : $\tilde{\mathbf{x}} = \tilde{\mathbf{y}}$ if and only if $\mathbf{x} = \mathbf{y}$ and $\mathbf{x}^* = \mathbf{y}^*$.
2. *Scalar multiplication* : $\tilde{\lambda}\tilde{\mathbf{x}} = \lambda\mathbf{x} + \epsilon(\lambda\mathbf{x}^* + \lambda^*\mathbf{x})$.
3. *Scalar product* : $\langle\tilde{\mathbf{x}},\tilde{\mathbf{y}}\rangle = \langle\mathbf{x},\mathbf{y}\rangle + \epsilon(\langle\mathbf{x},\mathbf{y}^*\rangle + \langle\mathbf{x}^*,\mathbf{y}\rangle)$.
4. *Cross product* : $\tilde{\mathbf{x}}\times\tilde{\mathbf{y}} = \mathbf{x}\times\mathbf{y} + \epsilon(\mathbf{x}\times\mathbf{y}^* + \mathbf{x}^*\times\mathbf{y})$.

In this definition, the notions $\langle\cdot,\cdot\rangle$ and $\times$ mean the standard scalar product and the standard cross product of $\mathbb{E}^3$, respectively.

If $\mathbf{x}\neq 0$, the norm of a dual vector $\tilde{\mathbf{x}}$ is defined by

$$||\tilde{\mathbf{x}}|| = ||\mathbf{x}|| + \epsilon\frac{\langle\mathbf{x},\mathbf{y}^*\rangle}{||\mathbf{x}||}.$$

A dual vector $\tilde{\mathbf{x}}$ with norm 1 is called a *dual unit vector.* Let $\tilde{\mathbf{x}} = \mathbf{x}+\epsilon\mathbf{x}^* \in \mathbb{D}^3$. Then the set

$$\tilde{\mathbb{S}}^2 = \{\tilde{\mathbf{x}} = \mathbf{x}+\epsilon\mathbf{x}^* \mid \|\tilde{\mathbf{x}}\| = (1,0);\ \mathbf{x},\mathbf{x}^*\in\mathbb{E}^3\}$$

is called the *dual unit sphere.*

The angle $\tilde{\theta} = \theta + \epsilon\theta^*$ between two dual unit vectors $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ is called *a dual angle* and defined by

$$\langle\tilde{\mathbf{x}},\tilde{\mathbf{y}}\rangle = \cos(\tilde{\theta}) = \cos(\theta) - \epsilon\theta^*\sin(\theta).$$

Let $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ and $\gamma^*(t) = (\gamma_1^*(t), \gamma_2^*(t), \gamma_3^*(t))$ be real valued curves in the Euclidean space $\mathbb{E}^3$. The $\tilde{\gamma}(t) = \gamma(t) + \epsilon\gamma^*(t)$ is a curve in the dual space $\mathbb{D}^3$ and is called a *dual space curve.* If the real valued functions $\gamma_i(t)$ and $\gamma_i^*(t)$ are differentiable, then the dual space curve

$$\begin{aligned}\tilde{\gamma} : I &\longrightarrow \mathbb{D}^3\\ t &\longmapsto \tilde{\gamma}(t) = (\gamma_1(t) + \epsilon\gamma_1^*(t), \gamma_2(t) + \epsilon\gamma_2^*(t), \gamma_3(t) + \epsilon\gamma_3^*(t))\\ &\qquad\quad = \gamma(t) + \epsilon\gamma^*(t)\end{aligned}$$

is differentiable in $\mathbb{D}^3$. We call the real part $\tilde{\gamma}$ *the inicatrix* of $\tilde{\gamma}(t)$. The *dual arc length* of the dual space curve $\tilde{\gamma}(t)$ from t_1 to t is defined by

$$\tilde{s} = \int_{t_1}^{t} ||\tilde{\gamma}'(t)||dt + \epsilon \int_{t_1}^{t} \langle \mathbf{t}, (\gamma^*(t))' \rangle dt = s + \epsilon s^*, \tag{1}$$

where $\mathbf{t}$ is a unit tangent vector of $\gamma(t)$. From now on, we will take the arc length s of $\gamma(t)$ as the parameter instead of t.

Denote by $\{\tilde{\mathbf{t}}, \tilde{\mathbf{n}}, \tilde{\mathbf{b}}\}$ the moving dual Frenet frame along the dual space curve $\tilde{\gamma}(s)$ in the dual space $\mathbb{D}^3$. Then $\tilde{\mathbf{t}}$, $\tilde{\mathbf{n}}$ and $\tilde{\mathbf{b}}$ are the dual tangent, the dual principal normal and the dual binormal vector fields, respectively. Then for the curve $\tilde{\gamma}$, the Frenet formulae are given by

$$\frac{d}{d\tilde{s}}\begin{bmatrix}\tilde{\mathbf{t}}\\ \tilde{\mathbf{n}}\\ \tilde{\mathbf{b}}\end{bmatrix} = \begin{bmatrix}\kappa\mathbf{n} + \epsilon(\kappa^*\mathbf{n} + \kappa\mathbf{n}^*)\\ -\kappa\mathbf{t} + \tau\mathbf{b} + \epsilon(-\kappa^*\mathbf{t} - \kappa\mathbf{t}^* + \tau^*\mathbf{b} + \tau\mathbf{b}^*)\\ -\tau\mathbf{n} - \epsilon(\tau^*\mathbf{n} + \tau\mathbf{n}^*)\end{bmatrix} = \begin{bmatrix} & \tilde{\kappa} & 0\\ -\tilde{\kappa} & 0 & \tilde{\tau}\\ 0 & -\tilde{\tau} & 0\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{t}}\\ \tilde{\mathbf{n}}\\ \tilde{\mathbf{b}}\end{bmatrix},$$

where $\tilde{\kappa} = \kappa + \epsilon\kappa^*$ is nowhere pure dual curvature and $\tilde{\tau} = \tau + \epsilon\tau^*$ is nowhere pure dual torsion. The above formulae are called *the Frenet formulae of dual curve* in $\mathbb{D}^3$ [7].

