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UNIFORMLY CONTINUOUS SUPERPOSITION OPERATORS IN THE SPACE OF FUNCTIONS OF BOUNDED n-DIMENSIONAL Φ-VARIATION

Abstract. We prove that if a superposition operator maps a subset of the space of all metric-vector-space-valued-functions of bounded n-dimensional Φ-variation into another such space, and is uniformly continuous, then the generating function of the operator is an affine function in the functional variable.

1. Introduction

Given two (non-empty) sets A and B, the notation B^A will stand for the set of all functions from A to B. As usual, if M, N are linear spaces, the notation $L(M, N)$ stands for the set of all linear maps from M to N.

Let A, B and C be non-empty sets. If $h : A \times C \to B$ is a given function, $X \subset C^A$ and $Y \subset B^A$ are linear spaces then the nonlinear superposition (Nemytskij) operator $\mathbf{H} : X \to Y$, generated by the function h, is defined as

$$(\mathbf{H}f)(\mathbf{t}) := h(\mathbf{t}, f(\mathbf{t})), \quad \mathbf{t} \in A.$$

This operator plays a central role in various mathematical fields, e.g. in the theory of nonlinear integral equations, and has been studied thoroughly. Perhaps, the most important problem concerning the theory of the superposition operator is to establish necessary and sufficient conditions guaranteeing that this operator maps a given function space into itself. These conditions are called *acting conditions* (e.g., (non-linear) boundedness, continuity, local or global Lipschitz conditions, etc.). On the other hand, superposition operators being the simplest operators between function spaces, another important problem is to determine if a certain given operator, that acts between some given function spaces, can be redefined via the notion of superposition,

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thus, e.g., it has been established that for some function spaces, any locally defined operator is a Nemytskij operator (cf. [12], [13] and [8]). We refer the reader to the celebrated book [1], by J. Appell and P. P. Zabrejko, in which most of the basic facts and results concerning superposition operators are exposed.

Throughout this paper, the letter n denotes a positive integer. Let $\mathbf{a} = (a_1, a_2, \dots, a_n)$ and $\mathbf{b} = (b_1, b_2, \dots, b_n)$ be points in $\mathbb{R}^n$. We will use the notation $\mathbf{a} < \mathbf{b}$ to mean that $a_i < b_i$ for each $i = 1, \dots, n$ and accordingly we define $\mathbf{a} = \mathbf{b}$, $\mathbf{a} \leq \mathbf{b}$, $\mathbf{a} \geq \mathbf{b}$ and $\mathbf{a} > \mathbf{b}$. If $\mathbf{a} < \mathbf{b}$, the set $\mathbf{J} := [\mathbf{a}, \mathbf{b}] = \prod_{i=1}^{n} [a_i, b_i]$ will be called an *n-dimensional closed interval.*

Given an n-dimensional closed interval $\mathbf{J}$, a metric vector space M and a φ-function Φ, we will denote by $BRV_\Phi^n(\mathbf{J}; M)$ the normed space of all functions of n-dimensional bounded Φ-variation on $\mathbf{J}$. Suppose that N is another linear metric space, $\mathcal{C}$ is a convex subset of M, Ψ is another φ-function and $h : \mathbf{J} \times \mathcal{C} \to N$ is a given function.

In this paper, we prove that if the superposition operator $\mathbf{H}$, generated by h, maps the set $R_{\mathcal{C}} = \{f \in BRV_\Phi^n(\mathbf{J}; \mathcal{M}) : f(\mathbf{J}) \subset \mathcal{C})\}$ into $BRV_\Psi^n(\mathbf{J}; N)$ and is uniformly continuous[1] then there is a linear operator $A : M \to N$ and a function $B \in N^{\mathbf{J}}$ such that

$$h(\mathbf{x}, y) = A(\mathbf{x})y + B(\mathbf{x}), \qquad \mathbf{x} \in \mathbf{J}, \quad y \in \mathcal{C}.$$

2. Functions of bounded n-dimensional Φ-variation

In this section, we present the definition and main basic aspects of the notion of n-dimensional Φ-variation for functions defined on rectangles of $\mathbb{R}^n$ that take values on a *metric semigroup* (cf. also [2, 3]).

