DE GRUYTER
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DEMONSTRATIO MATHEMATICA
Vol. XLVII No 1 2014

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CERTAIN MULTIPLIER VERSION OF THE RIEMANN DERANGEMENT THEOREM

Abstract. Aim of this paper is to consider a problem formulated in [6]. Namely, it has been proven that for any sequences $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$, $\lim\limits_{n\to\infty} x_n = \infty$ and $\{a_n\}_{n=1}^{\infty} \subset (0,\infty)$, $\lim\limits_{n\to\infty} a_n = 0$, for every interval $[a,b] \subset [-\infty,\infty]$, there exist a nondecreasing sequence $\{k_n\}_{n=1}^{\infty}$ of positive integers and a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of ± 1 signs such that the set of limit points of the series $\sum \varepsilon_n x_{k_n} a_n$ is equal to $[a,b]$.

In this paper, it will be proven that assumptions of the main theorem from [6] can be significantly weaken. It results from the fact that the assumptions forced for the sequence $\{x_n\}$ in [6] were caused by the method of theorem proving. Additionally, it will turn out that the way of proving the generalization of this theorem presented in here is identical with the one used in the constructive proof of the Riemann derangement theorem. Such manner of proving can be also applied to some generalizations of this theorem on series of terms in given finitely dimensional normed space.

At first, let us present the result determining the technical base of the main result of this paper.

Lemma 1. *Let* $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$, $\lim_{n\to\infty} x_n = \infty$ *and* $\{a_n\}_{n=1}^{\infty} \subset (0,\infty)$, $\lim_{n\to\infty} a_n = 0$. *Then there exists a nondecreasing sequence* $\{k_n\}_{n=1}^{\infty}$ *of positive integers satisfying the following conditions:*

i) $\lim\limits_{n\to\infty} k_n = \infty$,

ii) $\lim\limits_{n\to\infty} x_{k_n} a_n = 0$,

iii) $\sum\limits_{n=1}^{\infty} x_{k_n} a_n = \infty$.

2010 *Mathematics Subject Classification*: 40A05, 05A99.
Key words and phrases: Riemann derangement theorem.

DOI: 10.2478/dema-2014-0009

Proof. Without loss of generality we can assume that $\{x_n\}_{n=1}^{\infty} \subset (0,\infty)$. Let

$$K_r := \min\Big\{k \in \mathbb{N}: \ \sup\{x_r\, a_n: \ n \geq k\} \leq \frac{1}{r}\Big\}, \qquad r = 1,2,\ldots,$$

$$L_1 := \min\Big\{k \in \mathbb{N}: \ k \geq K_2 \ \text{ and } \sum_{n=1}^{k} x_1\, a_n \geq 1\Big\},$$

$$L_r := \min\Big\{k \in \mathbb{N}: \ k \geq \max\{K_{r+1}, 1 + L_{r-1}\} \ \text{ and } \sum_{n=L_{r-1}+1}^{k} x_r\, a_n \geq 1\Big\},$$

for $r = 2,3,\ldots$.

Let us take $k_n = r$, for $L_{r-1} < n \leq L_r$, $r = 1,2,\ldots$, where $L_0 := 0$. It guarantees that the condition *i*) is satisfied.

Then we also have

$$x_{k_n}\, a_n = x_r\, a_n \leq \frac{1}{r}, \qquad \text{for } \ L_{r-1} < n \leq L_r, \quad r = 2,3,\ldots,$$

since $L_{r-1} \geq K_r$. It means that the condition *ii*) holds.

Finally

$$\sum_{n=1}^{\infty} x_{k_n}\, a_n = \sum_{n=1}^{L_1} x_1\, a_n + \sum_{r=2}^{\infty} \sum_{n=L_{r-1}+1}^{L_r} x_r\, a_n \geq \sum_{n=1}^{L_1} x_1\, a_n + \sum_{r=2}^{\infty} 1 = \infty,$$

which implies condition *iii*). ∎

From the above result one can easily receive the expected generalization of Theorem 2 from [6].

Theorem 2. *Let $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$, $\lim_{n\to\infty} x_n = \infty$ and $\{a_n\}_{n=1}^{\infty} \subset (0,\infty)$, $\lim_{n\to\infty} a_n = 0$. We assume that $\{k_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of positive integers satisfying conditions i), ii) and iii) of Lemma 1. Let $I \subset \mathbb{R} \cup \{\pm\infty\}$ be a nonempty interval, closed in 2-point compactification of $\mathbb{R}$. Then there exists a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of ± 1 signs, such that the set of limit points of the series $\sum_{n=1}^{\infty} \varepsilon_n x_{k_n} a_n$ is equal to I.*

Proof. Few cases will be considered with regard to the form of interval I.

Let $I = [\alpha,\beta]$, $\alpha,\beta \in \mathbb{R}$, $\alpha \leq \beta$. Let us define the auxiliary increasing sequence $\{D_n\}_{n=1}^{\infty}$ of positive integers

$$D_1 := \min\Big\{d \in \mathbb{N}: \ \sum_{n=1}^{d} x_{k_n}\, a_n \geq \beta\Big\},$$

$$D_2 := \min\Big\{d \in \mathbb{N}: \ d \geq D_1 + 1 \ \text{ and } \sum_{n=1}^{D_1} x_{k_n}\, a_n - \sum_{n=D_1+1}^{d} x_{k_n}\, a_n \leq \alpha\Big\},$$

$$D_3 := \min\Big\{d \in \mathbb{N}:\ d \geq D_2 + 1 \ \text{ and } \sum_{n=1}^{D_1} x_{k_n} a_n - \sum_{n=D_1+1}^{D_2} x_{k_n} a_n + \sum_{n=D_2+1}^{d} x_{k_n} a_n \geq \beta\Big\},$$

etc.