3. Dual plane curves

For two real valued functions f and g, it is well-known that if f and g hold $f(t)^2 + g(t)^2 = 1$, then there exists a function θ from $\mathbb{R}$ to $\mathbb{R}$ such that $f(t) = \cos(\theta(t))$ and $g(t) = \sin(\theta(t))$. Similarly, we have the following lemma.

Lemma 3.1. *Let f and g be two differentiable dual functions satisfying $[f(\tilde{t})]^2 + [g(\tilde{t})]^2 = 1$. Then, there exists a function $\tilde{\theta}$ from $\mathbb{D}$ to $\mathbb{D}$ such that $f(\tilde{t}) = \cos(\tilde{\theta}(\tilde{t}))$ and $g(\tilde{t}) = \sin(\tilde{\theta}(\tilde{t}))$.*

Proof. Put $\tilde{t} = t + \epsilon t^*$. Then, the functions f and g are given by

$$f(\tilde{t}) = f(t) + \epsilon t^* f'(t) \quad \text{and} \quad g(\tilde{t}) = g(t) + \epsilon t^* g'(t), \tag{2}$$

where $f'(t)$ and $g'(t)$ are the derivations of f and g. Since $f(\tilde{t})^2 + g(\tilde{t})^2 = f(t)^2 + g(t)^2 + 2\epsilon t^*(f(t)f'(t) + g(t)g'(t)) = 1$, it follows that $f(t)^2 + g(t)^2 = 0$ and $t^*(f(t)f'(t) + g(t)g'(t)) = 0$. From the result of real valued functions, there is a real function θ such that $f(t) = \cos(\theta(t))$ and $g(t) = \sin(\theta(t))$. The function θ can be extended as a dual function $\tilde{\theta}(\tilde{t}) = \theta(t) + \epsilon t^*\theta'(t)$, from which, we have $f(\tilde{t}) = \cos(\tilde{\theta}(\tilde{t}))$ and $g(\tilde{t}) = \sin(\tilde{\theta}(\tilde{t}))$. ∎

Now, we can have the *explicit determination* of a dual plane curve in terms of its curvature.

Theorem 3.2. *A dual plane curve $\tilde{\gamma}(\tilde{s}) = (x(\tilde{s}), y(\tilde{s})) \in \mathbb{D}^2$ with the dual curvature $\tilde{\kappa}(\tilde{s})$ is locally expressed by*

$$(3) \qquad x(\tilde{s}) = \int_0^{\tilde{s}} \cos\Big(\int_0^{\tilde{t}} \tilde{\kappa}(\tilde{p})\, d\tilde{p}\Big) d\tilde{t}, \quad y(\tilde{s}) = \int_0^{\tilde{s}} \sin\Big(\int_0^{\tilde{t}} \tilde{\kappa}(\tilde{p})\, d\tilde{p}\Big) d\tilde{t}.$$

Proof. Let $\tilde{\mathbf{t}}(\tilde{s})$ be the unit dual tangent vector of $\tilde{\gamma}$. Then, from Lemma 2.1, we can set

$$(4) \qquad \tilde{\mathbf{t}}(\tilde{s}) = \Big(\cos(\tilde{\theta}(\tilde{s})), \sin(\tilde{\theta}(\tilde{s}))\Big),$$

for a dual function $\tilde{\theta}(\tilde{s})$. The dual Frenet equation implies that $\tilde{\kappa} = d\tilde{\theta}/d\tilde{s}$ with the principal normal vector field $\tilde{\mathbf{n}}(\tilde{s})$ given by

$$(5) \qquad \tilde{\mathbf{n}}(\tilde{s}) = \Big(-\sin(\tilde{\theta}(\tilde{s})), \cos(\tilde{\theta}(\tilde{s}))\Big).$$

By a suitable choice of a coordinate system, we can assume the initial point $\tilde{\gamma}(0) = (0, 0)$ and the initial tangent vector $\tilde{\mathbf{t}}(0) = (1, 0)$. Hence, together with $\tilde{\theta}(0) = 0$, the curve $\tilde{\gamma}(\tilde{s})$ is expressed by (3). ∎

Example 3.3. (Circle with the dual radius $1/\tilde{k}$) In (3), we put $\tilde{\kappa}$ by a dual number $\tilde{k} = k + \epsilon k^*$. Then, the dual components $x(\tilde{s})$ and $y(\tilde{s})$ are given by

$$(6) \qquad x(\tilde{s}) = \frac{1}{\tilde{k}} \sin(\tilde{k}\tilde{s}) \ \text{ and } \ y(\tilde{s}) = -\frac{1}{\tilde{k}} \cos(\tilde{k}\tilde{s}) + \frac{1}{\tilde{k}}.$$

By putting $\tilde{s} = s + \epsilon f(s)$ in (1), (6) is rewritten as

$$x(s) = \frac{1}{k} \sin(ks) + \epsilon \left((f(s) + \frac{k^*}{k} s) \cos(ks) - \frac{k^*}{k^2} \sin(ks) \right)$$

and

$$y(s) = \frac{1}{k} (1 - \cos(ks)) + \epsilon \left((f(s) + \frac{k^*}{k} s) \sin(ks) - \frac{k^*}{k^2} (1 - \cos(ks)) \right).$$

In other words, a dual circle in $\mathbb{D}^2$ with the dual radius $1/(k + \epsilon k^*)$ is expressed by a product curve of $x(s) \in \mathbb{D}$ and $y(s) \in \mathbb{D}$. Fig. 1 represents a curve with $\tilde{k} = 1 + \epsilon \frac{1}{10}$ and $f(s) = s$.

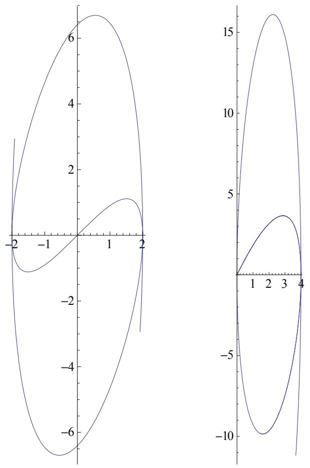

Fig. 1. A dual circle is a product curve of above two curves in $\mathbb{D}$.