Definition 2.1. A *metric semigroup* is a structure $(M, d, +)$ where $(M, +)$ is an abelian semigroup and d is a translation invariant metric on M.

In particular, the triangle inequality implies that, for all $u, v, p, q \in M$,

$$d(u, v) \leq d(p, q) + d(u + p, v + q), \quad \text{and}$$

$$d(u + p, v + q) \leq d(u, v) + d(p, q). \tag{2.1}$$

Following [2, 10], in this paper we use the following notations:

$\mathbb{N}$ (resp. $\mathbb{N}_0$) denotes the set of all positive integers (resp. non-negative integers) and, if $n \in \mathbb{N}$, a typical point in $\mathbb{R}^n$ is denoted as $\mathbf{x} = (x_1, x_2, \dots, x_n) := (x_i)_{i=1}^n$; however, the canonical unit vectors of $\mathbb{R}^n$ are denoted by $\mathbf{e_j}$; that

[1] That is, given $\epsilon > 0$ there is $\delta > 0$ such that $\|H(f) - H(g)\|_{BRV_\Psi^n(\mathbf{J};N)} < \epsilon$, for all $f, g \in R_{\mathcal{C}}$ such that $\|f - g\|_{BRV_\Phi^n(\mathbf{J};\mathcal{M})} < \delta$.

is, $\mathbf{e_j} := (e_r^{(j)})_{r=1}^n$ where,

$$e_r^j := \begin{cases} 0, & \text{if } r \neq j, \\ 1, & \text{if } r = j, \end{cases} \qquad (j = 1, 2, \ldots, n).$$

The zero n-tuple $(0, 0, \ldots, 0)$ is denoted by $\mathbf{0}$, and by $\mathbf{1}$ we mean the n-tuple $\mathbf{1} = (1, 1, \ldots, 1)$.

If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, with $\alpha_j \in \mathbb{N}_0$, is a n-tuple of non-negative integers then we call α a *multi-index*.

The euclidean volume of an n-dimensional closed interval $[\mathbf{a}, \mathbf{b}] = \prod_{i=1}^{n} [a_i, b_i]$ will be denoted by $Vol\,[\mathbf{a}, \mathbf{b}]$; that is, $Vol\,[\mathbf{a}, \mathbf{b}] = \prod_{i=1}^{n} (b_i - a_i)$.

In addition, for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we will use the notations

$$|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n \qquad \text{and} \qquad \alpha\mathbf{x} := (\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n).$$

We will denote by $\mathcal{N}$ the set of all strictly increasing continuous convex functions $\Phi : [0, +\infty) \to [0, +\infty)$ such that $\Phi(t) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty} \Phi(t) = +\infty$.

Also, $\mathcal{N}_\infty$ will denote the set of all functions $\Phi \in \mathcal{N}$, for which the Orlicz condition (also called ∞_1 condition) holds: $\lim_{t\to\infty} \dfrac{\Phi(t)}{t} = +\infty$.

Functions from $\mathcal{N}$ are often called φ-functions.

One says that a function $\Phi \in \mathcal{N}$ satisfies a condition Δ_2, and writes $\Phi \in \Delta_2$, if there are constants $K > 0$ and $t_0 > 0$ such that

$$\Phi(2t) \leq K\Phi(t), \quad \text{for all} \quad t \geq t_0. \tag{2.2}$$

Notice that these sets are related in a one to one fashion; indeed, if $\theta = (\theta_1, \ldots, \theta_n) \in \mathcal{E}(n)$ then we can define $\widetilde{\theta} := (1 - \theta_1, \theta_2, \ldots, \theta_n) \in \mathcal{O}(n)$, and this operation is clearly invertible.

In what follows, M is supposed to be a metric semigroup and $[\mathbf{a}, \mathbf{b}]$ an n-dimensional closed interval.

