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LINEAR APPROXIMATION AND ASYMPTOTIC EXPANSION ASSOCIATED WITH THE SYSTEM OF NONLINEAR FUNCTIONAL EQUATIONS

Abstract. This paper is devoted to the study of the following perturbed system of nonlinear functional equations

$$\text{(E)}\quad f_i(x) = \sum_{k=1}^{m}\sum_{j=1}^{n}\left[\varepsilon a_{ijk}\Psi\left(x, f_j(R_{ijk}(x)), \int_0^{X_{ijk}(x)} f_j(t)dt\right) + b_{ijk} f_j(S_{ijk}(x))\right] + g_i(x),$$

$x \in \Omega = [-b, b]$, $i = 1, \ldots, n$, where ε is a small parameter, a_{ijk}, b_{ijk} are the given real constants, R_{ijk}, S_{ijk}, $X_{ijk} : \Omega \to \Omega$, $g_i : \Omega \to \mathbb{R}$, $\Psi : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ are the given continuous functions and $f_i : \Omega \to \mathbb{R}$ are unknown functions. First, by using the Banach fixed point theorem, we find sufficient conditions for the unique existence and stability of a solution of (E). Next, in the case of $\Psi \in C^2(\Omega \times \mathbb{R}^2; \mathbb{R})$, we investigate the quadratic convergence of (E). Finally, in the case of $\Psi \in C^N(\Omega \times \mathbb{R}^2; \mathbb{R})$ and ε sufficiently small, we establish an asymptotic expansion of the solution of (E) up to order $N + 1$ in ε. In order to illustrate the results obtained, some examples are also given.

1. Introduction

In this paper, we consider the following system of nonlinear functional equations

$$(1.1)\quad f_i(x) = \sum_{k=1}^{m}\sum_{j=1}^{n}\left[\varepsilon a_{ijk}\Psi\left(x, f_j(R_{ijk}(x)), \int_0^{X_{ijk}(x)} f_j(t)dt\right) + b_{ijk} f_j(S_{ijk}(x))\right] + g_i(x),$$

$i = 1, \ldots, n$, $x \in \Omega = [-b, b]$, where a_{ijk}, b_{ijk} are the given real constants;

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R_{ijk}, S_{ijk}, $X_{ijk} : \Omega \to \Omega$, $g_i : \Omega \to \mathbb{R}$, $\Psi : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ are the given continuous functions and $f_i : \Omega \to \mathbb{R}$ are unknown functions, ε is a small parameter.

The existence of solutions for functional integral equations of the form (1.1) have been extensively studied by many authors via various techniques in functional analysis, topology and fixed point theory, such as using the Banach fixed point theorem, fixed point theorems of Krasnoselskii type, the Darbo fixed point theorem, the nonlinear alternative of Leray–Schauder and using the technique of the measure of noncompactness. There are many interesting results of solvability, asymptotic stability and some properties of solutions; for example, we refer to the [1], [4]–[17] and references therein.

It is well known that, integral equations and functional integral equations as above have attracted great interest in the field of nonlinear analysis not only because of their mathematical context but also because of their applications in various fields of science and technology, in engineering, mechanics, physics, economics, ... For the details of such applications, see for example, C. Corduneanu [2], K. Deimling [3].

In [17], system (1.1) is studied with $m = n = 2$, $\Psi = 0$ and S_{ijk} binomials of first degree. The solution is approximated by a uniformly convergent recurrent sequence and it is stable with respect to the functions g_i.

In [7], [8], [10], the existence and uniqueness of a solution of the functional equation

$$f(x) = a(x, f(S(x))),$$

in the functional space $BC[a, b]$, have been studied.

In [11]–[14], special cases of (1.1) have been studied corresponding the following form

$$f_i(x) = \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ijk}\left(x, f_j(S_{ijk}(x))\right) + g_i(x),$$

$i = 1, \ldots, n$, $x \in I \subset \mathbb{R}$, where I is a bounded or unbounded interval. By using the Banach fixed point theorem, the authors have established the existence, uniqueness and stability of the solution of (1.1) with respect to the functions g_i. Furthermore, the quadratic convergence and an asymptotic expansion of solutions are also investigated.

Applying a fixed point theorem of Krasnosel'skii type and giving the suitable assumptions, Dhage and Ntouyas [4], Purnaras [16] obtained some results on the existence of solutions to the following nonlinear functional integral equation

$$x(t) = q(t) + \int_0^{\mu(t)} k(t,s)f(s,x(\theta(s)))ds + \int_0^{\sigma(t)} v(t,s)g(s,x(\eta(s)))ds,$$
$$t \in [0,1],$$

where $0 \leq \mu(t) \leq t$; $0 \leq \sigma(t) \leq t$; $0 \leq \theta(t) \leq t$; $0 \leq \eta(t) \leq t$, for all $t \in [0,1]$.

Purnaras also showed that the technique used in [16] can be applied to yield existence results for the following equation

$$x(t) = q(t) + \int_{\alpha(t)}^{\mu(t)} k(t,s)f(s,x(\theta(s)))ds$$
$$+ \int_{\beta(t)}^{\lambda(t)} \widehat{k}(t,s)F\left(s, x(\nu(s)), \int_0^{\sigma(s)} k_0(s,v,x(\eta(v)))dv\right)ds,\ t \in [0,1].$$

Recently, using the technique of the measure of noncompactness and the Darbo fixed point theorem, Z. Liu et al. [9] have proved the existence and asymptotic stability of solutions for the equation

$$x(t) = f\left(t,\ x(t),\ \int_0^t u(t,s,x(a(s)),x(b(s)))\ ds\right),\ t \in \mathbb{R}_+.$$

Motivated by the above mentioned works, we introduce and investigate the more general nonlinear functional integral equation of the form (1.1).

This paper consists of five sections. In Section 2, by using the Banach fixed point theorem, we find sufficient conditions for the unique existence and stability of a solution of (1.1). In Section 3, in the case of $\Psi \in C^2(\Omega \times \mathbb{R}^2;\mathbb{R})$, we investigate the quadratic convergence of (1.1). In the case of $\Psi \in C^N(\Omega \times \mathbb{R}^2;\mathbb{R})$ and ε sufficiently small, an asymptotic expansion of the solution of (1.1) up to order $N+1$ in ε is established in Section 4. We end the paper with illustrated examples.

The results obtained here relatively generalize the ones in [1], [4]–[17].

2. The theorems on existence, uniqueness and stability of solutions

With $\Omega = [-b,b]$, we denote by $X = C(\Omega;\mathbb{R}^n)$ the Banach space of functions $f : \Omega \to \mathbb{R}^n$ continuous on Ω with respect to the norm

$$\|f\|_X = \sup_{x\in\Omega} \sum_{i=1}^n |f_i(x)|,\quad f = (f_1,\ldots,f_n) \in X.$$

For any non-negative integer r, we put

$$C^r(\Omega;\mathbb{R}^n) = \left\{ f \in C(\Omega;\mathbb{R}^n) : f_i^{(k)} \in C(\Omega;\mathbb{R}),\ 0 \leq k \leq r,\ 1 \leq i \leq n \right\}.$$

It is clear that $C^r(\Omega;\mathbb{R}^n)$ is the Banach space with respect to the norm

$$\|f\|_r = \max_{0\leq k\leq r} \sup_{x\in\Omega} \sum_{i=1}^n \left| f_i^{(k)}(x) \right|.$$

We write the system (1.1) in the form of an operational equation in X as follows

$$f = \varepsilon Af + Bf + g, \tag{2.1}$$

where

$$f = (f_1, \dots, f_n),\ Af = ((Af)_1, \dots, (Af)_n), Bf = ((Bf)_1, \dots, (Bf)_n),$$

with

$$\begin{cases} (Af)_i(x) = \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ijk} \Psi\left(x, f_j(R_{ijk}(x)), \int_0^{X_{ijk}(x)} f_j(t)dt\right), \\ (Bf)_i(x) = \sum_{k=1}^{m} \sum_{j=1}^{n} b_{ijk} f_j(S_{ijk}(x)), \quad x \in \Omega,\ i = 1, 2, \dots, n. \end{cases}$$

