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ON JORDAN TRIPLE α-*CENTRALIZERS OF SEMIPRIME RINGS

Abstract. Let R be a 2-torsion free semiprime ring equipped with an involution $*$. An additive mapping $T: R \longrightarrow R$ is called a left (resp. right) Jordan α-*centralizer associated with a function $\alpha: R \longrightarrow R$ if $T(x^2) = T(x)\alpha(x^*)$ (resp. $T(x^2) = \alpha(x^*)T(x)$) holds for all $x \in R$. If T is both left and right Jordan α-*centralizer of R, then it is called Jordan α-*centralizer of R. In the present paper it is shown that if α is an automorphism of R, and $T: R \longrightarrow R$ is an additive mapping such that $2T(xyx) = T(x)\alpha(y^*x^*) + \alpha(x^*y^*)T(x)$ holds for all $x, y \in R$, then T is a Jordan α-*centralizer of R.

1. Introduction

Throughout the present paper, unless otherwise mentioned, R will denote an associative ring having at least two elements with center $Z(R)$. However, R may not have unity. For any $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$. Recall that R is *prime* if $aRb = \{0\}$ implies that $a = 0$ or $b = 0$. A ring R is called *semi prime* if $aRa = \{0\}$ with $a \in R$ implies $a = 0$. An additive mapping $x \mapsto x^*$ on a ring R, satisfying $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$, is called an involution on R. A ring equipped with an involution is called a $*$-ring or ring with involution. Suppose that R is a $*$-ring. An additive mapping $T: R \longrightarrow R$ is called a left (resp. right) α-*centralizer associated with a function $\alpha: R \longrightarrow R$ if $T(xy) = T(x)\alpha(y^*)$ (resp. $T(xy) = \alpha(x^*)T(y)$) holds for all $x, y \in R$ and T is called a left (resp. right) Jordan α-*centralizer if $T(x^2) = T(x)\alpha(x^*)$ (resp. $T(x^2) = \alpha(x^*)T(x)$) holds for all $x \in R$. If T is both left and right Jordan α-*centralizer of R then it is called Jordan α-*centralizer of R. Note that

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for $\alpha = I_R$, the identity map on R, T is said to be a Jordan $*$-centralizer of R.

Let R be a $*$-ring and T a left Jordan α-*centralizer. Then $T(x^2) = T(x)\alpha(x^*)$ holds for all $x \in R$. Now linearizing on x, we find that $T(xy + yx) = T(x)\alpha(y^*) + T(y)\alpha(x^*)$ and replacing x by $xy + yx$, we arrive at $2T(xyx) = 2T(x)\alpha(y^*x^*)$. Further if R is 2-torsion free then $T(xyx) = T(x)\alpha(y^*x^*)$ holds for all $x, y \in R$. Motivated by this observation, the notion of left (resp. right) Jordan triple α-*centralizer in a $*$-ring was introduced as follows: An additive mapping $T : R \longrightarrow R$ is called a left (resp. right) Jordan triple α-*centralizer associated with a function $\alpha : R \longrightarrow R$ if $T(xyx) = T(x)\alpha(y^*x^*)$ (resp. $T(xyx) = \alpha(x^*y^*)T(x)$) holds for all $x, y \in R$. If T is both left and right Jordan triple α-*centralizer, then T is called a Jordan triple α-*centralizer of R. Hence, the above calculation shows that every left Jordan α-*centralizer of a 2-torsion free $*$-ring is a left Jordan triple α-*centralizer of R but not conversely. For example, let R be the ring of all 3×3 strictly upper triangular matrices over a ring S. We define $\alpha, * : R \to R$ as follows: $\alpha\left(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & a & -b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}$ and $\left(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}\right)^* = \begin{pmatrix} 0 & c & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$. Let $T : R \longrightarrow R$ be defined by $T\left(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

It is obvious to see that T is a left Jordan triple α-*centralizer of R which is not a left Jordan α-*centralizer.

