DE GRUYTER
OPEN

DEMONSTRATIO MATHEMATICA
Vol. XLVII No 1 2014

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FIXED POINTS OF F-WEAK CONTRACTIONS ON COMPLETE METRIC SPACES

Abstract. In this paper, we introduce the notion of an F-weak contraction and prove a fixed point theorem for F-weak contractions. Examples are given to show that our result is a proper extension of some results known in the literature.

1. Introduction and preliminaries

Recently, many results of the fixed point problems for maps on metric spaces have been proved [1], [2], [3], [6], [7], [9]. In [11], Wardowski has introduced the concept of an F-contraction as follows.

Definition 1.1. ([11], Definition 2.1) Let $\mathcal{F}$ be the family of all functions $F : (0, +\infty) \longrightarrow \mathbb{R}$ such that

(F1) F is strictly increasing, that is, for all $\alpha, \beta \in (0, +\infty)$ if $\alpha < \beta$ then $F(\alpha) < F(\beta)$;

(F2) For each sequence $\{\alpha_n\}$ of positive numbers, the following holds:

$$\lim_{n\to\infty} \alpha_n = 0 \text{ if and only if } \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \big(\alpha^k F(\alpha)\big) = 0$.

Let (X, d) be a metric space. A map $T : X \longrightarrow X$ is said to be an F-*contraction* on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \tau + F\big(d(Tx, Ty)\big) \le F\big(d(x, y)\big). \tag{1.1}$$

Example 1.2. [11] The following functions $F : (0, +\infty) \to \mathbb{R}$ are the elements of $\mathcal{F}$:

2010 *Mathematics Subject Classification*: Primary 47H10, 54H25; Secondary 54D99, 54E99.

Key words and phrases: F-contraction, F-weak contraction, fixed point theorem, complete metric space.

This work is partly discussed at The Dong Thap Seminar on Mathematical Analysis.

DOI: 10.2478/dema-2014-0012

(1) $F\alpha = \ln \alpha$,
(2) $F\alpha = \ln \alpha + \alpha$,
(3) $F\alpha = -\dfrac{1}{\sqrt{\alpha}}$,
(4) $F\alpha = \ln(\alpha^2 + \alpha)$.

By using the notion of F-contraction, the author has proved a fixed point theorem which generalizes Banach contraction principle in a different way than in the known results from the literature.

Theorem 1.3. ([11], Theorem 2.1) *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be an F-contraction. Then we have*

(1) *T has a unique fixed point x^*.*
(2) *For all $x \in X$, the sequence $\{T^n x\}$ is convergent to x^*.*

Remark 1.4. ([11], Remark 2.1) Let T be an F-contraction. Then $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ such that $Tx \neq Ty$. Also, T is a continuous map.

In this paper, we introduce the notion of an F-weak contraction and prove a fixed point theorem for F-weak contractions, which generalizes some results known from the literature. Examples are given to show that our result is a proper extension of [11, Theorem 2.1].

2. Main results

First we generalize the notion of an F-contraction into an F-weak contraction as follows.

Definition 2.1. Let (X, d) be a metric space. A map $T : X \longrightarrow X$ is said to be an *F-weak contraction* on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ satisfying $d(Tx, Ty) > 0$, the following holds:

$$\tau + F\big(d(Tx, Ty)\big) \leq F\left(\max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}\right). \tag{2.1}$$

Remark 2.2.

(1) Every F-contraction is an F-weak contraction.
(2) Let T be an F-weak contraction. From (2.1) we have, for all $x, y \in X$, $Tx \neq Ty$

$$F\big(d(Tx, Ty)\big) < \tau + F\big(d(Tx, Ty)\big) \leq F\left(\max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}\right).$$

Then, by (F1), we get

$$d(Tx,Ty) < \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\},$$

for all $x, y \in X$, $Tx \neq Ty$.

The following example shows that the inverse implication of Remark 2.2(1) does not hold.

