DE GRUYTER
OPEN

DEMONSTRATIO MATHEMATICA
Vol. XLVII No 1 2014

Figen Özpinar, Sermin Öztürk and Zeynep Fidan Koçak

OSCILLATION FOR CERTAIN IMPULSIVE PARTIAL DIFFERENCE EQUATIONS

Abstract. In this paper, we obtain some sufficient criteria for the oscillation of the solutions of linear impulsive partial difference equations with continuous variables.

1. Introduction

The impulsive differential equations are adequate mathematical models of various processes and phenomena studied in physics, chemical technology, population dynamics, technics and economics. These equations provide natural mathematical description of processes which are subject to short-time perturbations during their evolution. Various monographs [2, 4, 5, 6, 7, 9, 10], devoted to the impulsive differential equations were published in the recent years. At the same time, partial difference equations arise in applications, involving populations dynamics with spatial migrations, chemical reactions, mathematical physics and finite-difference schemes. Recently, there are many papers that devoted to the development of qualitative theory of partial difference equations with continuous variables, see e.g. [8, 11, 13].

Let $0 = x_0 < x_1 < \ldots < x_n < x_{n+1} < \ldots$, be fixed points with $\lim_{n\to\infty} x_n = \infty$, and let for $n \in \mathbb{N}$ $x_{n+r} = x_n + \tau$, where r is a fixed natural number, and $\tau > 0$ is a constant. Define $J_{imp} = \{x_n\}_{n=1}^{\infty}$, $\mathbb{R}^+ = [0, \infty)$, $J = \{(x, y) : x \in J_{imp}, y \in \mathbb{R}^+\}$.

In 2009, Agarwal and Karakoc studied the oscillation of solutions of impulsive partial difference equations with continuous variables of the type

$$\begin{aligned} p_1 z(x+a, y+b) + p_2 z(x+a, y) + p_3 z(x, y+b) - p_4 z(x, y) \\ + P(x, y) z(x-\tau, y-\sigma) = 0, \qquad (x, y) \in (\mathbb{R}^+ \times \mathbb{R}^+)\backslash J, \end{aligned}$$

2010 *Mathematics Subject Classification*: 39A11, 34K11, 34C10.
Key words and phrases: partial difference equation, impulsive PDEs, oscillation, continuous variables.

DOI: 10.2478/dema-2014-0007

$$z(x_n^+, y) - z(x_n^-, y) = L_n z(x_n^-, y), \qquad (x_n, y) \in J,$$

where $z(x^+, y) = \lim_{\substack{(q,s)\to(x,y) \\ q>x}} z(q,s)$, $z(x^-, y) = \lim_{\substack{(q,s)\to(x,y) \\ q<x}} z(q,s)$ [2].

In this paper, we shall consider the impulsive partial difference equations with continuous variables of the form

$$\begin{aligned}(1)\qquad & c_1 u(x+a, y+b) + c_2 u(x+a, y) + c_3 u(x, y+b) - c_4 u(x,y) \\ & + F(x,y)u(x-\tau, y) + G(x,y)u(x, y-\sigma) + H(x,y)u(x-\tau, y-\sigma) = 0, \\ & \qquad (x,y) \in (\mathbb{R}^+ \times \mathbb{R}^+)\backslash J,\end{aligned}$$

$$(2) \qquad u(x_n^+, y) - u(x_n^-, y) = L_n u(x_n^-, y), \qquad (x_n, y) \in J,$$

where $u(x^+, y) = \lim_{\substack{(q,s)\to(x,y) \\ q>x}} u(q,s)$, $u(x^-, y) = \lim_{\substack{(q,s)\to(x,y) \\ q<x}} u(q,s)$ and $F, G, H \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+ - \{0\})$, a, b, τ, σ are positive constants, c_i, $i = 1, 2, 3, 4$, are nonnegative constants.

Our results extend the oscillation results of Agarwal and Karakoc [2].

A function $u(x,y)$ in $[-\tau, \infty) \times [-\sigma, \infty)$ is said to be solution of (1)–(2) if

(i) for $(x,y) \in (\mathbb{R}^+ \times \mathbb{R}^+)\backslash J$, u is continuous and satisfies (1),
(ii) for $(x,y) \in J$, $u(x^+, y)$ and $u(x^-, y)$ exist, $u(x^-, y) = u(x,y)$, and satisfy (2).

A solution $u(x,y)$ of (1)–(2) is said to be eventually positive (or negative) if $u(x,y) > 0$ (or $u(x,y) < 0$) for all large x and y. It is said to be oscillatory if it is neither eventually positive nor eventually negative otherwise, it is called nonoscillatory.

2. Preparatory lemmas

Throughout this paper, we shall assume that

(A1) $\{L_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} L_n < \infty$,
(A2) $c_2, c_3 \geq c_4 > 0$,
(A3) $\tau = ka + \theta$, $\sigma = lb + \eta$, where k, l are nonnegative integers, $\theta \in [0, a)$ and $\eta \in [0, b)$.

We set, for any $(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+$,

$$P(x,y) = \min\{F(s,t), x \leq s \leq x+a, y \leq t \leq y+b\} \quad \text{and} \quad 0 < \limsup_{x,y\to\infty} P(x,y) < \infty,$$

$$Q(x,y) = \min\{G(s,t), x \leq s \leq x+a, y \leq t \leq y+b\} \quad \text{and} \quad 0 < \limsup_{x,y\to\infty} Q(x,y) < \infty,$$

$$R(x,y)=\min\{H(s,t), x\le s\le x+a, y\le t\le y+b\} \quad \text{and} \quad 0<\limsup_{x,y\to\infty} R(x,y)<\infty,$$

$$T(x,y)=\min\{P(x,y),Q(x,y),R(x,y)\},$$

$$E_P=\Big\{\lambda>0: c_4\prod_{x_0<x_m<x+a}(1+L_m)-\lambda P(x,y)>0, \text{eventually}\Big\},$$

$$E_Q=\Big\{\lambda>0: c_4\prod_{x_0<x_m<x+a}(1+L_m)-\lambda Q(x,y)>0, \text{eventually}\Big\},$$

$$E_R=\Big\{\lambda>0: c_4\prod_{x_0<x_m<x+a}(1+L_m)-\lambda R(x,y)>0, \text{eventually}\Big\},$$

$$E=\Big\{\lambda>0: c_4\prod_{x_0<x_m<x+a}(1+L_m)-\lambda T(x,y)>0, \text{eventually}\Big\}.$$

Here, the symbol $\prod_{x_0<x_m<s} a_m$ denotes the product of the members of the sequence $\{a_m\}$ over m such that $x_m\in(x_0,s)\cap J_{imp}$. If $(x_0,s)\cap J_{imp}=\varnothing$, or $x_0\ge s$ then we assume that $\prod_{x_0<x_m<s} a_m=1$.

Lemma 1. *Assume that $u(x,y)$ is an eventually positive solution of (1)–(2). Then for $(x,y)\in\mathbb{R}^+\times\mathbb{R}^+$ the function*

$$\omega(x,y)=\int_x^{x+a}\int_y^{y+b}\Big(\prod_{x_0<x_m<s}(1+L_m)^{-1}\Big)u(s,t)\mathrm{d}t\mathrm{d}s \tag{3}$$

is an eventually positive solution of the partial difference inequality

$$\begin{aligned}(4)\quad & c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b)\\ &-c_4\prod_{x_0<x_m<x+a}(1+L_m)\omega(x,y)+P(x,y)\omega(x-ka,y)+Q(x,y)\omega(x,y-lb)\\ &\qquad +R(x,y)\omega(x-ka,y-lb)\le 0.\end{aligned}$$

Proof. So that

$$\prod_{x_0<x_m<x}(1+L_m)^{-1}u(x,y)$$

is continuous for $(x,y)\in\mathbb{R}^+\times\mathbb{R}^+$,

$$\begin{aligned}(5)\quad & \frac{\partial\omega}{\partial x}\\ &=\int_y^{y+b}\Big[\prod_{x_0<x_m<x+a}(1+L_m)^{-1}u(x+a,t)-\prod_{x_0<x_m<x+a}(1+L_m)^{-1}u(x,t)\Big]\mathrm{d}t\\ &=\int_y^{y+b}\prod_{x_0<x_m<x}(1+L_m)^{-1}\Big[\prod_{x\le x_m<x+a}(1+L_m)^{-1}u(x+a,t)-u(x,t)\Big]\mathrm{d}t,\end{aligned}$$

$$\frac{\partial \omega}{\partial y} = \int_x^{x+a} \prod_{x_0<x_m<s} \left(1+L_m\right)^{-1} [u(s,y+b)-u(s,y)]\mathrm{d}s. \tag{6}$$