Definition 2.2. [2, 5, 11] Given a function $f : [\mathbf{a}, \mathbf{b}] \to M$, we define the *$n$-dimensional Vitali difference* of f over an n-dimensional interval $[\mathbf{x}, \mathbf{y}] \subseteq [\mathbf{a}, \mathbf{b}]$, by

$$\Delta_n(f, [\mathbf{x}, \mathbf{y}]) := d\Big(\sum_{\theta\in\mathcal{E}(n)} f(\theta\mathbf{x} + (1-\theta)\mathbf{y}), \sum_{\theta\in\mathcal{O}(n)} f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \Big), \tag{2.3}$$

where

$$\mathcal{E}(n) := \{\theta \in \mathbb{N}_0^n : \theta \leq \mathbf{1} \text{ and } |\theta| \text{ is even }\},$$
$$\mathcal{O}(n) := \{\theta \in \mathbb{N}_0^n : \theta \leq \mathbf{1} \text{ and } |\theta| \text{ is odd}\}.$$

This difference is usually associated to the names of Vitali, Lebesgue, Hardy, Krause, Fréchet and De la Vallée Poussin ([5, 4, 6]).

Now, in order to define the Φ-variation of a function $f : [\mathbf{a}, \mathbf{b}] \to M$, we consider *net* partitions of $[\mathbf{a}, \mathbf{b}]$; that is, partitions of the kind

$$\xi = \xi_1 \times \xi_2 \times \cdots \times \xi_n \text{ with } \xi_i := \{t_j^{(i)}\}_{j=0}^{k_i}, \quad i = 1, \dots, n, \tag{2.4}$$

where $\{k_i\}_{i=1}^n \subset \mathbb{N}$ and for each i, ξ_i is a partition of $[a_i, b_i]$. The set of all net partitions of an interval $[\mathbf{a}, \mathbf{b}]$ will be denoted by $\pi([\mathbf{a}, \mathbf{b}])$.

A point in a net partition ξ is called a *node* and it is of the form

$$\mathbf{t}_\alpha := (t_{\alpha_1}^{(1)}, t_{\alpha_2}^{(2)}, t_{\alpha_3}^{(3)}, \dots, t_{\alpha_n}^{(n)}),$$

where $\mathbf{0} \leq \alpha = (\alpha_i)_{i=1}^n \leq \kappa$, with $\kappa := (k_i)_{i=1}^n$.

For the sake of simplicity in notation, we will simply write $\xi = \{\mathbf{t}_\alpha\}$, to refer to all the nodes that form a given partition ξ.

A cell of an n-dimensional interval $[\mathbf{a}, \mathbf{b}]$ is an n-dimensional subinterval of the form $[\mathbf{t}_{\alpha-\mathbf{1}}, \mathbf{t}_\alpha]$, for $\mathbf{0} < \alpha \leq \kappa$.

Note that

$$\begin{aligned} \mathbf{t}_\mathbf{0} &= (t_0^{(1)}, t_0^{(2)}, \dots, t_0^{(n)}) = (a_1, a_2, \dots, a_n) \quad \text{and} \\ \mathbf{t}_\kappa &= (t_{k_1}^{(1)}, t_{k_2}^{(2)}, \dots, t_{k_n}^{(n)}) = (b_1, b_2, \dots, b_n). \end{aligned}$$

Definition 2.3. Let $f : [\mathbf{a}, \mathbf{b}] \to M$ and $\Phi \in \mathcal{N}$. The Φ-*variation, in the sense of Vitali-Riesz* of f is defined as

$$\rho_\Phi^n(f, [\mathbf{a}, \mathbf{b}]) := \sup_{\xi \in \pi[\mathbf{a}, \mathbf{b}]} \rho_\Phi^n(f, [\mathbf{a}, \mathbf{b}], \xi), \tag{2.5}$$

where

$$\rho_\Phi^n(f, [\mathbf{a}, \mathbf{b}], \xi) := \sum_{1 \leq \alpha \leq \kappa} \Phi\left(\frac{\Delta_n(f, [\mathbf{t}_{\alpha-\mathbf{1}}, \mathbf{t}_\alpha])}{Vol\,[\mathbf{t}_{\alpha-\mathbf{1}}, \mathbf{t}_\alpha]} \right) Vol\,[\mathbf{t}_{\alpha-\mathbf{1}}, \mathbf{t}_\alpha].$$

Now, just as in [2], we need to define *the truncation* of a point (an interval or a function) by a given multi-index $\mathbf{0} < \eta \leq \mathbf{1}$. Notice that in this case, the entries of such η are either 0 or 1.