Example 2.3. Let $T : [0,1] \longrightarrow [0,1]$ be given by

$$Tx = \begin{cases} \frac{1}{2}, & \text{if } x \in [0,1), \\ \frac{1}{4}, & \text{if } x = 1. \end{cases}$$

Since T is not continuous, T is not an F-contraction by Remark 1.4. For $x \in [0,1)$ and $y = 1$, we have

$$d(Tx, T1) = d\left(\frac{1}{2}, \frac{1}{4}\right) = \left|\frac{1}{2} - \frac{1}{4}\right| = \frac{1}{4} > 0$$

and

$$\max\left\{d(x,1), d(x,Tx), d(1,T1), \frac{d(x,T1)+d(1,Tx)}{2}\right\} \geq d(1,T1) = \frac{3}{4}.$$

Therefore, by choosing $F\alpha = \ln\alpha$, $\alpha \in (0,+\infty)$ and $\tau = \ln 3$, we see that T is an F-weak contraction.

Now we state the main result of the paper.

Theorem 2.4. *Let (X,d) be a complete metric space and $T : X \longrightarrow X$ be an F-weak contraction. If T or F is continuous, then we have*

(1) *T has a unique fixed point $x^* \in X$.*
(2) *For all $x \in X$, the sequence $\{T^n x\}$ is convergent to x^*.*

Proof. (1). Let $x \in X$ be arbitrary and fixed. We define $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$, where $x_0 = x$. If there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0+1} = x_{n_0}$, then $Tx_{n_0} = x_{n_0}$. This proves that x_{n_0} is a fixed point of T.

Now we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Then $d(x_{n+1}, x_n) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. It follows from (2.1) that for each $n \in \mathbb{N}$:

$$\begin{aligned}
(2.2)\qquad F\big(d(x_{n+1},x_n)\big) &= F\big(d(Tx_n,Tx_{n-1})\big)\\
&\le F\bigg(\max\Big\{d(x_n,x_{n-1}),d(x_n,Tx_n),d(x_{n-1},Tx_{n-1}),\\
&\qquad\qquad \frac{d(x_n,Tx_{n-1})+d(x_{n-1},Tx_n)}{2}\Big\}\bigg)-\tau\\
&= F\bigg(\max\Big\{d(x_n,x_{n-1}),d(x_n,x_{n+1}),d(x_{n-1},x_n),\frac{d(x_{n-1},x_{n+1})}{2}\Big\}\bigg)-\tau\\
&= F\bigg(\max\Big\{d(x_n,x_{n-1}),d(x_n,x_{n+1}),\frac{d(x_{n-1},x_{n+1})}{2}\Big\}\bigg)-\tau\\
&\le F\bigg(\max\Big\{d(x_n,x_{n-1}),d(x_n,x_{n+1}),\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\Big\}\bigg)-\tau\\
&= F\big(\max\big\{d(x_n,x_{n-1}),d(x_n,x_{n+1})\big\}\big)-\tau.
\end{aligned}$$

If there exists $n\in\mathbb{N}$ such that $\max\big\{d(x_n,x_{n-1}),d(x_n,x_{n+1})\big\}=d(x_{n+1},x_n)$ then (2.2) becomes

$$F\big(d(x_{n+1},x_n)\big)\le F\big(d(x_{n+1},x_n)\big)-\tau<F\big(d(x_{n+1},x_n)\big).$$

It is a contradiction. Therefore,

$$\max\big\{d(x_n,x_{n-1}),d(x_n,x_{n+1})\big\}=d(x_n,x_{n-1}),$$

for all $n\in\mathbb{N}$. Thus, from (2.2), we have

$$F\big(d(x_{n+1},x_n)\big)\le F\big(d(x_n,x_{n-1})\big)-\tau,$$

for all $n\in\mathbb{N}$. It implies that

$$F\big(d(x_{n+1},x_n)\big)\le F\big(d(x_1,x_0)\big)-n\tau, \tag{2.3}$$

for all $n\in\mathbb{N}$. Taking the limit as $n\to\infty$ in (2.3), we get

$$\lim_{n\to\infty}F(d(x_{n+1},x_n))=-\infty$$

that together with (F2) gives

$$\lim_{n\to\infty}d(x_{n+1},x_n)=0. \tag{2.4}$$

From (F3), there exists $k\in(0,1)$ such that

$$\lim_{n\to\infty}\Big(\big(d(x_{n+1},x_n)\big)^kF\big(d(x_{n+1},x_n)\big)\Big)=0. \tag{2.5}$$