Since $u(x,y)$ is an eventually positive solution of (1)–(2),

$$c_1 u(x+a,y+b) + c_2 u(x+a,y) + c_3 u(x,y+b) - c_4 u(x,y) < 0, \tag{7}$$

for $(x,y) \in (\mathbb{R}^+ \times \mathbb{R}^+)\backslash J$. From (A2) and (7)

$$u(x+a,y) - u(x,y) < 0 \qquad \text{and} \qquad u(x,y+b) - u(x,y) < 0,$$

eventually. Moreover since $0 < \prod_{x \le x_m < x+a} \left(1+L_m\right)^{-1} \le 1$, we obtain

$$\prod_{x \le x_m < x+a} \left(1+L_m\right)^{-1} u(x+a,y) - u(x,y) < 0. \tag{8}$$

Let $(x,y) \in J$ and, say, $x = x_n$. From (2) and (8), we have

$$\begin{aligned} u(x_n,y) &= \frac{1}{1+L_n} u(x_n^+,y) \ge \frac{1}{1+L_n} \prod_{x_n^+ \le x_m < x_n^+ + a} \left(1+L_m\right)^{-1} u(x_n^+ + a, y) \\ &= \prod_{x_n \le x_m < x_n + a} \left(1+L_m\right)^{-1} u(x_n + a, y). \end{aligned}$$

So, for all $(x,y) \in (\mathbb{R}^+ \times \mathbb{R}^+)$ $\frac{\partial \omega}{\partial x} \le 0$ and $\frac{\partial \omega}{\partial y} \le 0$. Therefore

$$\omega(x-\tau, y-\sigma) = \omega(x-(ka+\theta), y-(lb+\eta) \ge \omega(x-ka, y-lb). \tag{9}$$

Since integrating (1), $0 < \prod_{x_0 < x_m < x+a} \left(1+L_m\right)^{-1} \le 1$, we find

$$\begin{aligned} 0 = c_1 &\int_x^{x+a} \int_y^{y+b} u(s+a,t+b)\mathrm{d}t\mathrm{d}s + c_2 \int_x^{x+a} \int_y^{y+b} u(s+a,t)\mathrm{d}t\mathrm{d}s \\ &+ c_3 \int_x^{x+a} \int_y^{y+b} u(s,t+b)\mathrm{d}t\mathrm{d}s - c_4 \int_x^{x+a} \int_y^{y+b} u(s,t)\mathrm{d}t\mathrm{d}s \\ &+ \int_x^{x+a} \int_y^{y+b} F(s,t)u(s-\tau,t)\mathrm{d}t\mathrm{d}s + \int_x^{x+a} \int_y^{y+b} G(s,t)u(s,t-\sigma)\mathrm{d}t\mathrm{d}s \\ &+ \int_x^{x+a} \int_y^{y+b} H(s,t)u(s-\tau,t-\sigma)\mathrm{d}t\mathrm{d}s, \end{aligned}$$

which yields

$$0 \ge c_1 \int_x^{x+a} \int_y^{y+b} \prod_{x_0<x_m<s} \left(1+L_m\right)^{-1} u(s+a,t+b)\mathrm{d}t\mathrm{d}s \tag{10}$$

$$+c_2\int_x^{x+a}\int_y^{y+b}\prod_{x_0<x_m<s}\big(1+L_m\big)^{-1}u(s+a,t)\mathrm{d}t\mathrm{d}s$$
$$+c_3\int_x^{x+a}\int_y^{y+b}\prod_{x_0<x_m<s}\big(1+L_m\big)^{-1}u(s,t+b)\mathrm{d}t\mathrm{d}s-c_4\int_x^{x+a}\int_y^{y+b}u(s,t)\mathrm{d}t\mathrm{d}s$$
$$+\int_x^{x+a}\int_y^{y+b}F(s,t)\prod_{x_0<x_m<s}\big(1+L_m\big)^{-1}u(s-\tau,t)\mathrm{d}t\mathrm{d}s$$
$$+\int_x^{x+a}\int_y^{y+b}G(s,t)\prod_{x_0<x_m<s}\big(1+L_m\big)^{-1}u(s,t-\sigma)\mathrm{d}t\mathrm{d}s$$
$$+\int_x^{x+a}\int_y^{y+b}H(s,t)\prod_{x_0<x_m<s}\big(1+L_m\big)^{-1}u(s-\tau,t-\sigma)\mathrm{d}t\mathrm{d}s.$$

From (9), (10) and using the definition of ω, P, Q, R, we obtain eventually

$$\begin{aligned}&c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b)\\&-c_4\prod_{x_0<x_m<x+a}(1+L_m)\omega(x,y)+P(x,y)\omega(x-ka,y)+Q(x,y)\omega(x,y-lb)\\&\qquad+R(x,y)\omega(x-ka,y-lb)\le 0.\end{aligned}$$

Therefore, $\omega(x,y)$ is an eventually positive solution of inequality (4). ∎

3. Main results

Theorem 1. *Assume that there exist $X_0\ge 0, Y_0\ge 0$ such that if $k>l>0$,*

$$\begin{aligned}(11)\qquad &\inf_{\lambda\in E,x\ge X_0,y\ge Y_0}\frac{1}{\lambda}\Bigg\{\Big(\frac{c_2}{c_4}\Big)^k\prod_{i=1}^{k}\Big(\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)^{-1}\Big)\\&+\Big(\frac{c_3}{c_4}\Big)^l\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l}\\&+c_2^{k-l}\prod_{i=1}^{l}\Big(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}\big(1+L_m\big)^{-1}\Big)\\&\times\Big[\prod_{i=1}^{l}\Big(c_4\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)-\lambda T\big(x-ia,y-ib\big)\Big)\\&\times\prod_{j=1}^{k-l}\Big(c_4\prod_{x_0<x_m<x-(l+j-1)a}\big(1+L_m\big)-\lambda T\big(x-(l+j)a,y-lb\big)\Big)\Big]^{-1}\Bigg\}>1,\end{aligned}$$

if $l>k>0$,

$$\inf_{\lambda\in E, x\geq X_0, y\geq Y_0} \frac{1}{\lambda}\Bigg\{\left(\frac{c_2}{c_4}\right)^k \prod_{i=1}^{k}\Bigg(\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)^{-1}\Bigg)$$
$$+\left(\frac{c_3}{c_4}\right)^l \prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l}$$
$$+c_3^{l-k}\prod_{i=1}^{k}\Bigg(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}\big(1+L_m\big)^{-1}\Bigg)$$
$$\times\Bigg[\prod_{i=1}^{k}\Bigg(c_4\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)-\lambda T\big(x-ia,y-ib\big)\Bigg)$$
$$\times\prod_{j=1}^{l-k}\Bigg(c_4\prod_{x_0<x_m<x-(k-1)a}\big(1+L_m\big)-\lambda T\big(x-ka,y-(k+j)b\big)\Bigg)\Bigg]^{-1}\Bigg\}>1 \tag{12}$$

and if $k=l>0$,

$$\inf_{\lambda\in E, x\geq X_0, y\geq Y_0} \frac{1}{\lambda}\Bigg\{\left(\frac{c_2}{c_4}\right)^k \prod_{i=1}^{k}\Bigg(\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)^{-1}\Bigg)$$
$$+\left(\frac{c_3}{c_4}\right)^k \prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-k}$$
$$+\prod_{i=1}^{k}\Bigg(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}\big(1+L_m\big)^{-1}\Bigg)$$
$$\times\Bigg[\prod_{i=1}^{k}\Bigg(c_4\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)-\lambda T\big(x-ia,y-ib\big)\Bigg)\Bigg]^{-1}\Bigg\}>1. \tag{13}$$

Then every solution of (1)–(2) *is oscillatory.*

Proof. Suppose to the contrary that $u(x,y)$ is nonoscillatory solution of (1)–(2). Without loss of generality we may assume that $u(x,y)$ is an eventually positive solution of (1)–(2). Let $\omega(x,y)$ be as in Lemma 1.