- The truncation of a point $\mathbf{x} \in \mathbb{R}^n$ by a multi-index $\mathbf{0} < \eta \leq \mathbf{1}$, which is denoted by $\mathbf{x}\lfloor\eta$, is defined as the $|\eta|$-tuple that is obtained if we suppress from $\mathbf{x}$ the entries for which the corresponding entries of η are equal to 0. That is, $\mathbf{x}\lfloor\eta = (x_i : i \in \{1, 2, \dots, n\}, \eta_i = 1)$. For instance, if $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$ and $\eta = (0, 1, 1, 0, 1)$ then $\mathbf{x}\lfloor\eta = (x_2, x_3, x_5)$.
- The truncation of an n-dimensional interval $[\mathbf{a}, \mathbf{b}]$ by a multi-index $\mathbf{0} < \eta \leq \mathbf{1}$ is defined as $[\mathbf{a}, \mathbf{b}]\lfloor\eta := [\mathbf{a}\lfloor\eta, \mathbf{b}\lfloor\eta]$.

- Given a function $f : [\mathbf{a}, \mathbf{b}] \to M$, a multi-index $\mathbf{0} < \eta \leq \mathbf{1}$ and a point $\mathbf{z} \in [\mathbf{a}, \mathbf{b}]$, we define $f_\eta^{\mathbf{z}} : [\mathbf{a}, \mathbf{b}]\lfloor\eta \to M$, the truncation of f by η, by the formula

$$f_\eta^{\mathbf{z}}(\mathbf{x}\lfloor\eta) := f(\eta\mathbf{x} + (\mathbf{1} - \eta)\mathbf{z}), \ \ x \in [\mathbf{a}, \mathbf{b}].$$

Note that the function $f_\eta^{\mathbf{z}}$ depends only on the $|\eta|$ variables x_i for which $\eta_i = 1$.

Remark 2.4. Given a function $f : [\mathbf{a}, \mathbf{b}] \to M$ and a multi-index $\eta \neq \mathbf{0}$, the $|\eta|$-dimensional Vitali difference for $f_\eta^{\mathbf{a}}$ (cf. (2.3)) is given by

$$\Delta_{|\eta|}(f_\eta^{\mathbf{a}}, [\mathbf{x}, \mathbf{y}]) := d\Big(\sum_{\substack{\theta \in \mathcal{E}(n) \\ \theta \leq \eta}} f\left(\eta(\theta\mathbf{x} + (\mathbf{1} - \theta)\mathbf{y}) + (\mathbf{1} - \eta)\mathbf{a}\right), \sum_{\substack{\theta \in \mathcal{O}(n) \\ \theta \leq \eta}} f(\eta(\theta\mathbf{x} + (\mathbf{1} - \theta)\mathbf{y}) + (\mathbf{1} - \eta)\mathbf{a})\Big).$$

Definition 2.5. Let $\Phi \in \mathcal{N}$ and let $(M, d, +, \cdot)$ be a metric semigroup. A function $f : [\mathbf{a}, \mathbf{b}] \to M$ is said to be of *bounded* Φ*-variation* in the sense of Vitali–Hardy–Riesz, if the total Φ-variation

$$TRV_\Phi(f, [\mathbf{a}, \mathbf{b}]) := \sum_{\mathbf{0} \neq \eta \leq \mathbf{1}} \rho_\Phi^{|\eta|}(f_\eta^{\mathbf{a}}, [\mathbf{a}, \mathbf{b}]\lfloor\eta) \tag{2.6}$$

is finite. The set of all functions f that satisfy $TRV_\Phi(f, [\mathbf{a}, \mathbf{b}]) < +\infty$ will be denoted by $RV_\Phi^n([\mathbf{a}, \mathbf{b}]; M)$.