Example 3.4. (Dual Cornu spiral in $\mathbb{D}^2$) In Euclidean space, a plane curve whose the curvature κ is a linear function of the arc-length parameter s is called *Cornu spiral.* We call a dual plane curve whose curvature $\tilde{\kappa}$ is a linear function of the dual arc-length parameter $\tilde{s}$, a *dual Cornu spiral.* From Theorem 3.2, the dual Cornu spiral with the curvature $\tilde{\kappa}(\tilde{s}) = 2\tilde{s}$ is

$$(7) \quad x(s) = \int_0^s \cos(t^2)dt + f(s)\cos(s^2), \ y(s) = \int_0^s \sin(t^2)dt + f(s)\sin(s^2),$$

where $f(s) = s^*$ in (1) (see Fig. 2).

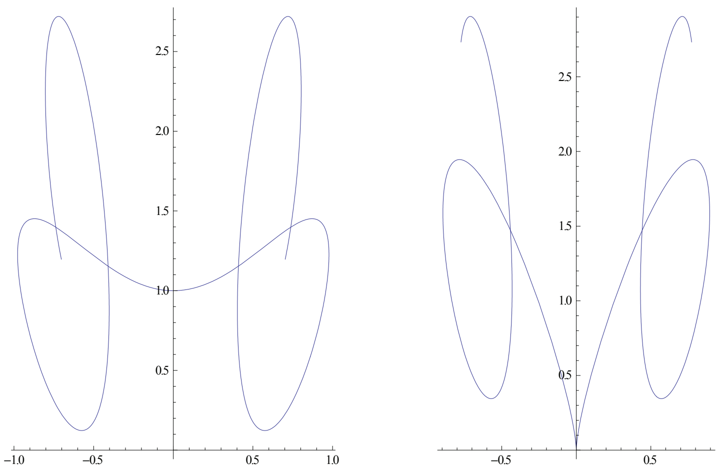

Fig. 2. A dual spiral is a product curve of above two curves in $\mathbb{D}$.

4. Dual helices and dual slant helices

Definition 4.1. Let $\tilde{\gamma}(\tilde{s})$ be a dual Frenet curve in $\mathbb{D}^3$ with non-zero dual curvature $\tilde{\kappa}$. The curve $\tilde{\gamma}(\tilde{s})$ is called a *general dual helix* if its tangent vector field makes a constant dual angle with a dual fixed line $\tilde{\ell}$. We call the direction of $\tilde{\ell}$ by *axis* of a general dual helix $\tilde{\gamma}(\tilde{s})$.

Consider a dual unit vector field $\tilde{V}(\tilde{s})$ along $\tilde{\gamma}(\tilde{s})$ given by

$$\tilde{V}(\tilde{s}) = \tilde{a}(\tilde{s})\tilde{\mathbf{t}}(\tilde{s}) + \tilde{b}(\tilde{s})\tilde{\mathbf{n}}(\tilde{s}) + \tilde{c}(\tilde{s})\tilde{\mathbf{b}}(\tilde{s}), \tag{8}$$

where $\tilde{a}(\tilde{s})$, $\tilde{b}(\tilde{s})$ and $\tilde{c}(\tilde{s})$ are dual functions. By taking the derivative of (8) with respect to $\tilde{s}$, $\tilde{V}(\tilde{s})$ is a constant vector along $\tilde{\gamma}(\tilde{s})$ if and only if

$$\begin{cases} \dfrac{d\tilde{a}}{d\tilde{s}}(\tilde{s}) - \tilde{b}(\tilde{s})\tilde{\kappa}(\tilde{s}) = 0, \\ \dfrac{d\tilde{b}}{d\tilde{s}}(\tilde{s}) + \tilde{a}(\tilde{s})\tilde{\kappa}(\tilde{s}) - \tilde{c}(\tilde{s})\tilde{\tau}(\tilde{s}) = 0, \\ \dfrac{d\tilde{c}}{d\tilde{s}}(\tilde{s}) + \tilde{b}(\tilde{s})\tilde{\tau}(\tilde{s}) = 0. \end{cases} \tag{9}$$

Let $\tilde{\gamma}(\tilde{s})$ be a general dual helix in $\mathbb{D}^3$ with unit axis $\tilde{V}$ and $\tilde{\theta}$ a constant dual angle between the tangent vector field of $\tilde{\gamma}(\tilde{s})$ and the axis $\tilde{V}$. Then, from (8) and (9), $\tilde{V}$ is given by

$$\tilde{V}(\tilde{s}) = \cos(\tilde{\theta})\tilde{\mathbf{t}}(\tilde{s}) + \sin(\tilde{\theta})\tilde{\mathbf{b}}(\tilde{s}) \tag{10}$$

and it satisfies that

$$\frac{\tilde{\kappa}}{\tilde{\tau}} = \tan(\tilde{\theta}). \tag{11}$$

Conversely, if (11) holds for a constant dual number $\tan(\tilde{\theta})$ then the vector (10) is a constant dual vector along a dual Frenet curve $\tilde{\gamma}(\tilde{s})$ satisfying $\langle \tilde{\mathbf{t}}, \tilde{V} \rangle = \cos(\tilde{\theta})$.