Recently, the main theorem of Vukman & Kosi-Ulbl [8, Theorem 1] was generalized by Daif et al. [4, Theorem 2.1] as follows: an additive mapping T on a 2-torsion free semiprime ring R, with associated endomorphism α on R satisfying $2T(xyx) = T(x)\alpha(yx) + \alpha(xy)T(x)$ for all $x, y \in R$, is a Jordan α-centralizer of R. In this paper, we improve this result in the setting of semiprime *-ring. In fact, we prove that an additive mapping T on a 2-torsion free semiprime *-ring R, satisfying $2T(xyx) = T(x)\alpha(y^*x^*) + \alpha(x^*y^*)T(x)$ for all $x, y \in R$ and automorphism α, is a Jordan α-*centralizer of R.

2. Main result

We begin with the following lemmas:

Lemma 2.1. [7, Lemma 1] *Let R be a 2-torsion free semiprime ring. Suppose that the identity $axb + bxc = 0$ holds for all $x \in R$ and for some $a, b, c \in R$. In this case $(a + c)xb = 0$ for all $x \in R$.*

Lemma 2.2. *Let R be a 2-torsion free semiprime $*$-ring, α an endomorphism of R such that α is onto and $T : R \longrightarrow R$ an additive mapping such that $2T(xyx) = T(x)\alpha(y^*x^*) + \alpha(x^*y^*)T(x)$ holds for all $x, y \in R$. Then*

(*i*) $2T(xyz + zyx) = T(x)\alpha(y^*z^*) + T(z)\alpha(y^*x^*) + \alpha(x^*y^*)T(z) + \alpha(z^*y^*)T(x)$, *for all $x, y, z \in R$,*
(*ii*) $(T(x^2) - T(x)\alpha(x^*))\alpha(y^*)\alpha(x^*) + \alpha(x^*)\alpha(y^*)(T(x^2) - \alpha(x^*)T(x)) = 0$, *for all $x \in R$,*
(*iii*) $2T(x^2) = T(x)\alpha(x^*) + \alpha(x^*)T(x)$, *for all $x, y \in R$.*

Proof. (*i*) Linearizing the hypothesis on x and comparing the relation so obtained with the hypothesis, we get (*i*).

(*ii*) Putting x^2 for z in part (*i*), we have

$$2T(xyx^2 + x^2yx) = T(x)\alpha(y^*{x^*}^2) + T(x^2)\alpha(y^*x^*) + \alpha(x^*y^*)T(x^2) + \alpha({x^*}^2y^*)T(x). \tag{2.1}$$

Replacing y by $xy + yx$ in the hypothesis, we get

$$2T(xyx^2 + x^2yx) = T(x)\alpha(x^*y^*x^*) + T(x)\alpha(y^*{x^*}^2) + \alpha({x^*}^2y^*)T(x) + \alpha(x^*y^*x^*)T(x). \tag{2.2}$$

Comparing (2.1) and (2.2), we get the required result.

(*iii*) Replacing y by y^* in(*ii*), we find that

$$(T(x^2) - T(x)\alpha(x^*))\alpha(y)\alpha(x^*) + \alpha(x^*)\alpha(y)(T(x^2) - \alpha(x^*)T(x)) = 0.$$

Now taking $a = T(x^2) - T(x)\alpha(x^*)$, $b = \alpha(x^*)$, $z = \alpha(y)$, $c = T(x^2) - \alpha(x^*)T(x)$ in the above relation, we find that $azb + bzc = 0$ for all $z \in R$. Then by Lemma 2.1, we get

$$(2T(x^2) - T(x)\alpha(x^*) - \alpha(x^*)T(x))\alpha(y)\alpha(x^*) = 0.$$

Now, if $A(x) = 2T(x^2) - T(x)\alpha(x^*) - \alpha(x^*)T(x)$ then the above relation can be written as

$$A(x)\alpha(y)\alpha(x^*) = 0, \text{ for all } x, y \in R. \tag{2.3}$$

Since α is an automorphism, replacing $\alpha(y)$ by $\alpha(x^*)\alpha(y)A(x)$ in the above relation, we find that