3. The normed space $BRV_\Phi^n([\mathbf{a}, \mathbf{b}]; M)$

So far, our choice of metric semigroups as range sets, for functions defined on $\mathbb{R}^n$, suffices adequately to define a notion of n-dimensional variation; however, as we need to study a *superposition operator problem* between linear normed spaces in which the presence of this notion is desired, it will be necessary to ask for additional structure on the range set M. The one that we will considerate is that of *vectorial metric space.*

Definition 3.1. By a metric vector space (`MVS`) we will understand a topological vector space $(\mathcal{M}, \tau)$ in which the topology τ is induced as a metric d that satisfies the following conditions:

(1) d is a translation invariant metric.
(2) $d(\alpha a, \alpha b) = |\alpha|\, d(a, b)$, for any $\alpha \in \mathbb{R}$ and $a, b \in \mathcal{M}$.

Note that any `MVS` is, in particular, a metric semigroup. In what follows, M is supposed to be an `MVS` and $[\mathbf{a}, \mathbf{b}]$ – an n-dimensional closed interval.

Remark 3.2. It readily follows from 2.1 that, given two functions $f, g : [\mathbf{a}, \mathbf{b}] \to M$, a multi-index $\eta \neq \mathbf{0}$ and an n-dimensional interval $[\mathbf{x}, \mathbf{y}] \subset [\mathbf{a}, \mathbf{b}]$, the $|\eta|$-dimensional Vitali difference (c.f. (2.3)) of the truncation $(f+g)^{\mathbf{a}}_{\eta}$ $(= f^{\mathbf{a}}_{\eta} + g^{\mathbf{a}}_{\eta})$ satisfies the inequality

$$\Delta_{|\eta|}\left((f+g)^{\mathbf{a}}_{\eta}, [\mathbf{x}, \mathbf{y}]\right) \leq \Delta_{|\eta|}\left(f^{\mathbf{a}}_{\eta}, [\mathbf{x}, \mathbf{y}]\right) + \Delta_{|\eta|}\left(g^{\mathbf{a}}_{\eta}, [\mathbf{x}, \mathbf{y}]\right). \tag{3.1}$$

Lemma 3.3. *The functional* $TRV_\Phi(\cdot, [\mathbf{a}, \mathbf{b}])$ *is convex.*

Proof. The lemma is a consequence of (3.1) and of the fact that Φ is a non decreasing convex function. ∎

Theorem 3.4. *The class* $RV^n_\Phi([\mathbf{a}, \mathbf{b}]; M)$ *is symmetric and convex.*

Proof. That $RV^n_\Phi([\mathbf{a}, \mathbf{b}]; M)$ is symmetric is a consequence of property (2) (since $d(-a, -b) \leq d(a, b)$) of Definition 3.1 while convexity follows from Lemma 3.3. ∎

As a consequence of Theorem 3.4, the linear space generated by $RV^n_\Phi([\mathbf{a}, \mathbf{b}]; M)$ is the set

$$\{f : [\mathbf{a}, \mathbf{b}] \to M \ : \ \lambda f \in RV^n_\Phi([\mathbf{a}, \mathbf{b}]; M) \ \text{ for some } \lambda > 0\},$$

which we will call the *space of functions of bounded* Φ*-variation in the sense of Vitali–Hardy–Riesz* and will denote as $BRV^n_\Phi([\mathbf{a}, \mathbf{b}]; M)$.

Lemma 3.5. *The set*

$$\Lambda := \{f \in BRV^n_\Phi([\mathbf{a}, \mathbf{b}]; M) \ / \ TRV_\Phi(f, [\mathbf{a}, \mathbf{b}]) \leq 1\}$$

is a convex, balanced and absorbent subset of $BRV^n_\Phi([\mathbf{a}, \mathbf{b}]; M)$.

Proof. To prove convexity suppose that $f, g \in \Lambda$ and let α, β be non-negative real numbers such that $\alpha + \beta = 1$. Then $TRV_\Phi(f, [\mathbf{a}, \mathbf{b}]) \leq 1$, $TRV_\Phi(g, [\mathbf{a}, \mathbf{b}]) \leq 1$ and by Lemma 3.3

$$\begin{aligned} TRV_\Phi(\alpha f + \beta g, [\mathbf{a}, \mathbf{b}]) &\leq \alpha TRV_\Phi(f, [\mathbf{a}, \mathbf{b}]) + \beta TRV_\Phi(g, [\mathbf{a}, \mathbf{b}]) \\ &\leq \alpha + \beta = 1. \end{aligned}$$

Hence Λ is convex.