We set the following notions

$$\|[\alpha_{ijk}]\| = \sum_{i=1}^{n} \sum_{k=1}^{m} \max_{1 \le j \le n} |\alpha_{ijk}|,$$

for any set $[\alpha_{ijk}] = \{\alpha_{ijk} \in \mathbb{R} : i,\ j = 1, \dots, n;\ k = 1, \dots, m\}$,

$$\|[F_{ijk}]\| = \sum_{i=1}^{n} \sum_{k=1}^{m} \max_{1 \le j \le n} \|F_{ijk}\|_\infty,$$

for any set $[F_{ijk}] = \{F_{ijk} \in C(\Omega; \mathbb{R}) : i,\ j = 1, \dots, n;\ k = 1, \dots, m\}$, where the symbol $\|\cdot\|_\infty$ denotes the supremum norm on $C(\Omega; \mathbb{R})$; and the following assumptions

(H_1) all the functions R_{ijk}, S_{ijk}, $X_{ijk} : \Omega \to \Omega$ are continuous,
(H_2) $g \in X$,
(H_3) $\|[b_{ijk}]\| < 1$,
(H_4) $\Psi : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the following condition: $\forall M > 0$, $\exists C_1(M) > 0$: $|\Psi(x, y_1, z_1) - \Psi(x, y_2, z_2)| \le C_1(M) \left(|y_1 - y_2| + |z_1 - z_2|\right)$ for all (x, y_1, z_1), $(x, y_2, z_2) \in \Omega \times [-M, M] \times [-bM, bM]$,
(H_5) $M > \frac{2\|g\|_X}{1 - \|[b_{ijk}]\|}$ and $0 < \varepsilon_0 < \frac{M\left(1 - \|[b_{ijk}]\|\right)}{2[(1+b)MC_1(M) + nM_0]\|[a_{ijk}]\|}$, where

$$M_0 = \sup\{|\Psi(x, 0, 0)| : x \in \Omega\}. \tag{2.2}$$

Given $M > 0$, we put

$$K_M = \{f \in X : \|f\|_X \le M\}.$$

The following lemmas are useful to establish our main results, the proof are not difficult so we omit it.

Lemma 2.1. *Let (H_1) and (H_3) hold. Then the linear operator $I - B : X \to X$ is invertible and*

$$\left\|(I-B)^{-1}\right\| \le \frac{1}{1 - \|[b_{ijk}]\|}. \ \blacksquare$$

By Lemma 2.1, we rewrite the functional equations system (2.1) as follows

$$f = (I-B)^{-1}(\varepsilon A f + g) \equiv Tf. \tag{2.3}$$

Lemma 2.2. *Let (H_1), (H_3), (H_4) hold. Then, for every $M > 0$ we have*

(i) $\|Af\|_X \le \|[a_{ijk}]\| \left[(1+b)\, C_1(M)\, \|f\|_X + nM_0\right], \quad \forall f \in K_M;$
(ii) $\left\|Af - A\bar{f}\right\|_X \le (1+b)\, C_1(M)\, \|[a_{ijk}]\|\, \left\|f - \bar{f}\right\|_X, \forall f, \bar{f} \in K_M.$ $\blacksquare$

Then, we have the following theorem.

Theorem 2.3. *Let (H_1)–(H_5) hold. Then, for every ε, with $|\varepsilon| \le \varepsilon_0$, the system (2.3) has a unique solution $f \in K_M$.*

Proof. It is evident that $T : X \to X$. Considering f, $\bar{f} \in K_M$, by Lemmas 2.1 and 2.2, we easily verify that

$$\|Tf\|_X \le \frac{1}{1 - \|[b_{ijk}]\|} \left[\varepsilon_0 \|[a_{ijk}]\| \left((1+b)\, M C_1(M) + nM_0\right) + \|g\|_X\right]; \tag{2.4}$$

$$\left\|Tf - T\bar{f}\right\|_X \le \frac{\varepsilon_0\, (1+b)\, C_1(M)\, \|[a_{ijk}]\|}{1 - \|[b_{ijk}]\|} \left\|f - \bar{f}\right\|_X. \tag{2.5}$$

Notice that, from (H_3)–(H_5) we have

$$\frac{1}{1 - \|[b_{ijk}]\|} \left[\varepsilon_0 \|[a_{ijk}]\| \left((1+b)\, M C_1(M) + nM_0\right) + \|g\|_X\right] \le M. \tag{2.6}$$

It follows from (2.4)–(2.6), that $T : K_M \to K_M$ is a contraction mapping. Then, using Banach fixed point theorem, there exists a unique function $f \in K_M$ such that $f = Tf$. $\blacksquare$

Remark 2.4. Theorem 2.3 gives a consecutive approximate algorithm

$$f^{(\nu)} = Tf^{(\nu-1)}, \nu = 1, 2, \ldots, \text{ where } f^{(0)} \in X \text{ is given.}$$

Then the sequence $\{f^{(\nu)}\}$ converges in X to the solution f of (2.3) and we have the error estimation

$$\left\|f^{(\nu)} - f\right\|_X \le \frac{\sigma^\nu}{1-\sigma} \left\|Tf^{(0)} - f^{(0)}\right\|_X \quad \text{for all } \nu \in \mathbb{N},$$

$$\text{where } \sigma = \frac{\varepsilon_0\, (1+b)\, C_1(M)\, \|[a_{ijk}]\|}{1 - \|[b_{ijk}]\|} < 1.$$

3. The second order algorithm

In this part, let $\Psi \in C^1(\Omega \times \mathbb{R}^2; \mathbb{R})$.

First, using the approximation

$$\Psi(x, u^{(\nu)}, v^{(\nu)}) \cong \Psi(x, u^{(\nu-1)}, v^{(\nu-1)}) + \frac{\partial \Psi}{\partial y}(x, u^{(\nu-1)}, v^{(\nu-1)}) \left(u^{(\nu)} - u^{(\nu-1)}\right) + \frac{\partial \Psi}{\partial z}(x, u^{(\nu-1)}, v^{(\nu-1)}) \left(v^{(\nu)} - v^{(\nu-1)}\right),$$

where $u^{(\nu)} = f_j^{(\nu)}(R_{ijk}(x))$, $v^{(\nu)} = \int_0^{X_{ijk}(x)} f_j^{(\nu)}(t)dt$, we obtain the following algorithm for system (1.1)

$$(3.1) \quad f_i^{(\nu)}(x) = \varepsilon \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ijk} \Psi\left(W_{ijk}^{(\nu)}(x)\right) + \varepsilon \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ijk} \frac{\partial \Psi}{\partial y}\left(W_{ijk}^{(\nu)}(x)\right)\left(f_j^{(\nu)}(R_{ijk}(x)) - f_j^{(\nu-1)}(R_{ijk}(x))\right) + \varepsilon \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ijk} \frac{\partial \Psi}{\partial z}\left(W_{ijk}^{(\nu)}(x)\right)\left(\int_0^{X_{ijk}(x)} \left(f_j^{(\nu)}(t) - f_j^{(\nu-1)}(t)\right) dt\right) + \sum_{k=1}^{m} \sum_{j=1}^{n} b_{ijk} f_j^{(\nu)}(S_{ijk}(x)) + g_i(x),$$

for all $x \in \Omega$, $1 \leq i \leq n$, and $\nu = 1, 2, \ldots$ where

$$(3.2) \quad W_{ijk}^{(\nu)}(x) = \left(x, f_j^{(\nu-1)}(R_{ijk}(x)), \int_0^{X_{ijk}(x)} f_j^{(\nu-1)}(t)dt\right),$$

and $f^{(0)} = (f_1^{(0)}, \ldots, f_n^{(0)}) \in K_M$ is given.