Set

$$S(\lambda)=\Bigg\{\lambda>0:\ c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b)$$
$$-\Bigg(c_4\prod_{x_0<x_m<x+a}\big(1+L_m\big)-\lambda T(x,y)\Bigg)\omega(x,y)\leq 0, \text{eventually}\Bigg\}.$$

Since $\frac{\partial\omega}{\partial x}<0$ and $\frac{\partial\omega}{\partial y}<0$, from (4) we obtain

$$c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b)-\Big(c_4\prod_{x_0<x_m<x+a}\big(1+L_m\big)-T(x,y)\Big)\omega(x,y)\le 0,$$

which implies that $1 \in S(\lambda)$. That is $S(\lambda)$ is nonempty. If $\lambda_0 \in S(\lambda)$, we have eventually

$$\begin{aligned}&c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b)\\&\qquad\le\Big(c_4\prod_{x_0<x_m<x+a}\big(1+L_m\big)-\lambda_0T(x,y)\Big)\omega(x,y).\end{aligned}$$

Therefore $c_4\prod_{x_0<x_m<x+a}\big(1+L_m\big)-\lambda_0T(x,y)>0$ eventually, i.e., $\lambda_0 \in E$. Thus $S(\lambda)\subset E$.

Let $\mu \in S(\lambda)$. Noting that $\frac{\partial\omega}{\partial x}<0$ and $\frac{\partial\omega}{\partial y}<0$ eventually, we obtain eventually

$$\begin{aligned}(14)\qquad&\left(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x+2a}\big(1+L_m\big)^{-1}\right)\omega(x+a,y+b)\\&\le c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b)\\&\le\left(c_4\prod_{x_0<x_m<x+a}\big(1+L_m\big)-\mu T(x,y)\right)\omega(x,y).\end{aligned}$$

and thus,

$$\begin{aligned}(15)\qquad&\left(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-1}\right)\omega(x,y)\\&\le\left(c_4\prod_{x_0<x_m<x}\big(1+L_m\big)-\mu T(x-a,y-b)\right)\omega(x-a,y-b).\end{aligned}$$

If $k>l>0$ then

$$\begin{aligned}(16)\qquad&\omega(x,y)\le\prod_{i=1}^{l}\left(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}\big(1+L_m\big)^{-1}\right)^{-1}\\&\times\prod_{i=1}^{l}\left(c_4\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)-\mu T(x-ia,y-ib)\right)\omega(x-la,y-lb).\end{aligned}$$

Since $\omega(x,y)>0$ for all large x and y then from (4)

$$c_2\omega(x+a,y)\le c_4\prod_{x_0<x_m<x+a}\big(1+L_m\big)\omega(x,y),$$

$$c_3\omega(x,y+b)\le c_4\prod_{x_0<x_m<x+a}\big(1+L_m\big)\omega(x,y),$$

eventually. Hence

$$\prod_{i=1}^{k} c_2\omega(x-(i-1)a,y) \le \prod_{i=1}^{k}\Big[c_4 \prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)\omega(x-ia,y)\Big] \tag{17}$$

$$c_2^k\omega(x,y) \le c_4^k\prod_{i=1}^{k}\Big[\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)\Big]\omega(x-ka,y)$$

and

$$\prod_{i=1}^{l} c_3\omega(x,y(i-1)b) \le \prod_{i=1}^{l}\Big[c_4 \prod_{x_0<x_m<x+a}\big(1+L_m\big)\omega(x,y-ib)\Big] \tag{18}$$

$$c_3^l\omega(x,y) \le c_4^l\prod_{x_0<x_m<x+a}\big(1+L_m\big)^l\omega(x,y-lb).$$

Moreover, from (14)

$$c_2\omega(x+a,y) \le \Big[c_4\prod_{x_0<x_m<x+a}\big(1+L_m\big)-\mu T(x,y)\Big]\omega(x,y)$$

and

$$\begin{aligned} &\omega(x-la,y-lb) \\ &\le \frac{1}{c_2}\Big[c_4\prod_{x_0<x_m<x-la}\big(1+L_m\big)-\mu T(x-(l+1)a,y-lb)\Big]\omega(x-(l+1)a,y-lb) \\ &\le \cdots \\ &\le \frac{1}{c_2^{k-l}}\prod_{j=1}^{k-l}\Bigg[c_4\prod_{x_0<x_m<x-(l+j-1)a}\big(1+L_m\big)-\mu T(x-(l+j)a,y-lb)\Bigg] \\ &\qquad\times\omega(x-ka,y-lb), \end{aligned} \tag{19}$$

eventually. Substituting (16)–(19) into (4), we obtain

$$\begin{aligned} &c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b)-c_4\prod_{x_0<x_m<x+a}\big(1+L_m\big)\omega(x,y) \\ &\quad+\Big(\frac{c_2}{c_4}\Big)^k P(x,y)\prod_{i=1}^{k}\Bigg(\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)^{-1}\Bigg)\omega(x,y) \\ &\quad+\Big(\frac{c_3}{c_4}\Big)^l Q(x,y)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l}\omega(x,y) \\ &\quad+c_2^{k-l}R(x,y)\prod_{i=1}^{l}\Bigg(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}\big(1+L_m\big)^{-1}\Bigg) \end{aligned}$$

$$\times\Bigg[\prod_{i=1}^{l}\Bigg(c_4\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)-\mu T(x-ia,y-ib)\Bigg)$$
$$\times\prod_{j=1}^{k-l}\Bigg(c_4\prod_{x_0<x_m<x-(l+j-1)a}\big(1+L_m\big)-\mu T(x-(l+j)a,y-lb)\Bigg)\Bigg]^{-1}\omega(x,y)\le 0,$$

eventually. Then

$$c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b)-\Bigg\{c_4\prod_{x_0<x_m<x+a}\big(1+L_m\big)$$
$$-\Bigg[\bigg(\frac{c_2}{c_4}\bigg)^kP(x,y)\prod_{i=1}^{k}\Bigg(\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)^{-1}\Bigg)$$
$$+\bigg(\frac{c_3}{c_4}\bigg)^lQ(x,y)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l}$$
$$+c_2^{k-l}R(x,y)\prod_{i=1}^{l}\Bigg(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}\big(1+L_m\big)^{-1}\Bigg)$$
$$\times\Bigg[\prod_{i=1}^{l}\Bigg(c_4\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)-\mu T(x-ia,y-ib)\Bigg)$$
$$\times\prod_{j=1}^{k-l}\Bigg(c_4\prod_{x_0<x_m<x-(l+j-1)a}\big(1+L_m\big)-\mu T(x-(l+j)a,y-lb)\Bigg)\Bigg]^{-1}\Bigg]\Bigg\}\omega(x,y)$$
$$\le 0,$$

eventually. Thus

$$c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b)-\Bigg\{c_4\prod_{x_0<x_m<x+a}\big(1+L_m\big)$$
$$-T(x,y)\Bigg[\bigg(\frac{c_2}{c_4}\bigg)^k\prod_{i=1}^{k}\Bigg(\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)^{-1}\Bigg)$$
$$+\bigg(\frac{c_3}{c_4}\bigg)^l\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l}$$
$$+c_2^{k-l}\prod_{i=1}^{l}\Bigg(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}\big(1+L_m\big)^{-1}\Bigg)$$

$$\times\Bigg[\sup_{x\geq X_0,y\geq Y_0}\prod_{i=1}^{l}\Bigg(c_4\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)-\mu T(x-ia,y-ib)\Bigg)$$
$$\times\prod_{j=1}^{k-l}\Bigg(c_4\prod_{x_0<x_m<x-(l+j-1)a}\big(1+L_m\big)-\mu T(x-(l+j)a,y-lb)\Bigg)\Bigg]^{-1}\Bigg]\Bigg\}\omega(x,y)$$
$$\leq 0,$$

eventually. Therefore

$$\lambda_0=\left(\frac{c_2}{c_4}\right)^k\prod_{i=1}^{k}\Bigg(\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)^{-1}\Bigg)+\left(\frac{c_3}{c_4}\right)^l\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l}$$
$$+c_2^{k-l}\prod_{i=1}^{l}\Bigg(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}\big(1+L_m\big)^{-1}\Bigg)$$
$$\times\Bigg[\sup_{x\geq X_0,y\geq Y_0}\prod_{i=1}^{l}\Bigg(c_4\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)-\mu T(x-ia,y-ib)\Bigg)$$
$$\times\prod_{j=1}^{k-l}\Bigg(c_4\prod_{x_0<x_m<x-(l+j-1)a}\big(1+L_m\big)-\mu T(x-(l+j)a,y-lb)\Bigg)\Bigg]^{-1}\in S(\lambda).$$