It follows from (2.3) that

$$\begin{aligned}
(2.6)\qquad \big(d(x_{n+1},x_n)\big)^k\Big(F\big(d(x_{n+1},x_n)\big)-F\big(d(x_1,x_0)\big)\Big)\\
\le -\big(d(x_{n+1},x_n)\big)^kn\tau\le 0,
\end{aligned}$$

for all $n \in \mathbb{N}$. By using (2.4), (2.5) and taking the limit as $n \to \infty$ in (2.6), we get

$$\lim_{n\to\infty}\Big(n\big(d(x_{n+1},x_n)\big)^k\Big)=0. \tag{2.7}$$

Then there exists $n_1 \in \mathbb{N}$ such that $n\big(d(x_{n+1},x_n)\big)^k \le 1$ for all $n \ge n_1$, that is,

$$d(x_{n+1},x_n) \le \frac{1}{n^{\frac{1}{k}}}, \tag{2.8}$$

for all $n \ge n_1$. For all $m > n \ge n_1$, by using (2.8) and the triangle inequality, we get

$$\begin{aligned} d(x_m,x_n) &\le d(x_m,x_{m-1})+\ldots+d(x_{n+1},x_n)\\ &< \sum_{i=n}^{\infty} d(x_{i+1},x_i) \le \sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}}. \end{aligned} \tag{2.9}$$

Since the series $\sum_{n=1}^{\infty} 1/n^{\frac{1}{k}}$ is convergent, taking the limit as $n \to \infty$ in (2.9), we get $\lim_{n,m\to\infty} d(x_m,x_n)=0$. This proves that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. We shall prove that x^* is a fixed point of T by two following cases.

Case 1. *T is continuous.* We have

$$d(x^*,Tx^*)=\lim_{n\to\infty} d(x_n,Tx_n)=\lim_{n\to\infty} d(x_n,x_{n+1})=0.$$

This proves that x^* is a fixed point of T.

Case 2. *F is continuous.* In this case, we consider two following subcases.

Subcase 2.1. For each $n \in \mathbb{N}$, there exists $i_n \in \mathbb{N}$ such that $x_{i_n+1} = Tx^*$ and $i_n > i_{n-1}$ where $i_0 = 1$. Then we have

$$x^*=\lim_{n\to\infty} x_{i_n+1}=\lim_{n\to\infty} Tx^*=Tx^*.$$

This proves that x^* is a fixed point of T.

Subcase 2.2. There exists $n_0 \in \mathbb{N}$ such that $x_{n+1} \ne Tx^*$ for all $n \ge n_0$. That is $d(Tx_n,Tx^*) > 0$ for all $n \ge n_0$. It follows from (2.1), (F1) and the triangle inequality that

$$\begin{aligned} &\tau + F\big(d(x_{n+1},Tx^*)\big) \qquad (2.10)\\ &= \tau + F\big(d(Tx_n,Tx^*)\big)\\ &\le F\bigg(\max\Big\{d(x_n,x^*),d(x_n,Tx_n),d(x^*,Tx^*),\frac{d(x_n,Tx^*)+d(x^*,Tx_n)}{2}\Big\}\bigg) \end{aligned}$$

$$= F\Big(\max\Big\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), \frac{d(x_n, Tx^*) + d(x^*, x_{n+1})}{2}\Big\}\Big)$$
$$\le F\Big(\max\Big\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), \frac{d(x_n, x^*) + d(x^*, Tx^*) + d(x^*, x_{n+1})}{2}\Big\}\Big).$$

If $d(x^*, Tx^*) > 0$ then by the fact

$$\lim_{n\to\infty} d(x_n, x^*) = \lim_{n\to\infty} d(x^*, x_{n+1}) = 0,$$

there exists $n_1 \in \mathbb{N}$ such that for all $n \ge n_1$, we have

$$\max\Big\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), \frac{d(x_n, x^*) + d(x^*, Tx^*) + d(x^*, x_{n+1})}{2}\Big\} = d(x^*, Tx^*).$$

By (2.10), we get

$$\tau + F\big(d(x_{n+1}, Tx^*)\big) \le F\big(d(x^*, Tx^*)\big), \tag{2.11}$$

for all $n \ge \max\{n_0, n_1\}$. Since F is continuous, taking the limit as $n \to \infty$ in (2.11), we obtain

$$\tau + F\big(d(x^*, Tx^*)\big) \le F\big(d(x^*, Tx^*)\big).$$

It is a contradiction. Therefore, $d(x^*, Tx^*) = 0$, that is, x^* is a fixed point of T.