Rewrite (3.1) as a linear system of functional equations

$$(3.3) \quad f_i^{(\nu)}(x) = (Bf^{(\nu)})_i(x) + \varepsilon \sum_{k=1}^{m} \sum_{j=1}^{n} \alpha_{ijk}^{(\nu)}(x) f_j^{(\nu)}(R_{ijk}(x)) + \varepsilon \sum_{k=1}^{m} \sum_{j=1}^{n} \beta_{ijk}^{(\nu)}(x) \int_0^{X_{ijk}(x)} f_j^{(\nu)}(t)dt + g_i^{(\nu)}(x),$$

for $x \in \Omega$, $i = 1, 2, \ldots, n$ and $\nu = 1, 2, \ldots$ with $\alpha_{ijk}^{(\nu)}(x)$, $\beta_{ijk}^{(\nu)}(x)$ and $g_i^{(\nu)}(x)$ depending on $f^{(\nu-1)}$ as follows

$$(3.4) \quad \alpha_{ijk}^{(\nu)}(x) = a_{ijk} \frac{\partial \Psi}{\partial y}\left(W_{ijk}^{(\nu)}(x)\right), \beta_{ijk}^{(\nu)}(x) = a_{ijk} \frac{\partial \Psi}{\partial z}\left(W_{ijk}^{(\nu)}(x)\right),$$

and

$$(3.5)\quad g_i^{(\nu)}(x) = g_i(x) + \varepsilon(Af^{(\nu-1)})_i(x) - \varepsilon \sum_{k=1}^{m}\sum_{j=1}^{n} \alpha_{ijk}^{(\nu)}(x) f_j^{(\nu-1)}(R_{ijk}(x)) - \varepsilon \sum_{k=1}^{m}\sum_{j=1}^{n} \beta_{ijk}^{(\nu)}(x) \int_0^{X_{ijk}(x)} f_j^{(\nu-1)}(t)dt.$$

Then, we have the following.

Theorem 3.1. *Let* (H_1)–(H_3) *hold and let* $\Psi \in C^1(\Omega \times \mathbb{R}^2; \mathbb{R})$. *If* $f^{(\nu-1)} \in X$ *satisfies*

$$\gamma_\nu = \|[b_{ijk}]\| + |\varepsilon| \left(\left\| [\alpha_{ijk}^{(\nu)}] \right\| + b \left\| [\beta_{ijk}^{(\nu)}] \right\| \right) < 1,$$

there exists a unique function $f^{(\nu)} \in X$ *being solution of system* (3.3)–(3.5).

Proof. We write system (3.3)–(3.5) in the form of an operational equation in $X = C(\Omega; \mathbb{R}^n)$

$$f^{(\nu)} = T_\nu f^{(\nu)},$$

where

$$(T_\nu f)_i(x) = (Bf)_i(x) + \varepsilon \sum_{k=1}^{m}\sum_{j=1}^{n} \alpha_{ijk}^{(\nu)}(x) f_j(R_{ijk}(x)) + \varepsilon \sum_{k=1}^{m}\sum_{j=1}^{n} \beta_{ijk}^{(\nu)}(x) \int_0^{X_{ijk}(x)} f_j(t)dt + g_i^{(\nu)}(x),$$

for $x \in \Omega$, $i = 1, 2, \ldots, n$ and $f = (f_1, \ldots, f_n) \in X$.

It is easy to check that $T_\nu : X \to X$ and

$$\left\| T_\nu f - T_\nu \bar{f} \right\|_X \le \gamma_\nu \left\| f - \bar{f} \right\|_X \quad \text{for all } f, \bar{f} \in X.$$

Using the Banach fixed point theorem, there exists a unique function $f^{(\nu)} \in X$ being a solution of system (3.3)–(3.5). ∎

Next, we make the following hypotheses:

(H_6) $\Psi \in C^2(\Omega \times \mathbb{R}^2; \mathbb{R})$,

(H_7) $\varepsilon_0 \|[a_{ijk}]\| \left[\frac{nM_0}{M} + (1+b)M_1 + \frac{1}{2}(1+b)^2 M_2 M \right] \le 1 - \|[b_{ijk}]\| - \frac{1}{M} \|g\|_X$,

where M_0 is given by (2.2) and

$$\begin{cases} M_1 = \sup\left\{ \left(\left| \frac{\partial \Psi}{\partial y} \right| + \left| \frac{\partial \Psi}{\partial z} \right| \right)(x,y,z) : (x,y,z) \in A_* \right\}, \\ M_2 = \sup\left\{ \left(\left| \frac{\partial^2 \Psi}{\partial y^2} \right| + \left| \frac{\partial^2 \Psi}{\partial y \partial z} \right| + \left| \frac{\partial^2 \Psi}{\partial z^2} \right| \right)(x,y,z) : (x,y,z) \in A_* \right\}, \end{cases}$$

with $A_* = \{(x,y,z) : x \in \Omega, |y| \le M, |z| \le bM\}$.

Theorem 3.2. *Let* (H_1)–(H_3), (H_6), (H_7) *hold, let* f *be the solution of system* (1.1) *and the sequence* $\{f^{(\nu)}\}$ *be defined by algorithm* (3.3)–(3.5).

(i) *If* $\|f^{(0)}\|_X \le M$, *then*

$$\left\|f^{(\nu)} - f\right\|_X \le \beta_M \left\|f^{(\nu-1)} - f\right\|_X^2, \forall \nu = 1, 2, \ldots \tag{3.6}$$

where

$$\beta_M = \frac{\frac{1}{2}\varepsilon_0 \, (1+b)^2 \, M_2 \, \|[a_{ijk}]\|}{1 - \|[b_{ijk}]\| - \varepsilon_0 \, (1+b) \, M_1 \, \|[a_{ijk}]\|} > 0. \tag{3.7}$$

(ii) *If the first term* $f^{(0)}$ *sufficiently near* f *such that* $\beta_M \|f^{(0)} - f\|_X < 1$, *then the sequence* $\{f^{(\nu)}\}$ *converges quadratically to* f *and furthermore*

$$\left\|f^{(\nu)} - f\right\|_X \le \frac{1}{\beta_M} \left(\beta_M \left\|f^{(0)} - f\right\|_X\right)^{2^\nu}, \forall \nu = 1, 2, \ldots \tag{3.8}$$

Proof. First, we verify that if $\|f^{(0)}\|_X \le M$, then

$$\left\|f^{(\nu)}\right\|_X \le M, \forall \nu = 1, 2, \ldots$$

Indeed, supposing

$$\|f^{(\nu-1)}\|_X \le M, \tag{3.9}$$

we deduce from (3.3) that

$$\left\|f^{(\nu)}\right\|_X \le \left(\|[b_{ijk}]\| + |\varepsilon| \left\|[\alpha_{ijk}^{(\nu)}]\right\| + |\varepsilon| \, b \left\|[\beta_{ijk}^{(\nu)}]\right\|\right) \left\|f^{(\nu)}\right\|_X + \left\|g^{(\nu)}\right\|_X. \tag{3.10}$$

On the other hand, we have

$$\begin{cases} \left|\alpha_{ijk}^{(\nu)}(x)\right| \le |a_{ijk}| \left|\dfrac{\partial \Psi}{\partial y}\left(W_{ijk}^{(\nu)}(x)\right)\right| \le M_1 \, |a_{ijk}|, \\ \left|\beta_{ijk}^{(\nu)}(x)\right| \le |a_{ijk}| \left|\dfrac{\partial \Psi}{\partial z}\left(W_{ijk}^{(\nu)}(x)\right)\right| \le M_1 \, |a_{ijk}|. \end{cases} \tag{3.11}$$

Hence, we deduce from (3.10), (3.11) that

$$\left\|f^{(\nu)}\right\|_X \le [\|[b_{ijk}]\| + \varepsilon_0 \, (1+b) \, M_1 \, \|[a_{ijk}]\|] \left\|f^{(\nu)}\right\|_X + \left\|g^{(\nu)}\right\|_X.$$

Note that (H_7) implies $\|[b_{ijk}]\| + \varepsilon_0 \, (1+b) \, M_1 \, \|[a_{ijk}]\| < 1$, so

$$\left\|f^{(\nu)}\right\|_X \le \frac{\|g^{(\nu)}\|_X}{1 - \|[b_{ijk}]\| - \varepsilon_0 \, (1+b) \, M_1 \, \|[a_{ijk}]\|}. \tag{3.12}$$

Now, we need an estimate on the term $\|g^{(\nu)}\|_X$.