$$A(x)\alpha(x^*)\alpha(y)A(x)\alpha(x^*) = 0.$$

Since α is onto, the semiprimeness of R forces that

$$A(x)\alpha(x^*) = 0, \text{ for all } x \in R. \tag{2.4}$$

Further multiplying from left by $\alpha(x^*)$ and from right by $A(x)$ to (2.3), we find that $\alpha(x^*)A(x)\alpha(y)\alpha(x^*)A(x) = 0$ for all $x, y \in R$ and now the semiprimeness of R gives that

$$\alpha(x^*)A(x) = 0, \text{ for all } x \in R. \tag{2.5}$$

Linearizing (2.4), we get $A(x+y)\alpha(x^*+y^*) = 0$. This can be expanded as $\{A(x)+A(y)+B(x,y)\}\{\alpha(x^*)+\alpha(y^*)\} = 0$ where $B(x,y)$ stands for $2T(xy+yx) - T(x)\alpha(y^*) - T(y)\alpha(x^*) - \alpha(x^*)T(y) - \alpha(y^*)T(x)$. This yields that

$$A(x)\alpha(y^*) + A(y)\alpha(x^*) + B(x,y)\alpha(x^*) + B(x,y)\alpha(y^*) = 0. \tag{2.6}$$

Replacing x by $-x$ in (2.6) and comparing the relation obtained with (2.6), we arrive at

$$A(x)\alpha(y^*) + B(x,y)\alpha(x^*) = 0.$$

Right multiplication of the above relation by $A(x)$ along with (2.5) gives $A(x)\alpha(y^*)A(x) = 0$. This implies that $A(x)\alpha(y)A(x) = 0$. By the surjectivity of α and the semiprimeness of R, it follows that $A(x) = 0$. This proves (iii). ∎

Now we are well equipped to prove our main result:

Theorem 2.3. *Let R be a 2-torsion free semiprime *-ring and α be an automorphism of R. Let $T : R \longrightarrow R$ be an additive mapping such that $2T(xyx) = T(x)\alpha(y^*x^*) + \alpha(x^*y^*)T(x)$ holds for all $x, y \in R$. Then T is a Jordan α-* centralizer.*

Proof. We show that $[T(x), \alpha(x^*)] = 0$ for all $x \in R$. Linearizing and using part (iii) of Lemma 2.2, we get

$$2T(xy+yx) = T(x)\alpha(y^*) + T(y)\alpha(x^*) + \alpha(x^*)T(y) + \alpha(y^*)T(x). \tag{2.7}$$

Replacing y by $2xyx$ and using the assumption of the theorem, we obtain

$$\begin{aligned} 4T(xyx^2 + x^2yx) = {} & 2T(x)\alpha(x^*y^*x^*) + T(x)\alpha(y^*{x^*}^2) \\ & + \alpha(x^*y^*)T(x)\alpha(x^*) + \alpha(x^*)T(x)\alpha(y^*x^*) \\ & + \alpha({x^*}^2y^*)T(x) + 2\alpha(x^*y^*x^*)T(x). \end{aligned} \tag{2.8}$$

Comparing (2.8) and (2.2), we get

$$\begin{aligned} T(x)\alpha(y^*{x^*}^2) + \alpha({x^*}^2y^*)T(x) - \alpha(x^*y^*)T(x)\alpha(x^*) & \\ - \alpha(x^*)T(x)\alpha(y^*x^*) = 0. & \end{aligned} \tag{2.9}$$

Replacing y by xy in the above relation, we get

$$\begin{aligned} T(x)\alpha(y^*{x^*}^3) + \alpha({x^*}^2y^*x^*)T(x) - \alpha(x^*y^*x^*)T(x)\alpha(x^*) & \\ - \alpha(x^*)T(x)\alpha(y^*{x^*}^2) = 0. & \end{aligned} \tag{2.10}$$