By two above cases, T has a fixed point x^*. Now, we prove that the fixed point of T is unique. Let x_1^*, x_2^* be two fixed points of T. Suppose to the contrary that $x_1^* \ne x_2^*$. Then $Tx_1^* \ne Tx_2^*$. It follows from (2.1) that

$$\tau + F\big(d(x_1^*, x_2^*)\big) = \tau + F\big(d(Tx_1^*, Tx_2^*)\big)$$
$$\le F\Big(\max\Big\{d(x_1^*, x_2^*), d(x_1^*, Tx_1^*), d(x_2^*, Tx_2^*), \frac{d(x_1^*, Tx_2^*) + d(x_2^*, Tx_1^*)}{2}\Big\}\Big)$$
$$= F\Big(\max\Big\{d(x_1^*, x_2^*), d(x_1^*, x_1^*), d(x_2^*, x_2^*), \frac{d(x_1^*, x_2^*) + d(x_2^*, x_1^*)}{2}\Big\}\Big)$$
$$= F\big(d(x_1^*, x_2^*)\big).$$

It is a contradiction. Then $d(x_1^*, x_2^*) = 0$, that is, $x_1^* = x_2^*$. This proves that the fixed point of T is unique.

(2). It follows from the proof of (1) that $\lim_{n\to\infty} T^n x = \lim_{n\to\infty} x_{n+1} = x^*$. ■

From Theorem 2.4, we get the following corollaries.

Corollary 2.5. *Let (X,d) be a complete metric space and $T : X \longrightarrow X$ satisfies*

$$(2.12)\qquad d(Tx,Ty) > 0 \Rightarrow \tau + F\big(d(Tx,Ty)\big) \leq F\big(ad(x,y) + bd(x,Tx) + cd(y,Ty) + e\big[d(x,Ty) + d(y,Tx)\big]\big),$$

for all $x, y \in X$ where $a, b, c \geq 0$ and $a + b + c + 2e < 1$. If T or F is continuous then

(1) *T has a unique fixed point $x^* \in X$.*
(2) *For all $x \in X$, the sequence $\{T^n x\}$ is convergent to x^*.*

Proof. For all $x, y \in X$, we have

$$\begin{aligned} &ad(x,y) + bd(x,Tx) + cd(y,Ty) + e\big[d(x,Ty) + d(y,Tx)\big] \\ &\quad \leq (a+b+c+2e)\max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\} \\ &\quad \leq \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}. \end{aligned}$$

Then, by (F1) we see that (2.1) is a consequence of (2.12). Then the corollary is proved. ∎

Remark 2.6. Since (2.12) is a consequence of (1.1) and T is continuous by Remark 1.4, we get Theorem 1.3 from Corollary 2.5.

The following example shows that Theorem 2.4 is a proper extension of Theorem 1.3.

Example 2.7. Let T be given as in Example 2.3. Since T is not an F-contraction for any F, Corollary 1.3 is not applicable to T. On the other hand, let F be given as in Example 2.3. Then T is an F-weak contraction. Therefore, Theorem 2.4 can be applicable to T and the unique fixed point of T is $\frac{1}{2}$.

Remark 2.8. When we consider the different types of F-weak contractions then we obtain the variety of known contractions in the literature. For example, see the following

(1) For all $x, y \in X$ and $a, b, c \geq 0, a + b + c < 1$, we have that

$$d(Tx,Ty) \leq ad(x,y) + bd(x,Tx) + cd(y,Ty)$$

implies

$$d(Tx,Ty) \leq (a+b+c)\max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Then if $d(Tx,Ty)>0$, we get

$$\tau+\ln(d(Tx,Ty)) \le \ln\left(\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\right\}\right),$$

where $\tau=\ln 1/(a+b+c)>0$. Then the contraction condition in [10] becomes the condition (2.1) with $F\alpha=\ln\alpha$ for all $\alpha>0$. This proves that Theorem 2.4 is a generalization of the main result of [10].