From (3.4) and (3.5), we obtain

$$(3.13)\qquad g_i^{(\nu)}(x) = g_i(x) + \varepsilon \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ijk} \Big[\Psi\left(W_{ijk}^{(\nu)}(x)\right) - \frac{\partial \Psi}{\partial y}\left(W_{ijk}^{(\nu)}(x)\right) f_j^{(\nu-1)}(R_{ijk}(x)) - \frac{\partial \Psi}{\partial z}\left(W_{ijk}^{(\nu)}(x)\right) \int_0^{X_{ijk}(x)} f_j^{(\nu-1)}(t)dt \Big].$$

On the other hand, the Taylor's expansion of the function $\Psi(x,0,0)$ at the point $W_{ijk}^{(\nu)}(x) = \left(x, f_j^{(\nu-1)}(R_{ijk}(x)), \int_0^{X_{ijk}(x)} f_j^{(\nu-1)}(t)dt\right)$, up to order 2, leads to

$$\Psi(x,0,0) = \Psi\left(W_{ijk}^{(\nu)}(x)\right) - \frac{\partial \Psi}{\partial y}\left(W_{ijk}^{(\nu)}(x)\right) f_j^{(\nu-1)}(R_{ijk}(x)) - \frac{\partial \Psi}{\partial z}\left(W_{ijk}^{(\nu)}(x)\right) \int_0^{X_{ijk}(x)} f_j^{(\nu-1)}(t)dt + \frac{1}{2}\frac{\partial^2 \Psi}{\partial y^2}\left(\bar{W}_{ijk}^{(\nu)}(x)\right)\left(f_j^{(\nu-1)}(R_{ijk}(x))\right)^2 + \frac{\partial^2 \Psi}{\partial y \partial z}\left(\bar{W}_{ijk}^{(\nu)}(x)\right) f_j^{(\nu-1)}(R_{ijk}(x)) \int_0^{X_{ijk}(x)} f_j^{(\nu-1)}(t)dt + \frac{1}{2}\frac{\partial^2 \Psi}{\partial z^2}\left(\bar{W}_{ijk}^{(\nu)}(x)\right)\left(\int_0^{X_{ijk}(x)} f_j^{(\nu-1)}(t)dt\right)^2,$$

where

$$\bar{W}_{ijk}^{(\nu)}(x) = \left(x, -\theta_{ijk} f_j^{(\nu-1)}(R_{ijk}(x)), -\theta_{ijk}\int_0^{X_{ijk}(x)} f_j^{(\nu-1)}(t)dt\right), \quad 0 < \theta_{ijk} < 1.$$

Therefore

$$(3.14)\qquad \left| \Psi\left(W_{ijk}^{(\nu)}(x)\right) - \frac{\partial \Psi}{\partial y}\left(W_{ijk}^{(\nu)}(x)\right) f_j^{(\nu-1)}(R_{ijk}(x)) - \frac{\partial \Psi}{\partial z}\left(W_{ijk}^{(\nu)}(x)\right) \int_0^{X_{ijk}(x)} f_j^{(\nu-1)}(t)dt \right|$$
$$= \left| \Psi(x,0,0) - \frac{1}{2}\frac{\partial^2 \Psi}{\partial y^2}\left(\bar{W}_{ijk}^{(\nu)}(x)\right)\left(f_j^{(\nu-1)}(R_{ijk}(x))\right)^2 - \frac{\partial^2 \Psi}{\partial y \partial z}\left(\bar{W}_{ijk}^{(\nu)}(x)\right) f_j^{(\nu-1)}(R_{ijk}(x)) \int_0^{X_{ijk}(x)} f_j^{(\nu-1)}(t)dt - \frac{1}{2}\frac{\partial^2 \Psi}{\partial z^2}\left(\bar{W}_{ijk}^{(\nu)}(x)\right)\left(\int_0^{X_{ijk}(x)} f_j^{(\nu-1)}(t)dt\right)^2 \right|$$

$$\leq M_0 + \frac{1}{2}M_2\left(f_j^{(\nu-1)}(R_{ijk}(x))\right)^2 + M_2\left|f_j^{(\nu-1)}(R_{ijk}(x))\int_0^{X_{ijk}(x)} f_j^{(\nu-1)}(t)dt\right|$$
$$+\frac{1}{2}M_2\left(\int_0^{X_{ijk}(x)} f_j^{(\nu-1)}(t)dt\right)^2$$
$$\leq M_0 + \frac{1}{2}M_2\left[\left|f_j^{(\nu-1)}(R_{ijk}(x))\right| + \left|\int_0^{X_{ijk}(x)} f_j^{(\nu-1)}(t)dt\right|\right]^2.$$

It follows from (3.13), (3.14) that

$$\left\|g^{(\nu)}\right\|_X \leq \|g\|_X + \varepsilon_0\left\|[a_{ijk}]\right\|\left[nM_0 + \frac{1}{2}(1+b)^2 M_2M^2\right]. \tag{3.15}$$

Hence, from (3.12), (3.15) and (H_7), we obtain

$$\left\|f^{(\nu)}\right\|_X \leq \frac{\|g\|_X + \varepsilon_0\left\|[a_{ijk}]\right\|\left[nM_0 + \frac{1}{2}(1+b)^2 M_2M^2\right]}{1 - \left\|[b_{ijk}]\right\| - \varepsilon_0(1+b)M_1\left\|[a_{ijk}]\right\|} \leq M.$$

Now, we shall estimate $\left\|f - f^{(\nu)}\right\|_X$.

Put $e^{(\nu)} = f - f^{(\nu)}$, we obtain from (1.1) and (3.1) the system

$$\begin{aligned}(3.16)\qquad e_i^{(\nu)}(x) &= f_i(x) - f_i^{(\nu)}(x)\\ &= (Be^{(\nu)})_i(x) + \varepsilon\sum_{k=1}^m\sum_{j=1}^n \alpha_{ijk}^{(\nu)}(x)e_j^{(\nu)}(R_{ijk}(x))\\ &+\varepsilon\sum_{k=1}^m\sum_{j=1}^n \beta_{ijk}^{(\nu)}(x)\int_0^{X_{ijk}(x)} e_j^{(\nu)}(t)dt\\ &+\varepsilon\sum_{k=1}^m\sum_{j=1}^n a_{ijk}\Bigg[\Psi\left(W_{ijk}(x)\right) - \Psi\left(W_{ijk}^{(\nu)}(x)\right)\\ &-\frac{\partial\Psi}{\partial y}\left(W_{ijk}^{(\nu)}(x)\right)e_j^{(\nu-1)}(R_{ijk}(x)) - \frac{\partial\Psi}{\partial z}\left(W_{ijk}^{(\nu)}(x)\right)\int_0^{X_{ijk}(x)} e_j^{(\nu-1)}(t)dt\Bigg],\end{aligned}$$

where $W_{ijk}^{(\nu)}(x)$ is given by (3.2) and

$$W_{ijk}(x) = \left(x, f_j(R_{ijk}(x)), \int_0^{X_{ijk}(x)} f_j(t)dt\right).$$

Using Taylor's expansion of the function $\Psi\left(x, f_j(Y), \int_0^Z f_j(t)dt\right)$ at the point $(x,\ f_j^{(\nu-1)}(Y),\ \int_0^Z f_j^{(\nu-1)}(t)dt)$, up to order 2, we obtain

$$\begin{aligned}
(3.17)\qquad \Psi\left(x, f_j(Y), \int_0^Z f_j(t)dt\right) &= \Psi\left(x, f_j^{(\nu-1)}(Y), \int_0^Z f_j^{(\nu-1)}(t)dt\right)\\
&+ \frac{\partial\Psi}{\partial y}\left(x, f_j^{(\nu-1)}(Y), \int_0^Z f_j^{(\nu-1)}(t)dt\right) e_j^{(\nu-1)}(Y)\\
&+ \frac{\partial\Psi}{\partial z}\left(x, f_j^{(\nu-1)}(Y), \int_0^Z f_j^{(\nu-1)}(t)dt\right) \int_0^Z e_j^{(\nu-1)}(t)dt\\
&+ \frac{1}{2}\frac{\partial^2\Psi}{\partial y^2}\left(\omega_j^{(\nu)}(x,Y,Z)\right)\left|e_j^{(\nu-1)}(Y)\right|^2\\
&+ \frac{\partial^2\Psi}{\partial y\partial z}\left(\omega_j^{(\nu)}(x,Y,Z)\right) e_j^{(\nu-1)}(Y)\int_0^Z e_j^{(\nu-1)}(t)dt\\
&+ \frac{1}{2}\frac{\partial^2\Psi}{\partial z^2}\left(\omega_j^{(\nu)}(x,Y,Z)\right)\left(\int_0^Z e_j^{(\nu-1)}(t)dt\right)^2,
\end{aligned}$$

where

$$\begin{aligned}
&\omega_j^{(\nu)}(x,Y,Z)\\
&\quad= \left(x, f_j^{(\nu-1)}(Y) + \theta_j e_j^{(\nu-1)}(Y), \int_0^Z \left[f_j^{(\nu-1)}(t) + \theta_j e_j^{(\nu-1)}(t)\right] dt\right),
\end{aligned}$$

$0 < \theta_j < 1$.