Right multiplying (2.9) by $\alpha(x^*)$ and using the fact that α is an endomorphism, one can easily obtain that

$$(2.11)\quad T(x)\alpha(y^*x^{*^3}) + \alpha(x^{*^2}y^*)T(x)\alpha(x^*) - \alpha(x^*y^*)T(x)\alpha(x^{*^2}) - \alpha(x^*)T(x)\alpha(y^*x^{*^2}) = 0.$$

Subtracting (2.10) from (2.11), we arrive at

$$\alpha(x^{*^2})\alpha(y^*)[T(x), \alpha(x^*)] - \alpha(x^*)\alpha(y^*)[T(x), \alpha(x^*)]\alpha(x^*) = 0.$$

Now replacing y by y^*, the last relation reduces to

$$(2.12)\quad \alpha(x^{*^2})\alpha(y)[T(x), \alpha(x^*)] - \alpha(x^*)\alpha(y)[T(x), \alpha(x^*)]\alpha(x^*) = 0.$$

Replacing $\alpha(y)$ by $T(x)\alpha(y^*)$ in (2.12), we find that

$$(2.13)\quad \alpha(x^{*^2})T(x)\alpha(y^*)[T(x), \alpha(x^*)] - \alpha(x^*)T(x)\alpha(y^*)[T(x), \alpha(x^*)]\alpha(x^*) = 0.$$

Left multiplication of (2.12) by $T(x)$ gives

$$(2.14)\quad T(x)\alpha(x^{*^2})\alpha(y^*)[T(x), \alpha(x^*)] - T(x)\alpha(x^*)\alpha(y^*)[T(x), \alpha(x^*)]\alpha(x^*) = 0.$$

Subtracting (2.13) from (2.14), we arrive at $[T(x), \alpha(x^{*^2})]\alpha(y^*)[T(x), \alpha(x^*)] - [T(x), \alpha(x^*)]\alpha(y^*)[T(x), \alpha(x^*)]\alpha(x^*) = 0$ and hence replacing y by y^*, we find that

$$[T(x), \alpha(x^{*^2})]\alpha(y)[T(x), \alpha(x^*)] - [T(x), \alpha(x^*)]\alpha(y)[T(x), \alpha(x^*)]\alpha(x^*) = 0.$$

In the above relation if we take $a = [T(x), \alpha(x^{*^2})],\ b = [T(x), \alpha(x^*)], c = -[T(x), \alpha(x^*)]\alpha(x^*)$ and $z = \alpha(y)$, then we have that $azb + bzc = 0$ for all $z \in R$. Hence again by Lemma 2.1, we get

$$\{[T(x), \alpha(x^{*^2})] - [T(x), \alpha(x^*)]\alpha(x^*)\}\alpha(y)[T(x), \alpha(x^*)] = 0.$$

But since $[T(x), \alpha(x^{*^2})] = \alpha(x^*)[T(x), \alpha(x^*)] + [T(x), \alpha(x^*)]\alpha(x^*)$, the above relation yields that $\alpha(x^*)[T(x), \alpha(x^*)]\alpha(y)[T(x), \alpha(x^*)] = 0$ for all $x, y \in R$. Replacing $\alpha(y)$ by $\alpha(y)\alpha(x^*)$ in the later relation, we find that

$$\alpha(x^*)[T(x), \alpha(x^*)]\alpha(y)\alpha(x^*)[T(x), \alpha(x^*)] = 0, \text{ for all } x, y \in R.$$

Now by the surjectivity of α and the semiprimeness of R, it follows that

$$(2.15)\quad \alpha(x^*)[T(x), \alpha(x^*)] = 0, \text{ for all } x \in R.$$

Replacing y by yx in the relation (2.9), we obtain

$$(2.16)\quad T(x)\alpha(x^*y^*x^{*^2}) + \alpha(x^{*^3}y^*)T(x) - \alpha(x^{*^2}y^*)T(x)\alpha(x^*) - \alpha(x^*)T(x)\alpha(x^*y^*x^*) = 0.$$