(2) For all $x,y\in X$ and $k\in[0,1)$, we have that

$$d(Tx,Ty)\le k\max\{d(x,Tx),d(y,Ty)\}$$

implies

$$d(Tx,Ty)\le k\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Then if $d(Tx,Ty)>0$, we get

$$\tau+\ln\big(d(Tx,Ty)\big) \le \ln\left(\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\right\}\right),$$

where $\tau=\ln 1/k>0$. Then the contraction condition in [4] becomes the condition (2.1) with $F\alpha=\ln\alpha$, for all $\alpha>0$. This proves that Theorem 2.4 is a generalization of the main result of [4].

(3) For all $x,y\in X$ and non-negative numbers $q(x,y),r(x,y),s(x,y)$ and $t(x,y)$ with

$$\sup_{x,y\in X}\{q(x,y)+r(x,y)+s(x,y)+2t(x,y)\}=\lambda<1$$

and $d(Tx,Ty)>0$, we have that

$$d(Tx,Ty)\le q(x,y)d(x,y)+r(x,y)d(x,Tx)+s(x,y)d(y,Ty) + t(x,y)\big[d(x,Ty)+d(y,Tx)\big]$$

implies

$$d(Tx,Ty)\le \lambda\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Then if $d(Tx,Ty)>0$, we get

$$\ln\frac{1}{\lambda}+\ln d(Tx,Ty) \le \ln\left(\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\right\}\right),$$

where $\tau = \ln 1/\lambda > 0$. Then the contraction condition (C) in [5, page 20] becomes the condition (2.1) with $F\alpha = \ln\alpha$ for all $\alpha > 0$. This proves that Theorem 2.4 is a generalization of [5, Theorem 2.5.(i)].

(4) For all $x, y \in X$ and non-negative numbers a, b, c, e, f with $a + b + c + e + f < 1$, we have that

$$d(Tx, Ty) \leq \frac{a+b}{2}\big[d(x, Tx) + d(y, Ty)\big] + \frac{c+e}{2}\big[d(x, Ty) + d(y, Tx)\big] + f d(x, y)$$

implies

$$d(Tx, Ty) \leq (a + b + c + e + f) \times \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

Then if $d(Tx, Ty) > 0$, we get

$$\ln \frac{1}{a+b+c+e+f} + \ln d(Tx, Ty) < \ln\left(\max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}\right),$$

where $\tau = \ln 1/(a + b + c + e + f) > 0$. Then the contraction condition (2) in [8, page 202] becomes the condition (2.1) with $F\alpha = \ln\alpha$ for all $\alpha > 0$. This proves that Theorem 2.4 is a generalization of [8, Theorem 1.(1).(a)].

References

[1] M. A. Alghamdi, A. Petrusel, N. Shahzad, *A fixed point theorem for cyclic generalized contractions in metric spaces*, Fixed Point Theory Appl. 122 (2012), 10 pages.

[2] T. V. An, K. P. Chi, E. Karapinar, T. D. Thanh, *An extension of generalized (ψ, φ)-weak contractions*, Int. J. Math. Math. Sci. 2012 (2012), 11 pages.

[3] V. Berinde, F. Vetro, *Common fixed points of mappings satisfying implicit contractive conditions*, Fixed Point Theory Appl. 105 (2012), 16 pages.

[4] R. M. T. Bianchini, *Su un problema di S. Reich aguardante la teoria dei punti fissi*, Boll. Un. Mat. Ital. 5 (1972), 103–108.

[5] L. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. 45 (1974), 267–273.

[6] H.-S. Ding, L. Li, S. Radenovic, *Coupled coincidence point theorems for generalized nonlinear contraction in partially ordered metric spaces*, Fixed Point Theory Appl. 96 (2012), 17 pages.

[7] W. S. Du, S. X. Zheng, *Nonlinear conditions for coincidence point and fixed point theorems*, Taiwanese J. Math. 16(3) (2012), 857–868.

[8] G. E. Hardy, T. D. Rogers, *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull. 16(2) (1973), 201–206.
[9] A. Latif, W. A. Albar, *Fixed point results in complete metric spaces*, Demonstratio Math. 41 (2008), 145–150.
[10] S. Reich, *Some remarks concerning contraction mappings*, Canad. Math. Bull. 14(1) (1971), 121–124.
[11] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl. 94 (2012), 11 pages.

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Received September 5, 2012; revised version September 10, 2012.