Substituting (3.17) into (3.16) where the arguments of f_j, $f_j^{(\nu-1)}$, $e_j^{(\nu-1)}$, $\omega_j^{(\nu)}$ appearing in (3.17) are replaced by $Y = R_{ijk}(x)$, $Z = X_{ijk}(x)$, we get

$$\begin{aligned}
(3.18)\qquad e_i^{(\nu)}(x) &= (Be^{(\nu)})_i(x) + \varepsilon\sum_{k=1}^{m}\sum_{j=1}^{n}\alpha_{ijk}^{(\nu)}(x)e_j^{(\nu)}(R_{ijk}(x))\\
&+ \varepsilon\sum_{k=1}^{m}\sum_{j=1}^{n}\beta_{ijk}^{(\nu)}(x)\int_0^{X_{ijk}(x)} e_j^{(\nu)}(t)dt\\
&+ \varepsilon\sum_{k=1}^{m}\sum_{j=1}^{n}a_{ijk}\Bigg[\frac{1}{2}\frac{\partial^2\Psi}{\partial y^2}\left(\omega_{ijk}^{(\nu)}(x)\right)\left|e_j^{(\nu-1)}(R_{ijk}(x))\right|^2\\
&+ \frac{\partial^2\Psi}{\partial y\partial z}\left(\omega_{ijk}^{(\nu)}(x)\right) e_j^{(\nu-1)}(R_{ijk}(x))\int_0^{X_{ijk}(x)} e_j^{(\nu-1)}(t)dt\\
&+ \frac{1}{2}\frac{\partial^2\Psi}{\partial z^2}\left(\omega_{ijk}^{(\nu)}(x)\right)\left|\int_0^{X_{ijk}(x)} e_j^{(\nu-1)}(t)dt\right|^2\Bigg],
\end{aligned}$$

where $\omega_{ijk}^{(\nu)}(x) = \omega_j^{(\nu)}(x, R_{ijk}(x), X_{ijk}(x))$.

Combining (3.9), (3.11), (3.18), the result is

$$\begin{aligned}
\left\|e^{(\nu)}\right\|_X \le{}& \left\|Be^{(\nu)}\right\|_X + \varepsilon_0 \sum_{i=1}^{n}\sum_{k=1}^{m} \max_{1\le j\le n}\sup_{x\in\Omega}\left|\alpha_{ijk}^{(\nu)}(x)\right| \sum_{j=1}^{n}\left|e_j^{(\nu)}(R_{ijk}(x))\right| \\
&+ \varepsilon_0 \sum_{i=1}^{n}\sum_{k=1}^{m} \max_{1\le j\le n}\sup_{x\in\Omega}\left|\beta_{ijk}^{(\nu)}(x)\right| \sum_{j=1}^{n}\left|\int_0^{X_{ijk}(x)} e_j^{(\nu)}(t)dt\right| \\
&+ \frac{1}{2}\varepsilon_0 M_2 \sum_{i=1}^{n}\sum_{k=1}^{m} \max_{1\le j\le n}|a_{ijk}|\sup_{x\in\Omega}\sum_{j=1}^{n}\left(\left|e_j^{(\nu-1)}(R_{ijk}(x))\right| + \left|\int_0^{X_{ijk}(x)} e_j^{(\nu-1)}(t)dt\right|\right)^2 \\
\le{}& \left[\|[b_{ijk}]\| + \varepsilon_0 (1+b) M_1 \|[a_{ijk}]\|\right] \left\|e^{(\nu)}\right\|_X \\
&+ \frac{1}{2}\varepsilon_0 (1+b)^2 M_2 \|[a_{ijk}]\| \left\|e^{(\nu-1)}\right\|_X^2 .
\end{aligned}$$

Consequently

$$\begin{aligned}
\text{(3.19)} \qquad \left\|e^{(\nu)}\right\|_X &\le \frac{\frac{1}{2}\varepsilon_0 \ (1+b)^2 M_2 \|[a_{ijk}]\|}{1 - \|[b_{ijk}]\| - \varepsilon_0 \ (1+b) M_1 \|[a_{ijk}]\|} \left\|e^{(\nu-1)}\right\|_X^2 \\
&\equiv \beta_M \left\|e^{(\nu-1)}\right\|_X^2 .
\end{aligned}$$

Hence, we obtain (3.6) by (3.7) and (3.19). Finally, from (3.6), (3.8) follows. ∎

Remark 3.3. If we choose μ_0 sufficient large such that

$$\beta_M \left\|g^{(\mu_0)} - f\right\|_X \le \beta_M \left\|Tg^{(0)} - g^{(0)}\right\|_X \frac{\sigma^{\mu_0}}{1-\sigma} < 1,$$

and choose $f^{(0)} = g^{(\mu_0)}$, then first term $f^{(0)}$ sufficiently near f such that $\beta_M \left\|f^{(0)} - f\right\|_X < 1$. ∎

4. Asymptotic expansion of solutions

In this part, we assume that the functions R_{ijk}, S_{ijk}, X_{ijk}, g, Ψ and the real numbers a_{ijk}, b_{ijk}, M satisfy the assumptions (H_1)–(H_5), respectively. We use the following notation

$$\Psi[f_j] = \Psi\left(x, f_j(R_{ijk}(x)), \int_0^{X_{ijk}(x)} f_j(t)dt\right).$$

Now, we assume that

(H_8) $\Psi \in C^N(\Omega \times \mathbb{R}^2; \mathbb{R})$.

We consider the perturbed system (2.1), where ε is a small parameter $|\varepsilon| \le \varepsilon_0$. Let us consider the finite sequence of functions $\{f^{[r]}\}$, $r =$

$0, 1, \ldots, N$, $f^{[r]} \in K_M$ (with suitable constants $M > 0$, $\varepsilon_0 > 0$) defined as follows:

$$f^{[r]} = (I - B)^{-1} P^{[r]}, r = 0, 1, \ldots, N, \tag{4.1}$$

where

$$P^{[r]} = \left(P_1^{[r]}, \ldots, P_n^{[r]}\right), r = 0, 1, \ldots, N,$$

and

$$P^{[0]} = g.$$

With $r = 1$:

$$P_i^{[1]} = (Af^{[0]})_i(x) = \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ijk} \pi_j^{[0]}[\Psi],$$

where

$$\begin{cases} \pi_j^{[0]}[\Psi] = \Psi[f_j^{[0]}] = \Psi\left(x, f_j^{[0]}(R_{ijk}(x)), Jf_j^{[0]}(X_{ijk}(x))\right), \\ Jf_j^{[0]}(X_{ijk}(x)) = \int_0^{X_{ijk}(x)} f_j^{[0]}(t)dt. \end{cases} \tag{4.2}$$

With $r = 2$:

$$P_i^{[2]} = \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ijk} \pi_j^{[1]}[\Psi],$$

where

$$\pi_j^{[1]}[\Psi] = \pi_j^{[0]}[D_2\Psi] f_j^{[1]} + \pi_j^{[0]}[D_3\Psi] Jf_j^{[1]},\ D_2\Psi = \frac{\partial \Psi}{\partial y},\ D_3\Psi = \frac{\partial \Psi}{\partial z}. \tag{4.3}$$

For $2 \leq r \leq N$,

$$P_i^{[r]} = \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ijk} \pi_j^{[r-1]}[\Psi],$$

where, $\pi_j^{[r]}[\Psi]$, $0 \leq r \leq N-1$ defined by the recurrence formulas

$$\pi_j^{[r]}[\Psi] = \sum_{s=0}^{r-1} \frac{r-s}{r} \left\{\pi_j^{[s]}[D_2\Psi] f_j^{[r-s]} + \pi_j^{[s]}[D_3\Psi] Jf_j^{[r-s]}\right\}. \tag{4.4}$$