Left multiplication of (2.9) by $\alpha(x^*)$ gives

$$\alpha(x^*)T(x)\alpha(y^*x^{*^2}) + \alpha(x^{*^3}y^*)T(x) - \alpha(x^{*^2}y^*)T(x)\alpha(x^*) - \alpha(x^{*^2})T(x)\alpha(y^*x^*) = 0. \tag{2.17}$$

Subtracting (2.17) from (2.16), we arrive at

$$[T(x), \alpha(x^*)]\alpha(y^*)\alpha(x^{*^2}) - \alpha(x^*)[T(x), \alpha(x^*)]\alpha(y^*)\alpha(x^*) = 0.$$

Using (2.15), the above relation reduces to

$$[T(x), \alpha(x^*)]\alpha(y^*)\alpha(x^{*^2}) = 0, \text{ for all } x, y \in R. \tag{2.18}$$

Again replacing y by y^*, we arrive at $[T(x), \alpha(x^*)]\alpha(y)\alpha(x^{*^2}) = 0$ for all $x, y \in R$. Now, replacing $\alpha(y)$ by $\alpha(y^*)T(x)$, we find that

$$[T(x), \alpha(x^*)]\alpha(y^*)T(x)\alpha(x^{*^2}) = 0, \text{ for all } x, y \in R. \tag{2.19}$$

Right multiplication of (2.18) by $T(x)$ gives

$$[T(x), \alpha(x^*)]\alpha(y^*)\alpha(x^{*^2})T(x) = 0, \text{ for all } x, y \in R. \tag{2.20}$$

Subtracting (2.19) from (2.20), we obtain

$$[T(x), \alpha(x^*)]\alpha(y^*)[T(x), \alpha(x^{*^2})] = 0.$$

The above relation can be rewritten by using the relation (2.15) as

$$[T(x), \alpha(x^*)]\alpha(y^*)[T(x), \alpha(x^*)]\alpha(x^*) = 0.$$

Now replacing y by yx in the above relation, we find that $[T(x), \alpha(x^*)]\alpha(x^*)\alpha(y^*)[T(x), \alpha(x^*)]\alpha(x^*) = 0$, and again replacing y by y^* it follows that $[T(x), \alpha(x^*)]\alpha(x^*)\alpha(y)[T(x), \alpha(x^*)]\alpha(x^*) = 0$. Now by the surjectivity of α and the semiprimeness of R, it follows that

$$[T(x), \alpha(x^*)]\alpha(x^*) = 0, \text{ for all } x \in R. \tag{2.21}$$

Linearizing and using the relation (2.15) gives

$$\alpha(x^*)[T(x), \alpha(y^*)] + \alpha(x^*)[T(y), \alpha(x^*)] + \alpha(x^*)[T(y), \alpha(y^*)] + \alpha(y^*)[T(x), \alpha(x^*)] + \alpha(y^*)[T(x), \alpha(y^*)] + \alpha(y^*)[T(y), \alpha(x^*)] = 0.$$

Now replacing x by $-x$ in the above relation and comparing relation so obtained with the above relation, we get

$$\alpha(x^*)[T(x), \alpha(y^*)] + \alpha(x^*)[T(y), \alpha(x^*)] + \alpha(y^*)[T(x), \alpha(x^*)] = 0. \tag{2.22}$$

Left multiplying the relation (2.22) by $[T(x), \alpha(x^*)]$ and then using (2.21), we get $[T(x), \alpha(x^*)]\alpha(y^*)[T(x), \alpha(x^*)] = 0$ and hence replacing y by y^*, it follows that $[T(x), \alpha(x^*)]\alpha(y)[T(x), \alpha(x^*)] = 0$ for all $x, y \in R$. Again by the surjectivity of α and the semiprimeness of R, it follows that $[T(x), \alpha(x^*)] = 0$, for all $x \in R$. Now combining this relation with part (*iii*) of Lemma 2.2, we get $T(x^2) = T(x)\alpha(x^*)$, for all $x \in R$ and $T(x^2) = \alpha(x^*)T(x)$, for all $x \in R$.

This means that T is a left and a right Jordan α-*centralizer. So T is a Jordan α-*centralizer of R. ■

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