However, on the other hand, from (11) there exists an $\alpha_1\in(1,\infty)$ such that

$$\inf_{\lambda\in E,x\geq X_0,y\geq Y_0}\frac{1}{\lambda}\Bigg\{\left(\frac{c_2}{c_4}\right)^k\prod_{i=1}^{k}\Bigg(\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)^{-1}\Bigg)$$
$$+\left(\frac{c_3}{c_4}\right)^l\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l}$$
$$+c_2^{k-l}\prod_{i=1}^{l}\Bigg(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}\big(1+L_m\big)^{-1}\Bigg)$$
$$\times\Bigg[\prod_{i=1}^{l}\Bigg(c_4\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)-\lambda T(x-ia,y-ib)\Bigg)$$
$$\times\prod_{j=1}^{k-l}\Bigg(c_4\prod_{x_0<x_m<x-(l+j-1)a}\big(1+L_m\big)-\lambda T(x-(l+j)a,y-lb)\Bigg)\Bigg]^{-1}\Bigg\}\geq\alpha_1>1.$$

Therefore, for $\lambda=\mu$, we obtain

$$\inf_{\lambda\in E, x\geq X_0, y\geq Y_0}\Bigg\{\left(\frac{c_2}{c_4}\right)^k\prod_{i=1}^{k}\Bigg(\prod_{x_0<x_m<x-(i-1)a}(1+L_m)^{-1}\Bigg)$$
$$+\left(\frac{c_3}{c_4}\right)^l\prod_{x_0<x_m<x+a}(1+L_m)^{-l}$$
$$+c_2^{k-l}\prod_{i=1}^{l}\Bigg(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}(1+L_m)^{-1}\Bigg)$$
$$\times\Bigg[\prod_{i=1}^{l}\Bigg(c_4\prod_{x_0<x_m<x-(i-1)a}(1+L_m)-\mu T(x-ia,y-ib)\Bigg)$$
$$\times\prod_{j=1}^{k-l}\Bigg(c_4\prod_{x_0<x_m<x-(l+j-1)a}(1+L_m)-\mu T(x-(l+j)a,y-lb)\Bigg)\Bigg]^{-1}\Bigg\}\geq\mu\alpha_1.$$

The above inequality implies

$$\tag{20}\left(\frac{c_2}{c_4}\right)^k\prod_{i=1}^{k}\Bigg(\prod_{x_0<x_m<x-(i-1)a}(1+L_m)^{-1}\Bigg)$$
$$+\left(\frac{c_3}{c_4}\right)^l\prod_{x_0<x_m<x+a}(1+L_m)^{-l}$$
$$+c_2^{k-l}\prod_{i=1}^{l}\Bigg(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}(1+L_m)^{-1}\Bigg)$$
$$\times\Bigg[\prod_{i=1}^{l}\Bigg(c_4\prod_{x_0<x_m<x-(i-1)a}(1+L_m)-\mu T(x-ia,y-ib)\Bigg)$$
$$\times\prod_{j=1}^{k-l}\Bigg(c_4\prod_{x_0<x_m<x-(l+j-1)a}(1+L_m)-\mu T(x-(l+j)a,y-lb)\Bigg)\Bigg]^{-1}$$
$$=\lambda_0\geq\mu\alpha_1.$$

It is clear that if $\lambda_0\in S(\lambda)$ then $\lambda_0^{'}\leq\lambda_0$ implies that $\lambda_0^{'}\in S(\lambda)$. Therefore $\mu\alpha_1\in S(\lambda)$. Repeating the above arguments with μ replaced by $\mu\alpha_1$, we can prove that there exists $\alpha_2\in(1,\infty)$ such that $\mu\alpha_1\alpha_2\in S(\lambda)$. Continuing in this way, we obtain $\mu\prod_{i=1}^{\infty}\alpha_i\in S(\lambda)$, where $\alpha_i\in(1,\infty)$. This implies that $S(\lambda)$ is unbounded.

On the other hand

$$\lim_{x,y\to\infty}\sup T(x,y)>0 \qquad\text{and}\qquad c_4\prod_{x_0<x_m<x+a}(1+L_m)-\lambda T(x,y)>0,$$

eventually. So E is unbounded. Because of $S(\lambda) \subset E$, then $S(\lambda)$ is also bounded. If $l > k > 0$ then from (15), we obtain

$$\omega(x,y) \le \prod_{i=1}^{k}\Big(c_1 + \frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}\big(1+L_m\big)^{-1}\Big)^{-1} \tag{21}$$
$$\times\prod_{i=1}^{k}\big(c_4\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)-\mu T(x-ia,y-ib)\big)\omega(x-ka,y-kb),$$

$$\Big(\frac{c_2}{c_4}\Big)^k\omega(x,y) \le \prod_{i=1}^{k}\Big[\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)\Big]\omega(x-ka,y), \tag{22}$$

$$\Big(\frac{c_3}{c_4}\Big)^l\omega(x,y) \le \prod_{x_0<x_m<x+a}\big(1+L_m\big)^l\omega(x,y-lb) \tag{23}$$

and

$$\omega(x-ka,y-kb) \le \frac{1}{c_3^{l-k}} \tag{24}$$
$$\times\prod_{j=1}^{l-k}\Big[c_4\prod_{x_0<x_m<x-(k-1)a}\big(1+L_m\big)-\mu T(x-ka,y-(k+j)b)\Big]\omega(x-ka,y-lb).$$

By (21)–(24), from (14), we have

$$c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b)-c_4\prod_{x_0<x_m<x+a}\big(1+L_m\big)\omega(x,y)$$
$$+\Big(\frac{c_2}{c_4}\Big)^kP(x,y)\prod_{i=1}^{k}\Big(\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)^{-1}\Big)\omega(x,y)$$
$$+\Big(\frac{c_3}{c_4}\Big)^lQ(x,y)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l}\omega(x,y)$$
$$+c_3^{l-k}R(x,y)\prod_{i=1}^{k}\Big(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}\big(1+L_m\big)^{-1}\Big)$$
$$\times\Big[\prod_{i=1}^{k}\Big(c_4\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)-\mu T(x-ia,y-ib)\Big)$$
$$\times\prod_{j=1}^{l-k}\Big(c_4\prod_{x_0<x_m<x-(k-l)a}\big(1+L_m\big)-\mu T(x-ka,y-(k+j)b)\Big)\Big]^{-1}\omega(x,y)\le 0$$

and hence

$$
\begin{aligned}
&c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b)-\Big\{c_4\prod_{x_0<x_m<x+a}(1+L_m)\\
&\quad -T(x,y)\Big[\Big(\frac{c_2}{c_4}\Big)^k\prod_{i=1}^{k}\Big(\prod_{x_0<x_m<x-(i-1)a}(1+L_m)^{-1}\Big)\\
&\quad +\Big(\frac{c_3}{c_4}\Big)^l\prod_{x_0<x_m<x+a}(1+L_m)^{-l}\\
&\quad +c_3^{l-k}\prod_{i=1}^{k}\Big(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}(1+L_m)^{-1}\Big)\\
&\quad \times\Big[\prod_{i=1}^{k}\Big(c_4\prod_{x_0<x_m<x-(i-1)a}(1+L_m)-\mu T(x-ia,y-ib)\Big)\\
&\quad \times\prod_{j=1}^{l-k}\Big(c_4\prod_{x_0<x_m<x-(k-1)a}(1+L_m)-\mu T(x-ka,y-(k+j)b)\Big)\Big]^{-1}\Big]\Big\}\omega(x,y)\\
&\qquad \le 0,
\end{aligned}
$$