We also note that $\pi_j^{[r]}[\Psi]$ is the first-order function with respect to $f_j^{[r]}$, $Jf_j^{[r]}$. In fact,

$$\pi_j^{[r]}[\Psi] = \pi_j^{[0]}[D_2\Psi] f_j^{[r]} + \pi_j^{[0]}[D_3\Psi] Jf_j^{[r]} + \text{ terms depending on } (j, r, \pi_j^{[s]}[D_2\Psi], \pi_j^{[s]}[D_3\Psi], f_j^{[s]}, Jf_j^{[s]}), s = 1, \ldots, r-1.$$

Put

$$h = f^{[0]} + \sum_{r=1}^{N} f^{[r]}\varepsilon^r \equiv f^{[0]} + U,$$

then

$$v = f_\varepsilon - \sum_{r=0}^{N} f^{[r]}\varepsilon^r \equiv f_\varepsilon - h$$

satisfies the system

$$(I - B)v = \varepsilon[A(v + h) - Ah] + E_\varepsilon, \tag{4.5}$$

where

$$E_\varepsilon = \varepsilon[A(f^{[0]} + U) - A(f^{[0]})] - \sum_{r=2}^{N} P^{[r]}\varepsilon^r.$$

Then, we have the following lemmas.

Lemma 4.1. *The functions* $\pi_j^{[r]}[\Psi]$, $0 \le r \le N-1$ *as above are defined by the following formulas:*

$$\pi_j^{[r]}[\Psi] = \frac{1}{r!}\left.\frac{\partial^r}{\partial\varepsilon^r}\Psi[h_j]\right|_{\varepsilon=0}, 0 \le r \le N-1.$$

Proof. (i) It is easy to see that

$$\begin{aligned}\frac{1}{0!}\left.\frac{\partial^0}{\partial\varepsilon^0}\Psi[h_j]\right|_{\varepsilon=0} &= \Psi[h_j]|_{\varepsilon=0} \\ &= \Psi\left(x, f_j^{[0]}(R_{ijk}(x)), Jf_j^{[0]}(X_{ijk}(x))\right) = \Psi[f_j^{[0]}] = \pi_j^{[0]}[\Psi].\end{aligned}$$

With $r = 1$, we shall show that

$$\pi_j^{[1]}[\Psi] = \frac{1}{1!}\left.\frac{\partial}{\partial\varepsilon}\Psi[h_j]\right|_{\varepsilon=0}. \tag{4.6}$$

We have

$$\frac{\partial}{\partial\varepsilon}\Psi[h_j] = D_2\Psi[h_j]\frac{\partial}{\partial\varepsilon}h_j + D_3\Psi[h_j]\frac{\partial}{\partial\varepsilon}Jh_j. \tag{4.7}$$

On the other hand, from the formulas

$$h_j = \sum_{r=0}^{N} f_j^{[r]}\varepsilon^r, \quad \frac{\partial}{\partial\varepsilon}h_j = \sum_{r=1}^{N} r f_j^{[r]}\varepsilon^{r-1}, \frac{\partial}{\partial\varepsilon}Jh_j = \sum_{r=1}^{N} rJf_j^{[r]}\varepsilon^{r-1},$$

we have

$$h_j|_{\varepsilon=0} = f_j^{[0]}, \left.\frac{\partial}{\partial\varepsilon}h_j\right|_{\varepsilon=0} = f_j^{[1]}, \left.\frac{\partial}{\partial\varepsilon}Jh_j\right|_{\varepsilon=0} = Jf_j^{[1]}. \tag{4.8}$$

Hence, it follows from (4.7), (4.8) that

$$\begin{aligned}\frac{1}{1!}\frac{\partial}{\partial\varepsilon}\Psi[h_j]\Big|_{\varepsilon=0} &= D_2\Psi[h_j]\frac{\partial}{\partial\varepsilon}h_j\Big|_{\varepsilon=0} + D_3\Psi[h_j]\frac{\partial}{\partial\varepsilon}Jh_j\Big|_{\varepsilon=0} \\ &= D_2\Psi[f_j^{[0]}]f_j^{[1]} + D_3\Psi[f_j^{[0]}]Jf_j^{[1]} \\ &= \pi_j^{[0]}[D_2\Psi]f_j^{[1]} + \pi_j^{[0]}[D_3\Psi]Jf_j^{[1]} = \pi_j^{[1]}[\Psi].\end{aligned}$$

Thus, (4.6) holds.

Suppose that we have defined the functions $\pi_j^{[s]}[\Psi]$, $0 \le s \le r-1$ from formulas (4.2), (4.3) and (4.4). Therefore, it follows from (4.7) that

$$\begin{aligned}\frac{\partial^r}{\partial\varepsilon^r}\Psi[h_j] &= \frac{\partial^{r-1}}{\partial\varepsilon^{r-1}}\left(\frac{\partial}{\partial\varepsilon}\Psi[h_j]\right) = \frac{\partial^{r-1}}{\partial\varepsilon^{r-1}}\left[D_2\Psi[h_j]\frac{\partial}{\partial\varepsilon}h_j + D_3\Psi[h_j]\frac{\partial}{\partial\varepsilon}Jh_j\right] \\ &= \sum_{s=0}^{r-1} C_{r-1}^s\left[\frac{\partial^s}{\partial\varepsilon^s}D_2\Psi[h_j]\frac{\partial^{r-s}}{\partial\varepsilon^{r-s}}h_j + \frac{\partial^s}{\partial\varepsilon^s}D_3\Psi[h_j]\frac{\partial^{r-s}}{\partial\varepsilon^{r-s}}Jh_j\right].\end{aligned}$$

We also note that

$$\frac{\partial^s}{\partial\varepsilon^s}h_j\Big|_{\varepsilon=0} = s!f_j^{[s]},\ \frac{\partial^s}{\partial\varepsilon^s}Jh_j\Big|_{\varepsilon=0} = s!Jf_j^{[s]}, 0 \le s \le r.$$

Hence

$$\begin{aligned}\frac{1}{r!}\frac{\partial^r}{\partial\varepsilon^r}\Psi[h_j]\Big|_{\varepsilon=0} &= \frac{1}{r!}\sum_{s=0}^{r-1} C_{r-1}^s\Bigg[\frac{\partial^s}{\partial\varepsilon^s}D_2\Psi[h_j]\Big|_{\varepsilon=0}\frac{\partial^{r-s}}{\partial\varepsilon^{r-s}}h_j\Big|_{\varepsilon=0} \\ &\qquad + \frac{\partial^s}{\partial\varepsilon^s}D_3\Psi[h_j]\Big|_{\varepsilon=0}\frac{\partial^{r-s}}{\partial\varepsilon^{r-s}}Jh_j\Big|_{\varepsilon=0}\Bigg] \\ &= \frac{1}{r!}\sum_{s=0}^{r-1} C_{r-1}^s s!\pi_j^{[s]}[D_2\Psi](r-s)!f_j^{[r-s]} \\ &\quad + \sum_{s=0}^{r-1} C_{r-1}^s s!\pi_j^{[s]}[D_3\Psi](r-s)!Jf_j^{[r-s]} \\ &= \frac{1}{r!}\sum_{s=0}^{r-1} C_{r-1}^s s!(r-s)!\left[\pi_j^{[s]}[D_2\Psi]f_j^{[r-s]} + \pi_j^{[s]}[D_3\Psi]Jf_j^{[r-s]}\right] \\ &= \sum_{s=0}^{r-1}\frac{r-s}{r}\left[\pi_j^{[s]}[D_2\Psi]f_j^{[r-s]} + \pi_j^{[s]}[D_3\Psi]Jf_j^{[r-s]}\right] = \pi_j^{[r]}[\Psi].\end{aligned}$$