On the other hand, from Definition 2.3 it readily follows that if $f_0 \equiv 0$ then $TRV_\Phi(f_0, [\mathbf{a}, \mathbf{b}]) = 0$, thus $f_0 \in \Lambda$ and therefore, by virtue of the convexity property of Λ just proved, Λ is balanced. Finally, the fact that Λ is absorbent follows from property (2) of Definition 3.1 and the convexity of Φ. ∎

By virtue of 3.5, the *Minkowski Functional* of Λ

$$p_\Lambda(f) := \inf\left\{t > 0 : TRV_\Phi\left(\frac{f}{t}, [\mathbf{a}, \mathbf{b}]\right) \leq 1\right\},$$

defines a seminorm on $BRV_\Phi^n([\mathbf{a},\mathbf{b}];M)$, and therefore

$$\|f\| := \|f\|_{BRV_\Phi^n([\mathbf{a},\mathbf{b}];M)} := d(f(\mathbf{a}),0) + p_\Lambda(f) \tag{3.2}$$

defines a norm on $BRV_\Phi^n([\mathbf{a},\mathbf{b}];M)$.

Lemma 3.6. *Let $f \in BRV_\Phi^n([\mathbf{a},\mathbf{b}];M)$,*

(i) *if $\|f\| \neq 0$ then $TRV_\Phi(f/\|f\|,[\mathbf{a},\mathbf{b}]) \le 1$;*
(ii) *if $0 \neq \|f\| \le 1$ then $TRV_\Phi(f,[\mathbf{a},\mathbf{b}]) \le \|f\|$.*

Proof. (i) From Definition 3.2 $p_\Lambda(f) \le \|f\|$.

If $p_\Lambda(f) < \|f\|$ then there is $\xi \in \Lambda$ such that $p_\Lambda(f) < \xi \le \|f\|$ and $TRV_\Phi\left(\frac{f}{\xi},[\mathbf{a},\mathbf{b}]\right) \le 1$. The convexity of Λ implies then that $\frac{f}{\|f\|} \in \Lambda$.

If $p_\Lambda(f) = \|f\|$ then there is a sequence $t_n \in \Lambda$ such that

$$t_n \to \|f\| \quad \text{and} \quad TRV_\Phi\left(\frac{f}{t_n},[\mathbf{a},\mathbf{b}]\right) \le 1.$$

It follows, by continuity, that

$$TRV_\Phi\left(\frac{f}{\|f\|},[\mathbf{a},\mathbf{b}]\right) \le 1.$$

(ii) Follows from (i) and the convexity of $TRV_\Phi(\cdot,[\mathbf{a},\mathbf{b}])$. ∎

Remark 3.7. Recall that if Φ is any φ-function then

$$\lim_{r\to 0^+} r\Phi^{-1}(c/r) = 0, \quad \forall\ c \in [0,+\infty). \tag{3.3}$$

4. Main result

In this section, we state and prove the main result of this paper concerning the action of a superposition operator between spaces of functions of bounded n-dimensional Φ-variation. Since we are going to deal with different φ-functions, and for the sake of clarity of exposition, we will denote the norm of $BRV_\Phi^n([\mathbf{a},\mathbf{b}];M)$ by $\|\cdot\|_{(\Phi,M)}$.

Theorem 4.1. *Suppose that $[\mathbf{a},\mathbf{b}] \subseteq \mathbb{R}^n$ is an n-dimensional interval, Φ and $\Psi \in \mathcal{N}$, M and N are* `MVS` *and $\mathcal{C}$ is a convex and closed subset of M. If a composition operator **H**, generated by the function $h : [\mathbf{a},\mathbf{b}] \times \mathcal{C} \to N$ which is continuous in the first variable, maps the set $\{f \in BRV_\Phi^n([\mathbf{a},\mathbf{b}];M) : f([\mathbf{a},\mathbf{b}]) \subset \mathcal{C})\}$ into $BRV_\Psi^n([\mathbf{a},\mathbf{b}];N)$ and is uniformly continuous then there are functions $A \in L(M,N)$ and $B : [\mathbf{a},\mathbf{b}] \to N$ such that*

$$h(\mathbf{t},u) = A(\mathbf{t})u + B(\mathbf{t}), \qquad \mathbf{t} \in [\mathbf{a},\mathbf{b}], \quad u \in \mathcal{C}. \tag{4.1}$$

In addition, if $0 \in \mathcal{C}$ then $B \in BRV_\Psi^n([\mathbf{a},\mathbf{b}];N)$.