eventually. Thus

$$
\begin{aligned}
&c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b)-\Big\{c_4\prod_{x_0<x_m<x+a}(1+L_m)\\
&\quad -T(x,y)\Big[\Big(\frac{c_2}{c_4}\Big)^k\prod_{i=1}^{k}\Big(\prod_{x_0<x_m<x-(i-1)a}(1+L_m)^{-1}\Big)\\
&\quad +\Big(\frac{c_3}{c_4}\Big)^l\prod_{x_0<x_m<x+a}(1+L_m)^{-l}\\
&\quad +c_3^{l-k}\prod_{i=1}^{k}\Big(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}(1+L_m)^{-1}\Big)\\
&\quad \times\Big[\sup_{x\ge X_0,y\ge Y_0}\prod_{i=1}^{k}\Big(c_4\prod_{x_0<x_m<x-(i-1)a}(1+L_m)-\mu T(x-ia,y-ib)\Big)\\
&\quad \times\prod_{j=1}^{l-k}\Big(c_4\prod_{x_0<x_m<x-(k-1)a}(1+L_m)-\mu T(x-ka,y-(k+j)b)\Big)\Big]^{-1}\Big]\Big\}\omega(x,y)\\
&\qquad \le 0,
\end{aligned}
$$

eventually. It is clear that

$$\lambda_0' = \left(\frac{c_2}{c_4}\right)^k \prod_{i=1}^{k} \Bigg(\prod_{x_0<x_m<x-(i-1)a} (1+L_m)^{-1} \Bigg) + \left(\frac{c_3}{c_4}\right)^l \prod_{x_0<x_m<x+a} (1+L_m)^{-l}$$
$$+ c_3^{l-k} \prod_{i=1}^{k} \Bigg(c_1 + \frac{2c_2c_3}{c_4} \prod_{x_0<x_m<x-(i-2)a} (1+L_m)^{-1} \Bigg)$$
$$\times \Bigg[\sup_{x\geq X_0, y\geq Y_0} \prod_{i=1}^{k} \Bigg(c_4 \prod_{x_0<x_m<x-(i-1)a} (1+L_m) - \mu T(x-ia, y-ib) \Bigg)$$
$$\times \prod_{j=1}^{l-k} \Bigg(c_4 \prod_{x_0<x_m<x-(k-1)a} (1+L_m) - \mu T(x-ka, y-(k+j)b) \Bigg) \Bigg]^{-1} \in S(\lambda).$$

In the same way as in the proof of case $k > l > 0$, we can show that $S(\lambda)$ is unbounded.

For $k = l > 0$, (16)–(18) hold eventually. From (14), we obtain

$$c_1\omega(x+a, y+b) + c_2\omega(x+a, y) + c_3\omega(x, y+b) - \Bigg\{ c_4 \prod_{x_0<x_m<x+a} (1+L_m)$$
$$- \Bigg[\left(\frac{c_2}{c_4}\right)^k P(x,y) \prod_{i=1}^{k} \Bigg(\prod_{x_0<x_m<x-(i-1)a} (1+L_m)^{-1} \Bigg)$$
$$+ \left(\frac{c_3}{c_4}\right)^k Q(x,y) \prod_{x_0<x_m<x+a} (1+L_m)^{-k}$$
$$+ R(x,y) \prod_{i=1}^{k} \Bigg(c_1 + \frac{2c_2c_3}{c_4} \prod_{x_0<x_m<x-(i-2)a} (1+L_m)^{-1} \Bigg)$$
$$\times \Bigg[\prod_{i=1}^{k} \Bigg(c_4 \prod_{x_0<x_m<x-(i-1)a} (1+L_m) - \mu T(x-ia, y-ib) \Bigg) \Bigg]^{-1} \Bigg] \Bigg\} \omega(x,y) \leq 0$$

then

$$c_1\omega(x+a, y+b) + c_2\omega(x+a, y) + c_3\omega(x, y+b) - \Bigg\{ c_4 \prod_{x_0<x_m<x+a} (1+L_m)$$
$$- T(x,y) \Bigg[\left(\frac{c_2}{c_4}\right)^k \prod_{i=1}^{k} \Bigg(\prod_{x_0<x_m<x-(i-1)a} (1+L_m)^{-1} \Bigg)$$
$$+ \left(\frac{c_3}{c_4}\right)^k \prod_{x_0<x_m<x+a} (1+L_m)^{-k}$$

$$+\prod_{i=1}^{k}\left(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}(1+L_m)^{-1}\right)$$
$$\times\left[\prod_{i=1}^{k}\left(c_4\prod_{x_0<x_m<x-(i-1)a}(1+L_m)-\mu T(x-ia,y-ib)\right)\right]^{-1}\Bigg]\Bigg\}\omega(x,y)\le 0,$$

eventually. The above inequality implies

$$c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b)-\Bigg\{c_4\prod_{x_0<x_m<x+a}(1+L_m)$$
$$-T(x,y)\Bigg[\left(\frac{c_2}{c_4}\right)^k\prod_{i=1}^{k}\left(\prod_{x_0<x_m<x-(i-1)a}(1+L_m)^{-1}\right)$$
$$+\left(\frac{c_3}{c_4}\right)^k\prod_{x_0<x_m<x+a}(1+L_m)^{-k}$$
$$+\prod_{i=1}^{k}\left(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}(1+L_m)^{-1}\right)$$
$$\times\left[\sup_{x\ge X_0,y\ge Y_0}\prod_{i=1}^{k}\left(c_4\prod_{x_0<x_m<x-(i-1)a}(1+L_m)-\mu T(x-ia,y-ib)\right)\right]^{-1}\Bigg]\Bigg\}$$
$$\times\,\omega(x,y)\le 0.$$

Hence

$$\lambda_0''=\left(\frac{c_2}{c_4}\right)^k\prod_{i=1}^{k}\left(\prod_{x_0<x_m<x-(i-1)a}(1+L_m)^{-1}\right)+\left(\frac{c_3}{c_4}\right)^k\prod_{x_0<x_m<x+a}(1+L_m)^{-k}$$
$$+\prod_{i=1}^{k}\left(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}(1+L_m)^{-1}\right)$$
$$\times\left[\sup_{x\ge X_0,y\ge Y_0}\prod_{i=1}^{k}\left(c_4\prod_{x_0<x_m<x-(i-1)a}(1+L_m)-\mu T(x-ia,y-ib)\right)\right]^{-1}\in S(\lambda).$$

In the same way as in the proof of case $k>l>0$, we can show that $S(\lambda)$ is unbounded. This completes the proof. ∎

Corollary 1. *Assume that if $k>l>0$,*

$$\liminf_{x,y\to\infty}T(x,y)=q \tag{25}$$
$$>(c_4L)^{k+1}\left[c_2^k+c_3^l+c_2^{k-l}\left(c_1+\frac{2c_2c_3}{c_4L}\right)^l\frac{(k+1)^{k+1}}{k^k}\right]^{-1},$$

if $l > k > 0$,

$$\liminf_{x,y\to\infty} T(x,y) = q > \big(c_4L\big)^{l+1}\left[c_2^k + c_3^l + c_3^{l-k}\left(c_1 + \frac{2c_2c_3}{c_4L}\right)^k \frac{\big(l+1\big)^{l+1}}{l^l}\right]^{-1}, \tag{26}$$

and if $k = l > 0$,

$$\liminf_{x,y\to\infty} T(x,y) = q > \big(c_4L\big)^{k+1}\left[c_2^k + c_3^k + \left(c_1 + \frac{2c_2c_3}{c_4L}\right)^k \frac{\big(k+1\big)^{k+1}}{k^k}\right]^{-1}, \tag{27}$$

where $L = \prod_{m=1}^{\infty}\big(1+L_m\big)$. *Then every solution of* (1)–(2) *is oscillatory.*