Lemma 4.1 is proved completely. ■

Lemma 4.2. *Let* (H_1)–(H_5), (H_8) *hold. Then there exists a constant* $\bar{C}_N^{(1)}$ *depending only on* N, $\|[a_{ijk}]\|$, $\|[b_{ijk}]\|$, $\|f^{[r]}\|_X$, $0 \le r \le N$ *such that*

$$\|E_\varepsilon\|_X \le \bar{C}_N^{(1)} |\varepsilon|^{N+1}.$$

Proof. In the case of $N = 1$, the proof of Lemma 4.2 is easy, hence we omit the details, so we only prove with $N \ge 2$. We have

$$\left(A(f^{[0]} + U) - A(f^{[0]})\right)_i (x) = \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ijk} \left[\Psi[f_j^{[0]} + U_j] - \Psi[f_j^{[0]}]\right], \tag{4.9}$$

in which

$$\Psi[f_j^{[0]}] = \Psi\left(x, f_j^{[0]}(R_{ijk}(x)), Jf_j^{[0]}(X_{ijk}(x))\right),$$
$$\Psi[f_j^{[0]} + U_j] = \Psi\left(x, f_j^{[0]}(R_{ijk}(x)) + U_j(R_{ijk}(x)), J(f_j^{[0]} + U_j)(X_{ijk}(x))\right).$$

By using Maclaurin's expansion of the function $\Psi[h_j]$ round the point $\varepsilon = 0$ up to order N, we obtain

$$\begin{aligned} \Psi[h_j] - \Psi[f_j^{[0]}] &= \Psi[f_j^{[0]} + U_j] - \Psi[f_j^{[0]}] \\ &= \sum_{r=1}^{N-1} \frac{1}{r!} \frac{\partial^r}{\partial \varepsilon^r} \Psi[h_j]\Big|_{\varepsilon=0} \varepsilon^r + \frac{\varepsilon^N}{N!} R_j^{[N]}[\Psi] \\ &= \sum_{r=1}^{N-1} \pi_j^{[r]}[\Psi]\varepsilon^r + \frac{\varepsilon^N}{N!} R_j^{[N]}[\Psi, \theta_1], \end{aligned} \tag{4.10}$$

where $d_j^{[r]}[\Psi]$, $0 \le r \le N-1$ are defined by (4.2), (4.3) and (4.4); $R_j^{[N]}[\Psi, \theta_1]$ is defined as follows

$$R_j^{[N]}[\Psi, \theta_1] = \frac{\partial^N}{\partial \varepsilon^N} \Psi[h_j]\Big|_{\varepsilon=\theta_1\varepsilon}, \tag{4.11}$$

with $0 < \theta_1 < 1$.

Substituting $\Psi[f_j^{[0]} + U_j] - \Psi[f_j^{[0]}]$ in (4.10) into (4.9), we obtain after some rearrangements in order of ε that

$$\begin{aligned} E_{\varepsilon i} &= \varepsilon(A(f^{[0]} + U) - A(f^{[0]}))_i - \sum_{r=2}^{N} P_i^{[r]}\varepsilon^r \\ &= \varepsilon \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ijk} \left[\Psi[f_j^{[0]} + U_j] - \Psi[f_j^{[0]}]\right] - \sum_{r=2}^{N} P_i^{[r]}\varepsilon^r \end{aligned} \tag{4.12}$$

$$= \sum_{r=1}^{N-1} \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ijk} \pi_j^{[r]}[\Psi] \varepsilon^{r+1} + \frac{\varepsilon^{N+1}}{N!} \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ijk} R_j^{[N]}[\Psi, \theta_1] - \sum_{r=2}^{N} P_i^{[r]} \varepsilon^r$$

$$= \sum_{r=1}^{N-1} P_i^{[r+1]} \varepsilon^{r+1} + \frac{\varepsilon^{N+1}}{N!} \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ijk} R_j^{[N]}[\Psi, \theta_1] - \sum_{r=2}^{N} P_i^{[r]} \varepsilon^r$$

$$= \frac{\varepsilon^{N+1}}{N!} \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ijk} R_j^{[N]}[\Psi, \theta_1].$$

By the boundedness of the functions $f^{[r]}$, $r = 0, 1, 2, \ldots, N$, $f^{[r]} \in K_M$, it implies from (4.11), (4.12) that

$$\|E_\varepsilon\|_X = |\varepsilon|^{N+1} \|R_N[\Phi, \varepsilon]\|_X \leq \bar{C}_N^{(1)} |\varepsilon|^{N+1}.$$

Lemma 4.2 is proved completely. ∎

Theorem 4.3. *Let* (H_1)*–*(H_5), (H_8) *hold. Then there exists a constant* $\varepsilon_1 > 0$ *such that, for every* $\varepsilon \in \mathbb{R}$, *with* $|\varepsilon| \leq \varepsilon_1$, *the system* (2.3) *has a unique solution* $f_\varepsilon \in K_M$ *satisfying the asymptotic estimation up to order* $N+1$ *as follows*

$$\left\| f_\varepsilon - \sum_{r=0}^{N} f^{[r]} \varepsilon^r \right\|_X \leq \frac{2}{1 - \|[b_{ijk}]\|} \bar{C}_N^{(1)} |\varepsilon|^{N+1},$$

where the functions $f^{[r]}$, $r = 0, 1, \ldots, N$ *are defined by* (4.1).

Proof. From (4.5), Lemmas 2.1 and 4.2, we have

$$\begin{aligned} \|v\|_X &\leq \left\|(I-B)^{-1}\right\| \left(|\varepsilon| \|A(v+h) - Ah\|_X + \|E_\varepsilon\|_X\right) \\ &\leq \frac{1}{1 - \|[b_{ijk}]\|} \left(\varepsilon_1 \|A(v+h) - Ah\|_X + \bar{C}_N^{(1)} |\varepsilon|^{N+1}\right). \end{aligned} \tag{4.13}$$

On the other hand

$$\begin{aligned} &\|v+h\|_X = \|f_\varepsilon\|_X \leq N_1, \ \|h\|_X \leq \sum_{r=0}^{N} \left\| f^{[r]} \right\|_X \equiv \bar{N}_1, \\ &\|Jv + Jh\|_X = \|Jf_\varepsilon\|_X \leq N_2, \|Jh\|_X \leq \sum_{r=0}^{N} \left\| Jf^{[r]} \right\|_X \equiv \bar{N}_2. \end{aligned} \tag{4.14}$$

It follows from (4.14) that

$$\|A(v+h) - Ah\|_X \leq C_2(M) \|[a_{ijk}]\| \|v\|_X, \tag{4.15}$$

where

$C_2(M)$

$$= \sup\left\{\left|\frac{\partial\Psi}{\partial y}(x,y,z)\right| + \left|\frac{\partial\Psi}{\partial z}(x,y,z)\right| : x \in \Omega, |y| \leq N_1 + \bar{N}_1, |z| \leq N_2 + \bar{N}_2\right\}.$$

From (4.13), (4.15), we obtain

$$\|v\|_X \leq \frac{1}{1-\|[b_{ijk}]\|}\left(\varepsilon_1 C_2(M)\,\|[a_{ijk}]\|\,\|v\|_X + \bar{C}_N^{(1)}\,|\varepsilon|^{N+1}\right).$$

Choose $0 < \varepsilon_1 < \varepsilon_0$, such that

$$\varepsilon_1 C_2(M)\,\|[a_{ijk}]\|\,\frac{1}{1-\|[b_{ijk}]\|} \leq 1/2. \tag{4.16}$$

So, (4.16) leads to

$$\|v\|_X \leq \frac{2}{1-\|[b_{ijk}]\|}\bar{C}_N^{(1)}\,|\varepsilon|^{N+1},$$

or

$$\left\|f_\varepsilon - \sum_{r=0}^{N} f^{[r]}\varepsilon^r\right\|_X \leq \frac{2}{1-\|[b_{ijk}]\|}\bar{C}_N^{(1)}\,|\varepsilon|^{N+1}.$$

Theorem 4.3 is proved completely. ∎

5. Examples

Let us give two following illustrated examples for the results obtained as above.