Thus, we have the Lancret theorem in dual 3-space $\mathbb{D}^3$ as follows:

Theorem 4.2. (Lancret Theorem in $\mathbb{D}^3$) *A dual Frenet curve in $\mathbb{D}^3$ is a general dual helix if and only if there exists a constant dual number $\tilde{c}$ such that $\tilde{\tau} = \tilde{c}\tilde{\kappa}$.*

Proposition 4.3. *Let $\tilde{\gamma}_0(\tilde{s}) = (\tilde{x}(\tilde{s}), \tilde{y}(\tilde{s}), 0)$ be a dual plane curve in $\mathbb{D}^2 \subset \mathbb{D}^3$ with non-zero dual curvature $\tilde{\kappa}_0(\tilde{s})$ and $\tilde{\mathbf{b}}_0 = (0,0,1) \in \mathbb{D}^3$. If $\tilde{\gamma}(\tilde{s})$ is an integral curve of $-\cos(\tilde{\theta})\tilde{\mathbf{n}}_0(\tilde{s}) + \sin(\tilde{\theta})\tilde{\mathbf{b}}_0$, then $\tilde{\gamma}(\tilde{s})$ is a general dual helix in $\mathbb{D}^3$.*

Proof. By the definition of integral curve, the tangent vector field $\tilde{\mathbf{t}}(\tilde{s})$ of $\tilde{\gamma}(\tilde{s})$ is given by $-\cos(\tilde{\theta})\tilde{\mathbf{n}}_0(\tilde{s}) + \sin(\tilde{\theta})\tilde{\mathbf{b}}_0$. By taking the derivative of $\tilde{\mathbf{t}}$ in terms of $\tilde{s}$, we have the dual curvature $\tilde{\kappa}(\tilde{s}) = \cos(\tilde{\theta})\tilde{\kappa}_0(\tilde{s})$ and the principal normal vector field $\tilde{\mathbf{n}}(\tilde{s}) = \tilde{\mathbf{t}}_0(\tilde{s})$ of $\tilde{\gamma}(\tilde{s})$. Also, the binormal vector field $\tilde{\mathbf{b}}(\tilde{s})$ of $\tilde{\gamma}(\tilde{s})$ is given by

$$\begin{aligned} \tilde{\mathbf{b}}(\tilde{s}) &= \tilde{\mathbf{t}}(\tilde{s}) \times \tilde{\mathbf{n}}(\tilde{s}) \\ &= \Big[-\cos(\tilde{\theta})\tilde{\mathbf{n}}_0(\tilde{s}) + \sin(\tilde{\theta})\tilde{\mathbf{b}}_0\Big] \times \tilde{\mathbf{t}}_0(\tilde{s}) \\ &= \sin(\tilde{\theta})\tilde{\mathbf{n}}_0(\tilde{s}) + \cos(\tilde{\theta})\tilde{\mathbf{b}}_0. \end{aligned} \tag{12}$$

The differentiation of $\tilde{\mathbf{b}}(\tilde{s})$ leads the dual torsion $\tilde{\tau}(\tilde{s}) = \sin(\tilde{\theta})\tilde{\kappa}_0$ of $\tilde{\gamma}(\tilde{s})$, from which we have our assertion. ∎

In the proof of Proposition 4.3, the dual principal normal vector field $\tilde{\mathbf{n}}$ of the curve $\tilde{\gamma}(\tilde{s})$ is calculated by $\tilde{\mathbf{t}}_0$. This means that $\tilde{\gamma}_0(\tilde{s})$ is an integral curve of the principal normal vector field $\tilde{\mathbf{n}}$ along $\tilde{\gamma}(\tilde{s})$. Hence, the converse of Proposition 4.3 can be stated as follows:

Proposition 4.4. *Let $\tilde{\gamma}(\tilde{s})$ be a general dual helix in $\mathbb{D}^3$. An integral curve $\tilde{\gamma}_0(\tilde{s})$ of the principal normal vector field $\tilde{\mathbf{n}}$ along $\tilde{\gamma}(\tilde{s})$ is a dual plane curve in $\mathbb{D}^2$.*

Proof. For a non-zero dual number $\tilde{c}$, assume that $\tilde{\tau} = \tilde{c}\tilde{\kappa}$. Put $V(\tilde{s}) = \tilde{c}\tilde{\mathbf{t}}(\tilde{s}) + \tilde{\mathbf{b}}(\tilde{s})$. Then V is a constant vector along $\tilde{\gamma}$ and is orthogonal to $\tilde{\mathbf{t}}_0$ for all $\tilde{s}$. This means that $\tilde{\gamma}_0$ lies in a dual plane curve orthogonal to V in $\mathbb{D}^3$. ∎

Together with Theorem 3.2, from Propositions 4.3 and 4.4, we have the explicit determanation of a general dual helix in $\mathbb{D}^3$ as follows:

Theorem 4.5. *A general dual helix $\tilde{\gamma} := \tilde{\gamma}(\tilde{s})$ in $\mathbb{D}^3$ with $\tilde{\tau} = \tilde{c}\tilde{\kappa}$ can be expressed by*

$$\frac{1}{A}\left(\int_0^{\tilde{s}} \sin\left(|A|\int_0^{\tilde{\sigma}} \tilde{\kappa}(\tilde{t})d\tilde{t}\right) d\tilde{\sigma}, -\int_0^{\tilde{s}} \cos\left(|A|\int_0^{\tilde{\sigma}} \tilde{\kappa}(\tilde{t})d\tilde{t}\right) d\tilde{\sigma}, \tilde{c}\tilde{s}\right), \tag{13}$$

where $A = \pm\sqrt{1+\tilde{c}^2}$.

Proof. Let $\tilde{\gamma}_0(\tilde{s})$ be an integral curve of the principal normal vector field $\tilde{\mathbf{n}}$ along $\tilde{\gamma}(\tilde{s})$. Then, $\tilde{\gamma}_0(\tilde{s})$ is a dual plane curve with the dual curvature $\tilde{\kappa}_0(\tilde{s}$ given by

$$\tilde{\kappa}_0(\tilde{s}) = \left\|\frac{d}{d\tilde{s}}\tilde{\mathbf{t}}_0\right\| = \left\|\frac{d}{d\tilde{s}}\tilde{\mathbf{n}}\right\| = \|\tilde{\kappa}\tilde{\mathbf{t}} - \tilde{\tau}\tilde{\mathbf{b}}\| = \tilde{\kappa}\|\tilde{\mathbf{t}} - \tilde{c}\tilde{\mathbf{b}}\| = \sqrt{1+\tilde{c}^2}\tilde{\kappa}(\tilde{s}).$$