Proof. Since

$$\max_{0<\lambda\le\frac{c_4L}{q}} \lambda\big(c_4L - \lambda q\big)^k = \frac{\big(c_4L\big)^{k+1}}{q}\frac{k^k}{\big(k+1\big)^{k+1}}$$

for $k > l > 0$, from (25)

$$\begin{aligned}
&\inf_{\lambda\in E, x\ge X_0, y\ge Y_0} \frac{1}{\lambda}\Bigg\{\left(\frac{c_2}{c_4}\right)^k \prod_{i=1}^{k}\Bigg(\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)^{-1}\Bigg)\\
&\quad + \left(\frac{c_3}{c_4}\right)^l \prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l}\\
&\quad + c_2^{k-l}\prod_{i=1}^{l}\Bigg(c_1 + \frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}\big(1+L_m\big)^{-1}\Bigg)\\
&\quad \times\Bigg[\prod_{i=1}^{l}\Bigg(c_4\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big) - \lambda T\big(x-ia, y-ib\big)\Bigg)\\
&\quad \times\prod_{j=1}^{k-l}\Bigg(c_4\prod_{x_0<x_m<x-(l+j-1)a}\big(1+L_m\big) - \lambda T\big(x-(l+j)a, y-lb\big)\Bigg)\Bigg]^{-1}\Bigg\}\\
&\ge \frac{1}{\lambda}\left\{\left(\frac{c_2}{c_4L}\right)^k + \left(\frac{c_3}{c_4L}\right)^l + c_2^{k-l}\left(c_1 + \frac{2c_2c_3}{c_4L}\right)^l\big(c_4L - \lambda q\big)^{-k}\right\}\\
&\ge \frac{q}{c_4L}\left(\left(\frac{c_2}{c_4L}\right)^k + \left(\frac{c_3}{c_4L}\right)^l\right) + \frac{c_2^{k-l}\big(c_1 + \frac{2c_2c_3}{c_4L}\big)^l}{\max\lambda\big(c_4L - \lambda q\big)^k}
\end{aligned}$$

$$= \frac{q}{c_4 L}\left(\left(\frac{c_2}{c_4 L}\right)^k + \left(\frac{c_3}{c_4 L}\right)^l\right) + c_2^{k-l}\left(c_1 + \frac{2c_2c_3}{c_4 L}\right)^l \frac{q}{\left(c_4 L\right)^{k+1}} \frac{\left(k+1\right)^{k+1}}{k^k}$$

$$\geq \frac{q}{\left(c_4 L\right)^{k+1}}\left[c_2^k + c_3^l + c_2^{k-l}\left(c_1 + \frac{2c_2c_3}{c_4 L}\right)^l \frac{\left(k+1\right)^{k+1}}{k^k}\right] > 1.$$

Therefore from Theorem 1, every solution of (1)–(2) is oscillatory. In the same way, we can prove the case of $l > k > 0$ and $k = l > 0$. ∎

Theorem 2. *Assume that there exist $X_0 \geq 0, Y_0 \geq 0$ such that*

(28)
$$\inf_{\lambda \in E_P, x \geq X_0, y \geq Y_0} \frac{c_2^k}{\lambda}\left[\prod_{i=1}^{k}\left(c_4 \prod_{x_0 < x_m < x-(i-1)a} (1+L_m) - \lambda P(x-ia, y)\right)\right]^{-1} > 1.$$

Then every solution of (1)–(2) *is oscillatory.*

Proof. Assume that $u(x,y)$ is an eventually positive solution of (1)–(2). From Lemma 1, there exists an eventually positive solution $\omega(x,y)$ of (4). Then we have

$$\begin{aligned}(29)\qquad & c_1\omega(x+a, y+b) + c_2\omega(x+a, y) + c_3\omega(x, y+b) \\ & - c_4 \prod_{x_0 < x_m < x+a} \big(1 + L_m\big)\omega(x,y) + P(x,y)\omega(x-ka, y) \leq 0,\end{aligned}$$

eventually. Let

$$\begin{aligned} S_P(\lambda) = \big\{\lambda > 0 :\ & c_1\omega(x+a, y+b) + c_2\omega(x+a, y) + c_3\omega(x, y+b) \\ & - \big(c_4 \prod_{x_0 < x_m < x+a} (1 + L_m) - \lambda P(x,y)\big)\omega(x,y) \leq 0, \text{eventually}\big\}.\end{aligned}$$

Since $\frac{\partial\omega}{\partial x} < 0$ and $\frac{\partial\omega}{\partial y} < 0$ then from (29), we obtain

$$\begin{aligned} & c_1\omega(x+a, y+b) + c_2\omega(x+a, y) + c_3\omega(x, y+b) \\ & \qquad - \left(c_4 \prod_{x_0 < x_m < x+a} \big(1 + L_m\big) - P(x,y)\right)\omega(x,y) \leq 0,\end{aligned}$$

eventually. This implies that $1 \in S_P(\lambda)$. That is $S_P(\lambda)$ is nonempty. If $\lambda_0 \in S_P(\lambda)$ then we have eventually

$$\begin{aligned} 0 < {} & c_1\omega(x+a, y+b) + c_2\omega(x+a, y) + c_3\omega(x, y+b) \\ & \leq \left(c_4 \prod_{x_0 < x_m < x+a} \big(1 + L_m\big) - \lambda_0 P(x,y)\right)\omega(x,y).\end{aligned}$$

Hence $c_4 \prod_{x_0<x_m<x+a} \big(1+L_m\big) - \lambda_0 P(x,y) > 0$, i.e., $\lambda_0 \in E_P$. Thus $S_P(\lambda) \subset E_P$. Since

$$\limsup_{x\to\infty, y\to\infty} P(x,y) > 0 \qquad \text{and} \qquad c_4 \prod_{x_0<x_m<x+a} \big(1+L_m\big) - \lambda P(x,y) > 0$$

then E_P is bounded. Since $S_p(\lambda) \subset E_P$ then $S_p(\lambda)$ is also bounded. Let $\mu \in S_P(\lambda)$. Noting that $\frac{\partial\omega}{\partial x} < 0$ and $\frac{\partial\omega}{\partial y} < 0$ eventually, we have

$$\begin{aligned} & c_2\omega(x+a,y) < \Big[c_4 \prod_{x_0<x_m<x+a} \big(1+L_m\big) - \mu P(x,y)\Big]\omega(x,y), \\ (30)\quad & c_2^k\omega(x,y) < \prod_{i=1}^{k}\Big[c_4 \prod_{x_0<x_m<x-(i-1)a} \big(1+L_m\big) - \mu P(x-ia,y)\Big]\omega(x+a,y). \end{aligned}$$

From (29) and (30), we obtain

$$\begin{aligned} & c_1\omega(x+a,y+b) + c_2\omega(x+a,y) + c_3\omega(x,y+b) - c_4 \prod_{x_0<x_m<x+a} (1+L_m)\omega(x,y) \\ & \quad + c_2^k\Big[\sup_{x\ge X_0, y\ge Y_0} \prod_{i=1}^{k}\Big(c_4 \prod_{x_0<x_m<x-(i-1)a} \big(1+L_m\big) - \mu P(x-ia,y)\Big)\Big]^{-1} \\ & \quad \times P(x,y)\omega(x,y) \le 0, \end{aligned}$$

eventually. Hence

$$\begin{aligned} & c_2^k\Big[\sup_{x\ge X_0, y\ge Y_0} \prod_{i=1}^{k}\Big(c_4 \prod_{x_0<x_m<x-(i-1)a} \big(1+L_m\big) - \mu P(x-ia,y)\Big)\Big]^{-1} \\ & \quad = \lambda_0 \in S_P(\lambda). \end{aligned}$$

On the other hand, from (28) there exists $\alpha_1 \in (1,\infty)$ such that

$$\begin{aligned} & \inf_{\lambda\in E_P, x\ge X_0, y\ge Y_0} \frac{c_2^k}{\lambda}\Big[\prod_{i=1}^{k}\Big(c_4 \prod_{x_0<x_m<x-(i-1)a} \big(1+L_m\big) - \lambda P(x-ia,y)\Big)\Big]^{-1} \\ & \quad \ge \alpha_1 > 1. \end{aligned}$$

Therefore for $\lambda = \mu$,

$$\begin{aligned} \lambda_0 = {} & c_2^k\Big[\sup_{x\ge X_0, y\ge Y_0} \prod_{i=1}^{k}\Big(c_4 \prod_{x_0<x_m<x-(i-1)a} \big(1+L_m\big) - \mu P(x-ia,y)\Big)\Big]^{-1} \\ & \ge \mu\alpha_1 \in S_p(\lambda). \end{aligned}$$