5.1. Example 1. Let us consider system (1.1) with $n = 2$, $m = 1$, $\Psi(x,y,z) = \cos y \sin z$:

$$\begin{cases} f_1(x) = \varepsilon a_{11}\cos\left(f_1\left(\frac{x}{3}+\frac{2}{3}\right)\right)\sin\left(\int_0^x f_1(t)dt\right) \\ \qquad + \varepsilon a_{12}\cos\left(f_2\left(\frac{x}{3}-\frac{2}{3}\right)\right)\sin\left(\int_0^{x^3} f_2(t)dt\right) \\ \qquad + b_{11}f_1(\frac{2x}{3}+\frac{1}{3}) + b_{12}f_2(\frac{x}{2}-\frac{1}{2}) + g_1(x), \\ f_2(x) = \varepsilon a_{21}\cos\left(f_1\left(\frac{x}{4}+\frac{3}{4}\right)\right)\sin\left(\int_0^{x^5} f_1(t)dt\right) \\ \qquad + \varepsilon a_{22}\cos\left(f_2\left(\frac{x}{4}-\frac{3}{4}\right)\right)\sin\left(\int_0^x f_2(t)dt\right) \\ \qquad + b_{21}f_1\left(\frac{x}{2}+\frac{1}{2}\right) + b_{22}f_2\left(\frac{2x}{3}-\frac{1}{3}\right) + g_2(x), \\ x \in \Omega = [-1,1], \end{cases} \tag{5.1}$$

where $\varepsilon > 0$ is small enough; $a_{ijk} \equiv a_{ij}$, $b_{ijk} \equiv b_{ij}$ are constants and all the functions g_1, $g_2 : \Omega \to \mathbb{R}$; $R_{ijk} \equiv R_{ij}$, $S_{ijk} \equiv S_{ij}$, $X_{ijk} \equiv X_{ij} : \Omega \to \Omega$ are continuous defined respectively as follows

$$\begin{cases} a_{ij} \in \mathbb{R}; \\ b_{ij} \ \in \mathbb{R} \ \text{ such that} \\ \|[b_{ij}]\| = \sum_{i=1}^{2} \max\limits_{1\le j\le 2} |b_{ij}| = \max\limits_{1\le j\le 2} |b_{1j}| + \max\limits_{1\le j\le 2} |b_{2j}| < 1; \end{cases}$$

$$\begin{cases} g_1,\, g_2 \in C(\Omega;\mathbb{R}); \\ [R_{ij}(x)] = \begin{bmatrix} R_{11}(x) & R_{12}(x) \\ R_{21}(x) & R_{22}(x) \end{bmatrix} = \begin{bmatrix} \frac{x}{3}+\frac{2}{3} & \frac{x}{3}-\frac{2}{3} \\ \frac{x}{4}+\frac{3}{4} & \frac{x}{4}-\frac{3}{4} \end{bmatrix}; \\ [S_{ij}(x)] = \begin{bmatrix} S_{11}(x) & S_{12}(x) \\ S_{21}(x) & S_{22}(x) \end{bmatrix} = \begin{bmatrix} \frac{2x}{3}+\frac{1}{3} & \frac{x}{2}-\frac{1}{2} \\ \frac{x}{2}+\frac{1}{2} & \frac{2x}{3}-\frac{1}{3} \end{bmatrix}; \\ [X_{ij}(x)] = \begin{bmatrix} X_{11}(x) & X_{12}(x) \\ X_{21}(x) & X_{22}(x) \end{bmatrix} = \begin{bmatrix} x & x^3 \\ x^5 & x \end{bmatrix}. \end{cases}$$

It is obvious that (H_1)–(H_5) hold with $M > \frac{2\|g\|_X}{1-\|[b_{ij}]\|}$ and $0 < \varepsilon_0 < \frac{1-\|[b_{ij}]\|}{4\|[a_{ij}]\|}$. So we conclude that, for every ε with $|\varepsilon| < \varepsilon_0$, equation (5.1) has a unique solution $f \in K_M$.

On the other hand, because $\Psi(x,y,z) = \Psi(y,z) = \cos y \sin z$, (H_6) and (H_8) are also satisfied. Therefore, if $M > \frac{2\|g\|_X}{1-\|[b_{ij}]\|}$, $0 < \varepsilon_0 < \frac{1-\|[b_{ij}]\|-\frac{1}{M}\|g\|_X}{2(2+3M)\|[a_{ij}]\|}$ and $\varepsilon > 0$ is small enough, then we obtain the results as in Theorems 3.2 and 4.3.

5.2. Example 2. Consider system (1.1) with $n = m = 2$, $\Psi(x,y,z) = \Phi(z)$, $\Phi \in C^1(\mathbb{R})$:

$$(5.2)\quad \begin{cases} f_1(x) = \varepsilon a_{11}\Phi\left(\int_0^x f_1(t)dt\right) + \varepsilon a_{12}\Phi\left(\int_0^{x^3} f_2(t)dt\right) + b_{111} f_1\left(\frac{x+1}{2}\right) \\ \qquad + b_{112} f_1(\cos \pi x) + b_{122} f_2\left(\frac{2x+1}{3}\right) + g_1(x), \\ f_2(x) = b_{211} f_1\left(\frac{x-1}{2}\right) + b_{221} f_2(\sin \pi x) + b_{222} f_2\left(\frac{2x-1}{3}\right) + g_2(x), \\ x \in \Omega = [-1,1], \end{cases}$$

where $\varepsilon > 0$ is small enough; $a_{ijk} \equiv a_{ij}$, b_{ijk} are constants and all the functions g_1, $g_2 : \Omega \to \mathbb{R}$; $R_{ijk} \equiv R_{ij}$, S_{ijk}, $X_{ijk} \equiv X_{ij} : \Omega \to \Omega$ are continuous defined respectively as follows

$$\begin{cases} a_{ij} \in \mathbb{R} \text{ such that} \\ [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \end{bmatrix}; \\ b_{ijk} \in \mathbb{R} \text{ such that} \\ [b_{ijk}] = \begin{bmatrix} b_{111} & b_{121} & b_{112} & b_{122} \\ b_{211} & b_{221} & b_{212} & b_{222} \end{bmatrix} = \begin{bmatrix} b_{111} & 0 & b_{112} & b_{122} \\ b_{211} & b_{221} & 0 & b_{222} \end{bmatrix}, \\ \|[b_{ijk}]\| = \sum_{i=1}^{2} \sum_{k=1}^{2} \max_{1 \leq j \leq 2} |b_{ijk}| \\ \qquad = |b_{111}| + |b_{222}| + \max\{|b_{112}|, |b_{122}|\} + \max\{|b_{211}|, |b_{221}|\} < 1; \end{cases}$$

$$\begin{cases} g_1, g_2 \in C(\Omega; \mathbb{R}); \\ [S_{ijk}(x)] = \begin{bmatrix} S_{111}(x) & S_{121}(x) & S_{112}(x) & S_{122}(x) \\ S_{211}(x) & S_{221}(x) & S_{212}(x) & S_{222}(x) \end{bmatrix} \\ \qquad = \begin{bmatrix} \frac{x+1}{2} & 0 & \cos \pi x & \frac{2x+1}{3} \\ \frac{x-1}{2} & \sin \pi x & 0 & \frac{2x-1}{3} \end{bmatrix}; \\ [X_{ij}(x)] = \begin{bmatrix} X_{11}(x) & X_{12}(x) \\ X_{21}(x) & X_{22}(x) \end{bmatrix} = \begin{bmatrix} x & x^3 \\ 0 & 0 \end{bmatrix}. \end{cases}$$

It is also clear to see that (H_1)–(H_5) hold with $M > \frac{2\|g\|_X}{1-\|[b_{ijk}]\|}$ and $0 < \varepsilon_0 < \frac{M(1-\|[b_{ijk}]\|)}{4\|[a_{ij}]\| \left[M \sup_{|z| \leq M} |\Phi'(z)| + |\Phi(0)| \right]}$. So, for every ε such that $|\varepsilon| < \varepsilon_0$, equation (5.2) has a unique solution $f \in K_M$.

Furthermore, if $\Phi \in C^2(\mathbb{R})$ or $\Phi \in C^N(\mathbb{R})$, $M > \frac{2\|g\|_X}{1-\|[b_{ijk}]\|}$ and

$$0 < 2\varepsilon_0 \|[a_{ij}]\| \left[\frac{|\Phi(0)|}{M} + \sup_{|z| \leq M} |\Phi'(z)| + M \sup_{|z| \leq M} |\Phi''(z)| \right]$$
$$< 1 - \|[b_{ijk}]\| - \frac{1}{M} \|g\|_X$$

and $\varepsilon > 0$ is small enough, the results of Theorems 3.2 and 4.3 are obtained.

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