Thus, from Theorem 3.2 and Proposition 4.3, we have the position vector (13). ∎

Corollary 4.6. *Let $(x(\tilde{s}), y(\tilde{s}), 0)$ be a unit speed dual plane curve and $\tilde{a}$ and $\tilde{b}$ dual numbers satisfying $\tilde{a}^2 + \tilde{b}^2 = 1$. Then $\tilde{\gamma}(\tilde{s}) = (\tilde{a}y(\tilde{s}), -\tilde{a}x(\tilde{s}), \tilde{b}\tilde{s})$ is* *a dual slant helix in* $\mathbb{D}^3$*. Moreover, its curvature* $\tilde{\kappa}$ *and torsion* $\tilde{\tau}$ *are respectively given by* $|\tilde{a}|\sqrt{\left(\frac{d}{d\tilde{s}}x(\tilde{s})\right)^2+\left(\frac{d}{d\tilde{s}}y(\tilde{s})\right)^2}$ *and* $\tilde{b}\sqrt{\left(\frac{d}{d\tilde{s}}x(\tilde{s})\right)^2+\left(\frac{d}{d\tilde{s}}y(\tilde{s})\right)^2}$.

From Example 3.3 and Corollary 4.6, we can give an example of a dual slant helix in $\mathbb{D}^3$.

Example 4.7. (Dual circular helix) Putting $\tilde{\theta}=\theta+\epsilon\theta^*$, $\tilde{a}=\cos(\tilde{\theta})$ and $\tilde{a}=\sin(\tilde{\theta})$, a dual circle $(x(s),y(s))$ in Example 2.3 induces a general dual helix $\tilde{\gamma}$ given by $(\cos(\tilde{\theta})y(s),-\cos(\tilde{\theta})x(s),\sin(\tilde{\theta})(s+\epsilon f(s)))$. The position vector $(X(s),Y(s),Z(s))$ of $\tilde{\gamma}$ is expressed by

$$\begin{aligned}
X(s) &= \frac{1}{k}\cos(\theta)\sin(ks)\\
&\quad+\epsilon\left[-\frac{1}{k}\theta^*\sin(ks)+\cos(\theta)\left((f(s)+\frac{k^*}{k})\cos(ks)-\frac{k^*}{k^2}\sin(ks)\right)\right],\\
Y(s) &= \frac{1}{k}\cos(\theta)\cos(ks)\\
&\quad+\epsilon\left[-\frac{1}{k}\theta^*\cos(ks)+\cos(\theta)\left((f(s)+\frac{k^*}{k})\sin(ks)-\frac{k^*}{k^2}\cos(ks)\right)\right],\\
Z(s) &= s\sin(\theta)+\epsilon\left(\sin(\theta)f(s)+\theta^* s\cos(\theta)\right).
\end{aligned}$$

Moreover, its dual curvature $\tilde{\kappa}$ and dual torsion $\tilde{\tau}$ are given by $|\cos(\tilde{\theta})|\tilde{k}$ and $\sin(\tilde{\theta})\tilde{k}$, respectively. This curve is called a *dual circular helix* with constant dual curvature and constant dual torsion. Also, this curve is a triple product curve of three curves $X(s)$, $Y(s)$ and $Z(s)$ in $\mathbb{D}$. Fig. 3 represents a dual circular helix with $f(s)=\cos(s)$.

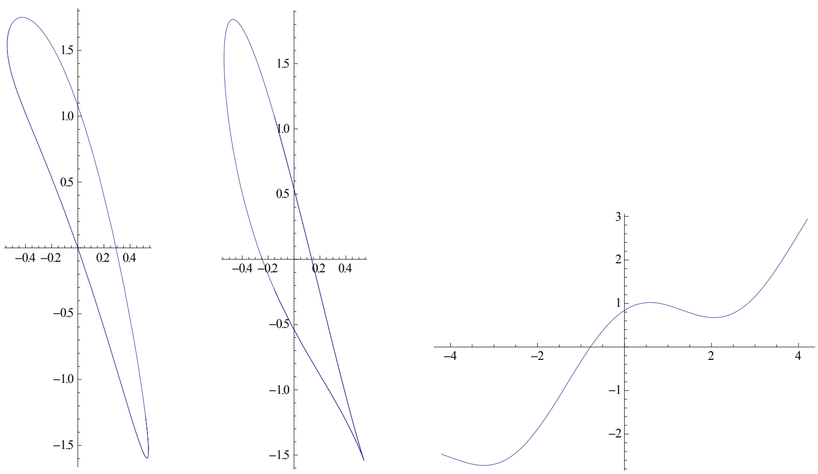

Fig. 3. A dual circular helix is a triple product curve of above three curves in $\mathbb{D}$.

For a given dual curve $\tilde{\gamma}(\tilde{s})$ in $\mathbb{D}^3$ with the dual Frenet frame $\{\tilde{\mathbf{t}}, \tilde{\mathbf{n}}, \tilde{\mathbf{b}}\}$, an integral curve $\tilde{\Gamma}(\tilde{s})$ of

$$-\cos\left(\int_{\tilde{s}_0}^{\tilde{s}} \tilde{\tau} d\tilde{s}\right) \tilde{\mathbf{n}}(\tilde{s}) + \sin\left(\int_{\tilde{s}_0}^{\tilde{s}} \tilde{\tau} d\tilde{s}\right) \tilde{\mathbf{b}}(\tilde{s})$$

is called a *principal-donor curve* of $\tilde{\gamma}(\tilde{s})$. Also, $\tilde{\gamma}(\tilde{s})$ is called a *principal-direction curve* of $\tilde{\Gamma}(\tilde{s})$.

Remark 4.8. We can check easily that $\tilde{\gamma}(\tilde{s})$ is an integral curve of the principal normal vector field along $\tilde{\Gamma}(\tilde{s})$.

Definition 4.9. Let $\tilde{\gamma}(\tilde{s})$ be a dual Frenet curve in $\mathbb{D}^3$ with non-zero dual curvature $\tilde{\kappa}$ and non-zero dual torsion $\tilde{\tau}$. The curve $\tilde{\gamma}(\tilde{s})$ is called a *dual slant helix* if its principal normal vector field makes a constant dual angle with a dual fixed line $\tilde{\ell}$. We call the direction of $\tilde{\ell}$ by *slant axis* of a dual slant helix $\tilde{\gamma}(\tilde{s})$.