Repeating the above arguments with μ replaced by $\mu\alpha_1$, we can prove that there exists $\alpha_2 \in (1,\infty)$ such that $\mu\alpha_1\alpha_2 \in S_P(\lambda)$. Continuing in this way, we obtain $\mu\prod_{i=1}^{\infty}\alpha_i \in S_P(\lambda)$, where $\alpha_i \in (1,\infty)$. This implies that $S_P(\lambda)$

is unbounded. This contradicts the boundness of $S_P(\lambda)$. The proof is complete. ∎

Theorem 3. *Assume that there exist $X_0 \geq 0, Y_0 \geq 0$ such that*

(31)
$$\inf_{\lambda\in E_Q, x\geq X_0, y\geq Y_0} \frac{c_3^l}{\lambda}\left[\prod_{i=1}^{l}\left(c_4\prod_{x_0<x_m<x+a}\left(1+L_m\right)-\lambda Q(x,y-ib)\right)\right]^{-1}>1.$$

Then every solution of (1)–(2) *is oscillatory.*

The proof of Theorem 3 is similar to the proof of Theorem 2.

Corollary 2. *If*

$$\liminf_{x,y\to\infty} P(x,y)=p>\frac{\left(c_4L\right)^{k+1}}{c_2^k}\frac{k^k}{(k+1)^{k+1}} \tag{32}$$

or

$$\liminf_{x,y\to\infty} Q(x,y)=q>\frac{\left(c_4L\right)^{l+1}}{c_3^l}\frac{l^l}{(l+1)^{l+1}} \tag{33}$$

then every solution of (1)–(2) *is oscillatory.*

Proof. From (32), we obtain

$$\begin{aligned}\inf_{\lambda\in E_P, x\geq X_0, y\geq Y_0} &\frac{c_2^k}{\lambda}\left[\prod_{i=1}^{k}\left(c_4\prod_{x_0<x_m<x-(i-1)a}\left(1+L_m\right)-\lambda P(x-ia,y)\right)\right]^{-1}\\ &\geq \inf_{0<\lambda\leq\frac{c_4L}{p}}\frac{c_2^k}{\lambda\left(c_4L-\lambda p\right)^k}=c_2^k\frac{p}{\left(c_4L\right)^{k+1}}\frac{(k+1)^{k+1}}{k^k}>1.\end{aligned}$$

So (28) holds. Similarly, if (33) holds then (31) holds. Thus, by Theorem 2 and Theorem 3, every solution of (1)–(2) is oscillatory. ∎

Theorem 4. *Assume that there exist $X_0 \geq 0, Y_0 \geq 0$ such that if $k>l>0$,*

$$\begin{aligned}\inf_{\lambda\in E_R, x\geq X_0, y\geq Y_0} &\frac{c_2^{k-l}}{\lambda}\prod_{i=1}^{l}\left(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}\left(1+L_m\right)^{-1}\right)\\ &\times\left[\prod_{i=1}^{l}\left(c_4\prod_{x_0<x_m<x-(i-1)a}\left(1+L_m\right)-\lambda R\left(x-ia,y-ib\right)\right)\right.\\ &\times\left.\prod_{j=1}^{k-l}\left(c_4\prod_{x_0<x_m<x-(l+j-1)a}\left(1+L_m\right)-\lambda R\left(x-(l+j)a,y-lb\right)\right)\right]^{-1}>1,\end{aligned} \tag{34}$$

if $l>k>0$,

$$\inf_{\lambda\in E_R, x\geq X_0, y\geq Y_0} \frac{{c_3^{l-k}}^{k}}{\prod}_{i=1}\left(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}\big(1+L_m\big)^{-1}\right)$$
$$\times\left[\prod_{i=1}^{k}\left(c_4\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)-\lambda R\big(x-ia,y-ib\big)\right)\right.$$
$$\left.\times\prod_{j=1}^{l-k}\left(c_4\prod_{x_0<x_m<x-(k-1)a}\big(1+L_m\big)-\lambda R\big(x-ka,y-(k+j)b\big)\right)\right]^{-1}>1 \tag{35}$$

and if $k=l>0$,

$$\inf_{\lambda\in E_R, x\geq X_0, y\geq Y_0} \frac{1}{\lambda}\prod_{i=1}^{k}\left(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}\big(1+L_m\big)^{-1}\right)$$
$$\times\left[\prod_{i=1}^{k}\left(c_4\prod_{x_0<x_m<x-(i-1)a}\big(1+L_m\big)-\lambda R\big(x-ia,y-ib\big)\right)\right]^{-1}>1. \tag{36}$$

Then every solution of (1)–(2) *is oscillatory.*

Proof. Assume that $u(x,y)$ is an eventually positive solution of (1)–(2). From Lemma 1, there exists an eventually positive solution $\omega(x,y)$ of (4). Then we have

$$\begin{aligned} &c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b)\\ &\quad -c_4\prod_{x_0<x_m<x+a}(1+L_m)\omega(x,y)+R(x,y)\omega(x-ka,y-lb)\leq 0, \end{aligned} \tag{37}$$

eventually. Set

$$\begin{aligned} S_R(\lambda)=\Bigg\{\lambda>0:\ &c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b)\\ &-\left(c_4\prod_{x_0<x_m<x+a}\big(1+L_m\big)-R(x,y)\right)\omega(x,y)\leq 0,\text{eventually}\Bigg\}. \end{aligned}$$

It is easy to prove that $S_R(\lambda)\subset E_R$, S_R is nonempty and E_R is bounded. Let $\mu\in S_R(\lambda)$. Noting that $\frac{\partial\omega}{\partial x}<0$ and $\frac{\partial\omega}{\partial y}<0$, eventually, we have

$$\begin{aligned} &\left(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-1}\right)\omega(x,y)\\ &\qquad\leq\big(c_4\prod_{x_0<x_m<x}\big(1+L_m\big)-\mu R(x-a,y-b)\big)\omega(x-a,y-b), \end{aligned}$$

for $k>l>0$,

$$\omega(x,y) \le \prod_{i=1}^{l}\left(c_1 + \frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}(1+L_m)^{-1}\right)^{-1} \tag{38}$$
$$\times \prod_{i=1}^{l}\big(c_4\prod_{x_0<x_m<x-(i-1)a}(1+L_m) - \mu R(x-ia,y-ib)\big)\omega(x-la,y-lb)$$

and

$$\omega(x-la,y-lb) \le \tag{39}$$
$$\frac{1}{c_2^{k-l}}\prod_{j=1}^{k-l}\left[c_4\prod_{x_0<x_m<x-(l+j-1)a}(1+L_m)-\mu R(x-(l+j)a,y-lb)\right]\omega(x-ka,y-lb).$$

From (37) and (39), we obtain

$$c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b)-c_4\prod_{x_0<x_m<x+a}(1+L_m)\omega(x,y)$$
$$+\,c_2^{k-l}\prod_{i=1}^{l}\left(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}(1+L_m)^{-1}\right)$$
$$\times\left[\sup_{x\ge X_0,y\ge Y_0}\prod_{i=1}^{l}\left(c_4\prod_{x_0<x_m<x-(i-1)a}(1+L_m)-\mu R(x-ia,y-ib)\right)\right.$$
$$\left.\times\prod_{j=1}^{k-l}\left(c_4\prod_{x_0<x_m<x-(l+j-1)a}(1+L_m)-\mu R(x-(l+j)a,y-lb)\right)\right]^{-1}$$
$$\times R(x,y)\omega(x,y)\le 0.$$

Hence,

$$c_2^{k-l}\prod_{i=1}^{l}\left(c_1+\frac{2c_2c_3}{c_4}\prod_{x_0<x_m<x-(i-2)a}(1+L_m)^{-1}\right)$$
$$\times\left[\sup_{x\ge X_0,y\ge Y_0}\prod_{i=1}^{l}\left(c_4\prod_{x_0<x_m<x-(i-1)a}(1+L_m)-\mu R(x-ia,y-ib)\right)\right.$$
$$\left.\times\prod_{j=1}^{k-l}\left(c_4\prod_{x_0<x_m<x-(l+j-1)a}(1+L_m)-\mu R(x-(l+j)a,y-lb)\right)\right]^{-1}$$
$$=\lambda_0\in S_R(\lambda).$$