We take

$$\varepsilon_n = \begin{cases} 1, & \text{for } D_{2k}+1 \leq n \leq D_{2k+1},\ k = 0, 1, \dots, \\ -1, & \text{for } D_{2k-1}+1 \leq n \leq D_{2k},\ k = 1, 2, \dots, \end{cases}$$

where $D_0 := 0$.

Let $I = [\alpha, \infty)$, $\alpha \in \mathbb{R}$. We proceed analogically like in previous case. One should only, in determining the numbers D_{2k-1}, replace the number β by the number $\alpha + k$, for every $k \in \mathbb{N}$.

Similarly, if $I = \mathbb{R}$ then defining the numbers D_{2k-1}, we replace the number β by k, whereas in determining the numbers D_{2k}, the number α is replaced by $-k$, for every $k \in \mathbb{N}$.

Furthermore, if $I = \{\infty\}$ then in determining the numbers D_{2k-1}, we replace the number β by k, whereas in defining the numbers D_{2k}, the number α is replaced by $k - 1$, for every $k \in \mathbb{N}$.

In the cases: $I = (-\infty, \alpha]$, $\alpha \in \mathbb{R}$ or $I = \{-\infty\}$, we take the sequence $\{\varepsilon_n\}_{n=1}^{\infty}$, such that $\varepsilon_n := -\varepsilon'_n$, where $\{\varepsilon'_n\}_{n=1}^{\infty}$ is the sequence of ± 1 determined for intervals $I = [\alpha, \infty)$ and $I = \{\infty\}$, respectively.

Condition *iii*) of Lemma 1 guarantees the correctness of the above definitions. ■

Similarity of the above proof to the constructive proof of the Riemann derangement theorem is striking. Let us recall that the given conditionally convergent series $\sum_{n=1}^{\infty} a_n$ of real terms is divided into two subseries $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ of positive and negative terms, respectively. If $I \subset \mathbb{R} \cup \{\pm\infty\}$ is an interval, like in assumptions of Theorem 2, then we proceed similarly like in the above proof. For example, if $I = [\alpha, \beta]$, $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$, then we determine an auxiliary sequence $\{D_n\}_{n=1}^{\infty} \subset \mathbb{N}$ in the following way

$$D_1 := \min\Big\{d \in \mathbb{N}:\ \sum_{n=1}^{d} b_n \geq \beta\Big\},$$

$$D_2 := \min\Big\{d \in \mathbb{N}:\ \sum_{n=1}^{D_1} b_n + \sum_{n=1}^{d} c_n \leq \alpha\Big\},$$

$$D_3 := \min\Big\{d \in \mathbb{N}:\ d \geq D_1 + 1 \ \text{ and } \ \sum_{n=1}^{d} b_n + \sum_{n=1}^{D_2} c_n \geq \beta\Big\},$$

$$D_4 := \min\Big\{d \in \mathbb{N}:\ d \geq D_2 + 1 \ \text{ and } \ \sum_{n=1}^{D_3} b_n + \sum_{n=1}^{d} c_n \leq \alpha\Big\},$$

etc.

The respective permutation p is defined as follows

$$\begin{cases} a_{p(n)} = b_n, & \text{for } 1 \leq n \leq D_1, \\ a_{p(D_1+n)} = c_n, & \text{for } 1 \leq n \leq D_2, \\ a_{p(D_{2k}+n)} = b_n, & \text{for } D_{2k-1} + 1 \leq n \leq D_{2k+1},\ k \in \mathbb{N}, \\ a_{p(D_{2k+1}+n)} = c_n, & \text{for } D_{2k} + 1 \leq n \leq D_{2k+2}, \quad k \in \mathbb{N}. \end{cases}$$

Of course, the set of accumulation points of the received rearranged series $\sum_{n=1}^{\infty} a_{p(n)}$ is equal to I. Correctness of such proceedings results from the convergence to zero of the sequences $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ and from the divergence of the series $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$.

Problem. Additionally, let us notice that the above way of reasoning applies also to the special series in finitely dimensional spaces. For example, let us assume that $u, v \in \mathbb{R}^2$ are nonparallel vectors (the nonzero vectors in consequence). Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be the conditionally convergent series of real terms. Let $\sum_{n=1}^{\infty} w_n$ be the series formed from all the elements of the sequence $a_1 u$, $b_1 v$, $a_2 u$, $b_2 v$, $a_3 u$, $b_3 v$, ... Then for every $x \in \mathbb{R}^2$ (more general, for any x from the closure of the set X equal to any $n-$gon or any disc in $\mathbb{R}^2$) there exists a permutation p of $\mathbb{N}$ such that $\sum_{n=1}^{\infty} w_{p(n)} = x$ (or the set of the limit points of the series $\sum_{n=1}^{\infty} w_{p(n)}$ is equal to X, respectively). The Author is not absolutely convinced that the closure of any open and connected set can be taken as the above set X.

Final remark

Subject matter concerning the Riemann derangement theorem appears to be constantly inspiring, which is shown in the literature from the last few years [1–3, 5, 6].

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Received April 27, 2012; revised version August 30, 2012.