A characterization of the dual slant helices by its dual curvature and dual torsion is the same as that of the slant helices in $\mathbb{E}^3$.

Theorem 4.10. [5] *Let $\tilde{\gamma}$ be a unit speed dual space curve with $\tilde{\kappa}(\tilde{s}) \neq 0$. Then $\tilde{\gamma}$ is a slant dual helix if and only if*

$$\frac{\tilde{\kappa}^2}{(\tilde{\kappa}^2 + \tilde{\tau}^2)^{\frac{3}{2}}} \left(\frac{\tilde{\tau}}{\tilde{\kappa}}\right)' = \text{constant}.$$

From Remark 4.8, the following is clear.

Proposition 4.11. *A dual curve $\tilde{\gamma}(\tilde{s})$ in $\mathbb{D}^3$ is a general dual helix if and only if its principal-donor curve $\tilde{\Gamma}(\tilde{s})$ is a dual slant helix.*

From Theorem 4.5 and Proposition 4.11, we can have the explicit determination of a dual slant helix. For this, we need the following lemma.

Lemma 4.12. *If a dual space curve $\tilde{\Gamma}(\tilde{s})$ in $\mathbb{D}^3$ is a principal-donor curve of a dual space curve $\tilde{\gamma}(\tilde{s})$ with the dual curvature $\tilde{\kappa}(\tilde{s})$ and the dual torsion $\tilde{\tau}(\tilde{s})$, then the dual curvature $\tilde{K}(\tilde{s})$ and the dual torsion $\tilde{T}(\tilde{s})$ of the curve $\tilde{\Gamma}$ are given by*

$$(14) \qquad \tilde{K}(\tilde{s}) = \tilde{\kappa}(\tilde{s}) \left|\cos\left(\int \tilde{\tau}(\tilde{s}) d\tilde{s}\right)\right| \quad \textit{and} \quad \tilde{T}(\tilde{s}) = \tilde{\kappa}(\tilde{s}) \sin\left(\int \tilde{\tau}(\tilde{s}) d\tilde{s}\right).$$

Moreover, it satisfies that

$$(15) \qquad \tilde{\kappa}(\tilde{s}) = \sqrt{\tilde{K}^2 + \tilde{T}^2} \quad \textit{and} \quad \int \tilde{\tau}(\tilde{s}) d\tilde{s} = \sin^{-1}\left(\frac{\tilde{T}(\tilde{s})}{\sqrt{\tilde{K}(\tilde{s})^2 + \tilde{T}(\tilde{s})^2}}\right).$$

Proof. From the definition of the principal-donor curve, the tangent vector field $\tilde{\mathbf{T}}$ of $\tilde{\Gamma}$ is given by

$$-\cos\left(\int\tilde{\tau}d\tilde{s}\right)\tilde{\mathbf{n}}(\tilde{s})+\sin\left(\int\tilde{\tau}d\tilde{s}\right)\tilde{\mathbf{b}}(\tilde{s}).$$

By taking the derivative of $\tilde{\mathbf{T}}$ with respect to $\tilde{s}$ and using the Frenet formulae of the dual curves in $\mathbb{D}^3$, it follows that

$$\tilde{K}(\tilde{s})\tilde{\mathbf{N}}(\tilde{s})=\tilde{\kappa}(\tilde{s})\cos\left(\int\tilde{\tau}d\tilde{s}\right)\tilde{\mathbf{t}}(\tilde{s}),$$

from which, $\tilde{K}(\tilde{s})=\tilde{\kappa}(\tilde{s})|\cos\left(\int\tilde{\tau}d\tilde{s}\right)|$ and $\tilde{\mathbf{N}}=\tilde{\mathbf{t}}$. Also, the binormal vector field $\tilde{\mathbf{B}}$ is given by

$$\begin{aligned}\tilde{\mathbf{T}}\times\tilde{\mathbf{N}}&=\left\{-\cos\left(\int\tilde{\tau}d\tilde{s}\right)\tilde{\mathbf{n}}(\tilde{s})+\sin\left(\int\tilde{\tau}d\tilde{s}\right)\tilde{\mathbf{b}}(\tilde{s})\right\}\times\left\{\tilde{\mathbf{t}}(\tilde{s})\right\}\\&=\sin\left(\int\tilde{\tau}d\tilde{s}\right)\tilde{\mathbf{n}}(\tilde{s})+\cos\left(\int\tilde{\tau}d\tilde{s}\right)\tilde{\mathbf{b}}(\tilde{s}).\end{aligned}$$

By taking the derivative of $\tilde{\mathbf{B}}$ with respect to $\tilde{s}$ and using the Frenet formulae of the dual curves in $\mathbb{D}^3$, we have the last assertion of (14). We can easily check (15). Thus, the proof is completed. ∎

Theorem 4.13. *Let $\tilde{\Gamma}(\tilde{s})$ be a dual slant helix with the dual curvature $\tilde{K}$ and the dual torsion $\tilde{T}$. Then $\tilde{\Gamma}(\tilde{s})=(X(\tilde{s}),Y(\tilde{s}),Z(\tilde{s}))$ can be expressed by*

$$\begin{aligned}(16)\qquad X(\tilde{s})=-\int_0^{\tilde{s}}&\frac{\tilde{K}(\tilde{\sigma})}{\sqrt{\tilde{T}(\tilde{\sigma})^2+\tilde{T}(\tilde{\sigma})^2}}\cos\left(|A|\int_0^{\tilde{\sigma}}\sqrt{\tilde{K}(\tilde{t})^2+\tilde{T}(\tilde{t})^2d\tilde{t}}\right)\\&+\frac{\tilde{c}\tilde{T}(\tilde{\sigma})}{A\sqrt{\tilde{K}(\tilde{\sigma})^2+\tilde{T}(\tilde{\sigma})^2}}\sin\left(|A|\int_0^{\tilde{\sigma}}\sqrt{\tilde{K}(\tilde{t})^2+\tilde{T}(\tilde{t})^2d\tilde{t}}\right)d\tilde{\sigma},\end{aligned}$$