On the other hand, from (34) there exists an $\alpha_1 \in (1,\infty)$ such that

$$\inf_{\lambda\in E_R, x\geq X_0, y\geq Y_0} \frac{c_2^{k-l}}{\lambda} \prod_{i=1}^{l}\Big(c_1 + \frac{2c_2c_3}{c_4} \prod_{x_0<x_m<x-(i-2)a} (1+L_m)^{-1}\Big)$$
$$\times\Big[\prod_{i=1}^{l}\Big(c_4 \prod_{x_0<x_m<x-(i-1)a} (1+L_m) - \lambda R(x-ia, y-ib)\Big)$$
$$\times\prod_{j=1}^{k-l}\Big(c_4 \prod_{x_0<x_m<x-(l+j-1)a} (1+L_m) - \lambda R(x-(l+j)a, y-lb)\Big)\Big]^{-1}$$
$$\geq \alpha_1 > 1.$$

Hence, for $\lambda = \mu$

$$\lambda_0 = c_2^{k-l} \prod_{i=1}^{l}\Big(c_1 + \frac{2c_2c_3}{c_4} \prod_{x_0<x_m<x-(i-2)a} (1+L_m)^{-1}\Big)$$
$$\times\Big[\prod_{i=1}^{l}\Big(c_4 \prod_{x_0<x_m<x-(i-1)a} (1+L_m) - \mu R(x-ia, y-ib)\Big)$$
$$\times\prod_{j=1}^{k-l}\Big(c_4 \prod_{x_0<x_m<x-(l+j-1)a} (1+L_m) - \mu R(x-(l+j)a, y-lb)\Big)\Big]^{-1}$$
$$\geq \mu\alpha_1 \in S_R(\lambda).$$

Repeating the above arguments with μ replaced by $\mu\alpha_1$, we get $\mu\alpha_1\alpha_2 \in S_R(\lambda)$, where $\alpha_2 \in (1,\infty)$. Continuing in this way, we have $\mu\prod_{i=1}^{\infty}\alpha_i \in S_R(\lambda)$, where $\alpha_i \in (1,\infty)$. This implies that $S_R(\lambda)$ is unbounded. This contradicts the boundness of $S_R(\lambda)$. The proof is complete. ■

Corollary 3. *Assume that if $k > l > 0$,*

$$\liminf_{x,y\to\infty} R(x,y) = r > (c_4L)^{k+1} c_2^{l-k}\Big(c_1 + \frac{2c_2c_3}{c_4L}\Big)^{-l}\frac{k^k}{(k+1)^{k+1}}, \tag{40}$$

if $l > k > 0$,

$$\liminf_{x,y\to\infty} R(x,y) = r > (c_4L)^{l+1} c_3^{k-l}\Big(c_1 + \frac{2c_2c_3}{c_4L}\Big)^{-k}\frac{l^l}{(l+1)^{l+1}}, \tag{41}$$

and if $k = l > 0$,

$$\liminf_{x,y\to\infty} R(x,y) = r > (c_4L)^{k+1}\Big(c_1 + \frac{2c_2c_3}{c_4L}\Big)^{-k}\frac{k^k}{(k+1)^{k+1}}, \tag{42}$$

where $L = \prod_{m=1}^{\infty}(1+L_m)$. Then every solution of (1)–(2) is oscillatory.

Proof. For $k > l > 0$, from (40), we have

$$\inf_{\lambda\in E_R, x\geq X_0, y\geq Y_0} \frac{c_2^{k-l}}{\lambda} \prod_{i=1}^{l} \Big(c_1 + \frac{2c_2c_3}{c_4} \prod_{x_0<x_m<x-(i-2)a} \big(1+L_m\big)^{-1}\Big)$$

$$\times \Big[\prod_{i=1}^{l} \Big(c_4 \prod_{x_0<x_m<x-(i-1)a} \big(1+L_m\big) - \lambda R\big(x-ia, y-ib\big)\Big)$$

$$\times \prod_{j=1}^{k-l} \Big(c_4 \prod_{x_0<x_m<x-(l+j-1)a} \big(1+L_m\big) - \lambda R\big(x-(l+j)a, y-lb\big)\Big)\Big]^{-1}$$

$$\geq \inf_{\lambda\in E_R, x\geq X_0, y\geq Y_0} \frac{c_2^{k-l}\Big(c_1+\dfrac{2c_2c_3}{c_4L}\Big)^l}{\lambda(c_4L-\lambda r)^k} = r\frac{c_2^{k-l}}{\big(c_4L\big)^{k+1}}\Big(c_1+\frac{2c_2c_3}{c_4L}\Big)^l\frac{(k+1)^{k+1}}{k^k}$$

$$> 1.$$

Thus (34) is fulfilled. Similarly, if (41) and (42) hold, (35) and (36) are fulfilled, respectively. Hence by Theorem 4, every solution of (1)–(2) is oscillatory. ■

References

[1] R. P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, 1992.

[2] R. P. Agarwal, F. Karakoc, *Oscillation of impulsive partial difference equations with continuous variables*, Math. Comput. Modelling 50 (2009), 1262–1278.

[3] R. P. Agarwal, P. J. Y. Wong, *Advanced Topics in Difference Equations*, Kluwer Academic, Dordrecht, 1997.

[4] D. D. Baĭnov, M. B. Dimitrova, A. B. Dishlies, *Oscillation of bounded solutions of impulsive differential–difference of second order*, Appl. Math. Comput. 114 (2000), 61–68.

[5] D. D. Baĭnov, E. Minchev, *Forced oscillation of solutions of impulsive nonlinear parabolic differential–difference equations*, J. Korean Math. Soc. 35(4) (1998), 881–890.

[6] D. D. Baĭnov, E. Minchev, *Oscillation of solutions of impulsive nonlinear parabolic differential–difference equations*, Internat. J. Theoret. Phys. 35(1) (1996), 207–215.

[7] D. D. Baĭnov, P. S. Simeonov. *Impulsive Differential Equations Asymtotic Properties of the Solutions*, World Scientific, Singapore, 1995.

[8] B. T. Cui, Y. Liu, *Oscillatory for partial difference equations with continuous variables*, J. Comput. Appl. Math. 154 (2003), 373–391.

[9] V. Lakshmikantham, D. D. Baĭnov, P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1998.

[10] E. Minchev, *Oscillation of solutions of impulsive nonlinear hyperbolic differential–difference equations*, Math. Balcanica 12(1–2) (1998), 215–224.

[11] B. G. Zhang, *Oscillation criteria of partial difference equations with continuous variables*, Acta Math. Sinica 42(3) (1999), 487–494, (in Chinese).
[12] B. G. Zhang, B. M. Liu, *Oscillation criteria of certain nonlinear partial difference equations*, Comput. Math. Appl. 38 (1999), 107–112.
[13] B. G. Zhang, B. M. Liu, *Necessary and sufficient conditions for oscillation of partial difference equations with continuous variables*, Comput. Math. Appl. 38 (1999), 163–167.
[14] B. G. Zhang, Y. Zhou, *Qualitative Analysis of Delay Partial Difference Equations*, Hindawi Publishing Corporation, New York, 2007.
[15] Y. Zhou, *Existence of bounded and unbounded nonoscillatory solutions of nonlinear partial difference equations*, J. Math. Anal. Appl. 332(2) (2007), 1267–1277.

F. Özpinar
BOLVADIN VOCATIONAL SCHOOL
AFYON KOCATEPE UNIVERSITY
03300 AFYONKARAHISAR, TURKEY
E-mail: fozpinar@aku.edu.tr

S. Öztürk
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND ARTS
ANS CAMPUS
AFYON KOCATEPE UNIVERSITY
03200 AFYONKARAHISAR, TURKEY
E-mail: ssahin@aku.edu.tr

Z. F. Koçak
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND LITERATURE
MUGLA UNIVERSITY
MUGLA, TURKEY
E-mail: zkocak@mu.edu.tr

Received February 29, 2012.