$$\begin{aligned}(17)\qquad Y(\tilde{s})=-\int_0^{\tilde{s}}&\frac{\tilde{K}(\tilde{\sigma})}{\sqrt{\tilde{T}(\tilde{\sigma})^2+\tilde{T}(\tilde{\sigma})^2}}\sin\left(|A|\int_0^{\tilde{\sigma}}\sqrt{\tilde{K}(\tilde{t})^2+\tilde{T}(\tilde{t})^2d\tilde{t}}\right)\\&-\frac{\tilde{c}\tilde{T}(\tilde{\sigma})}{A\sqrt{\tilde{K}(\tilde{\sigma})^2+\tilde{T}(\tilde{\sigma})^2}}\cos\left(|A|\int_0^{\tilde{\sigma}}\sqrt{\tilde{K}(\tilde{t})^2+\tilde{T}(\tilde{t})^2d\tilde{t}}\right)d\tilde{\sigma},\end{aligned}$$

and

$$(18)\qquad Z(\tilde{s})=\frac{1}{A}\int_0^{\tilde{s}}\frac{\tilde{T}(\tilde{\sigma})}{\sqrt{\tilde{K}(\tilde{\sigma})^2+\tilde{T}(\tilde{\sigma})^2}}d\tilde{\sigma},$$

for a dual number $\tilde{c}$, where $A=\pm\sqrt{1+\tilde{c}^2}$.

Proof. From Proposition 4.11, we construct a dual slant helix in $\mathbb{D}^3$ using (13) in Theorem 4.5. The dual Frenet frame of a general dual helix (13) is calculated by

$$\begin{cases} \tilde{\mathbf{t}}(\tilde{s}) = \dfrac{1}{A}\left(\sin\left(|A|\displaystyle\int_0^{\tilde{s}} \tilde{\kappa}(\tilde{t})d\tilde{t}\right), -\cos\left(|A|\displaystyle\int_0^{\tilde{s}} \tilde{\kappa}(\tilde{t})d\tilde{t}\right), \tilde{c}\right), \\ \tilde{\mathbf{n}}(\tilde{s}) = \left(\cos\left(|A|\displaystyle\int_0^{\tilde{s}} \tilde{\kappa}(\tilde{t})d\tilde{t}\right), \sin\left(|A|\displaystyle\int_0^{\tilde{s}} \tilde{\kappa}(\tilde{t})d\tilde{t}\right), 0\right), \\ \tilde{\mathbf{b}}(\tilde{s}) = \dfrac{1}{A}\left(-\tilde{c}\sin\left(|A|\displaystyle\int_0^{\tilde{s}} \tilde{\kappa}(\tilde{t})d\tilde{t}\right), \tilde{c}\cos\left(|A|\displaystyle\int_0^{\tilde{s}} \tilde{\kappa}(\tilde{t})d\tilde{t}\right), 1\right). \end{cases}$$

Since a dual slant helix $\tilde{\Gamma}(\tilde{s})$ is a principal-donor curve of a general dual helix, the tangent vector field $\tilde{\mathbf{T}}(\tilde{s})$ of $\tilde{\Gamma}(\tilde{s})$ is

$$-\cos\Big(\int_0^{\tilde{s}} \tilde{\tau} d\tilde{s}\Big)\tilde{\mathbf{n}}(\tilde{s}) + \sin\Big(\int_0^{\tilde{s}} \tilde{\tau} d\tilde{s}\Big)\tilde{\mathbf{b}}(\tilde{s}).$$

Thus, $\tilde{\Gamma}(\tilde{s})$ can be expressed by

$$\tilde{\Gamma}(\tilde{s}) = \int_0^{\tilde{s}} \tilde{\mathbf{T}}(\tilde{t})d\tilde{t} = (X(\tilde{s}), Y(\tilde{s}), Z(\tilde{s})),$$

where

$$X(\tilde{s}) = -\int_0^{\tilde{s}} \cos\left(\int_0^{\tilde{\sigma}} \tilde{\tau}(\tilde{t})d\tilde{t}\right)\cos\left(|A|\int_0^{\tilde{\sigma}} \tilde{\kappa}(\tilde{t})d\tilde{t}\right) + \frac{\tilde{c}}{A}\sin\left(\int_0^{\tilde{\sigma}} \tilde{\tau}(\tilde{t})d\tilde{t}\right)\sin\left(|A|\int_0^{\tilde{\sigma}} \tilde{\kappa}(\tilde{t})d\tilde{t}\right)d\tilde{\sigma}, \tag{19}$$

$$Y(\tilde{s}) = -\int_0^{\tilde{s}} \cos\left(\int_0^{\tilde{\sigma}} \tilde{\tau}(\tilde{t})d\tilde{t}\right)\sin\left(|A|\int_0^{\tilde{\sigma}} \tilde{\kappa}(\tilde{t})d\tilde{t}\right) - \frac{\tilde{c}}{A}\sin\left(\int_0^{\tilde{\sigma}} \tilde{\tau}(\tilde{t})d\tilde{t}\right)\cos\left(|A|\int_0^{\tilde{\sigma}} \tilde{\kappa}(\tilde{t})d\tilde{t}\right)d\tilde{\sigma}, \tag{20}$$

$$Z(\tilde{s}) = \frac{1}{A}\int_0^{\tilde{s}} \sin\left(\int_0^{\tilde{\sigma}} \tilde{\tau}(\tilde{t})d\tilde{t}\right)d\tilde{\sigma}. \tag{21}$$

By applying (15) into $X(\tilde{s})$, $Y(\tilde{s})$ and $Z(\tilde{s})$, we get our assertion. ∎

Lastly, we give an example dual slant helix $\tilde{\Gamma}$ in $\mathbb{D}^3$.

EXAMPLE 4.14. (Dual circular slant helix) In Example 4.7, we showed a circular dual helix in $\mathbb{D}^3$ which is a principal-donor curve of a dual circle in $\mathbb{D}^2$. When a dual slant helix $\tilde{\Gamma}$ in $\mathbb{D}^3$ is a principal-donor curve of a dual circular helix $\tilde{\gamma}$, we call the curve $\tilde{\Gamma}$ a *dual circular slant helix*. Since the dual circular helix $\tilde{\gamma}$ has the constant dual curvature $\tilde{k}$ and the constant dual torsion $\tilde{\tau} = \tilde{c}\tilde{k}$ for a dual number $\tilde{c}$, from (19), (20) and (21), we can give the position vector of $\tilde{\Gamma}$ as follows:

$$
\begin{aligned}
&X(\tilde{s})\\
&= -\int_0^{\tilde{s}} \cos(\tilde{c}\tilde{k}\tilde{\sigma}) \cos\left(\sqrt{1+\tilde{c}^2}\tilde{k}\tilde{\sigma}\right) + \frac{\tilde{c}}{\sqrt{1+\tilde{c}^2}} \sin(\tilde{c}\tilde{k}\tilde{\sigma}) \sin\left(\sqrt{1+\tilde{c}^2}\tilde{k}\tilde{\sigma}\right) d\tilde{\sigma}\\
&= \frac{2\tilde{c}}{\tilde{k}} \sin(\tilde{c}\tilde{k}\tilde{\sigma}) \cos\left(\sqrt{1+\tilde{c}^2}\tilde{k}\tilde{\sigma}\right) - \frac{1+2\tilde{c}^2}{\sqrt{1+\tilde{c}^2}\tilde{k}} \cos(\tilde{c}\tilde{k}\tilde{\sigma}) \sin\left(\sqrt{1+\tilde{c}^2}\tilde{k}\tilde{\sigma}\right),\\
&Y(\tilde{s})\\
&= -\int_0^{\tilde{s}} \cos(\tilde{c}\tilde{k}\tilde{\sigma}) \sin\left(\sqrt{1+\tilde{c}^2}\tilde{k}\tilde{\sigma}\right) - \frac{\tilde{c}}{\sqrt{1+\tilde{c}^2}} \sin(\tilde{c}\tilde{k}\tilde{\sigma}) \cos\left(\sqrt{1+\tilde{c}^2}\tilde{k}\tilde{\sigma}\right) d\tilde{\sigma}\\
&= \frac{2\tilde{c}}{\tilde{k}} \sin(\tilde{c}\tilde{k}\tilde{\sigma}) \sin\left(\sqrt{1+\tilde{c}^2}\tilde{k}\tilde{\sigma}\right)\\
&\quad + \frac{1+2\tilde{c}^2}{\sqrt{1+\tilde{c}^2}\tilde{k}} \left\{\cos(\tilde{c}\tilde{k}\tilde{\sigma}) \cos\left(\sqrt{1+\tilde{c}^2}\tilde{k}\tilde{\sigma}\right) + 1\right\},
\end{aligned}
$$

and

$$
Z(\tilde{s}) = \frac{1}{\sqrt{1+\tilde{c}^2}} \int_0^{\tilde{s}} \sin(\tilde{c}\tilde{k}\tilde{\sigma}) d\tilde{\sigma} = -\frac{1}{\tilde{c}\sqrt{1+\tilde{c}^2}\tilde{k}} \left\{\cos(\tilde{c}\tilde{k}\tilde{s}+1)\right\}.
$$

If $\tilde{c}=1$ and $\tilde{k}=1+\epsilon$, the dual circular slant $\tilde{\Gamma}$ is given by

$$
\begin{aligned}
X(s+\epsilon f(s)) &= 2\sin(s)\cos(\sqrt{2}s) - \frac{3}{\sqrt{2}}\cos(s)\sin(\sqrt{2}s)\\
&\quad + \epsilon\left\{(2-\frac{3}{\sqrt{2}})(s+f(s))\left(\cos(s)\cos(\sqrt{2}s) - \sqrt{2}\sin(s)\sin(\sqrt{2}s)\right)\right.\\
&\qquad \left. -2\cos(\sqrt{2}s)\sin(s) + \frac{3}{\sqrt{2}}\cos(s)\sin(\sqrt{2}s)\right\},
\end{aligned}
$$

$$
\begin{aligned}
Y(s+\epsilon f(s)) &= 2\sin(s)\sin(\sqrt{2}s) + \frac{3}{\sqrt{2}}\left(\cos(s)\cos(\sqrt{2}s) - 1\right)\\
&\quad + \epsilon\left\{(s+f(s))\left(\sin(\sqrt{2}s)\cos(s) - \frac{\sqrt{2}}{2}\sin(s)\cos(\sqrt{2}s)\right)\right.\\
&\qquad \left. -2\sin(s)\sin(\sqrt{2}s) - \frac{3}{\sqrt{2}}\left(\cos(s)\cos(\sqrt{2}s) - 1\right)\right\}
\end{aligned}
$$

and

$$
Z(s+\epsilon f(s)) = \frac{1}{\sqrt{2}}(1-\cos(s)) + \frac{\epsilon}{2}\left\{\cos(s) - 1 + (s+f(s))\sin(s)\right\}.
$$

If we put $f(s)=s$, the curve $\tilde{\Gamma}$ is represented in Fig. 4.

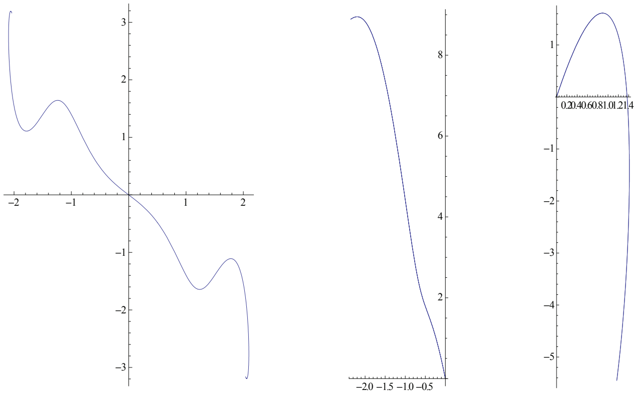

Fig. 4. A dual circular slant helix is a triple product curve of above three curves in $\mathbb{D}$.

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