Proof. Since $\mathbf{H}$ is uniformly continuous, given $\epsilon > 0$ there is $\delta > 0$ such that $\|\mathbf{H}(f_1) - \mathbf{H}(f_2)\|_\Psi \leq \epsilon$ whenever $f_1, f_2 \in \{f \in BRV_\Phi^n([\mathbf{a},\mathbf{b}]; M) : f([\mathbf{a},\mathbf{b}]) \subset \mathcal{C})\}$ satisfy $\|f_1 - f_2\|_\Phi \leq \delta$.

It follows that the *modulus of continuity* of $\mathbf{H}$:

$$\omega(p) := \sup\left\{\|\mathbf{H}(f_1) - \mathbf{H}(f_2)\|_{\Psi} : \|f_1 - f_2\|_{\Phi} \leq p\right\}\ (p > 0)$$

is well defined, continuous at zero, $\omega(0) = 0$ and

$$\|\mathbf{H}f_1 - \mathbf{H}f_2\|_\Psi \leq \omega(\|f_1 - f_2\|_\Phi). \tag{4.2}$$

From inequality (4.2) and (3.6) we get:

$$\begin{aligned}\rho_\Phi^n\left(\frac{\mathbf{H}(f_1) - \mathbf{H}(f_2)}{\omega(\|f_1 - f_2\|_\Phi)}, [\mathbf{a},\mathbf{b}]\right) &\leq TRV_\Phi\left(\frac{\mathbf{H}(f_1) - \mathbf{H}(f_2)}{\omega(\|f_1 - f_2\|_\Phi)}, [\mathbf{a},\mathbf{b}]\right)\\ &\leq TRV_\Phi\left(\frac{\mathbf{H}(f_1) - \mathbf{H}(f_2)}{\|Hf_1 - \mathbf{H}f_2\|_\Psi}\frac{\|Hf_1 - \mathbf{H}f_2\|_\Psi}{\omega(\|f_1 - f_2\|_\Phi)}, [\mathbf{a},\mathbf{b}]\right)\\ &\leq \frac{\|Hf_1 - \mathbf{H}f_2\|_\Psi}{\omega(\|f_1 - f_2\|_\Phi)} TRV_\Phi\left(\frac{\mathbf{H}(f_1) - \mathbf{H}(f_2)}{\|Hf_1 - \mathbf{H}f_2\|_\Psi}, [\mathbf{a},\mathbf{b}]\right) \leq 1.\end{aligned} \tag{4.3}$$

Thus

$$\begin{aligned}1 &\geq \rho_\Phi^n\left(\frac{\mathbf{H}(f_1) - \mathbf{H}(f_2)}{\omega(\|f_1 - f_2\|_\Phi)}, [\mathbf{a},\mathbf{b}]\right)\\ &\geq \Phi\left(\frac{\Delta_n\left(\frac{\mathbf{H}(f_1) - \mathbf{H}(f_2), [\mathbf{t}_1,\mathbf{t}_2]}{\omega(\|f_1 - f_2\|_\Phi)}\right)}{Vol\,[\mathbf{t}_1,\mathbf{t}_2]}\right) Vol\,[\mathbf{t}_1,\mathbf{t}_2],\end{aligned}$$

which implies

$$\Phi^{-1}\left(\frac{1}{Vol\,[\mathbf{t}_1,\mathbf{t}_2]}\right) Vol\,[\mathbf{t}_1,\mathbf{t}_2] \geq \Delta_n\left(\frac{\mathbf{H}(f_1) - \mathbf{H}(f_2), [\mathbf{t}_1,\mathbf{t}_2]}{\omega(\|f_1 - f_2\|_\Phi)}\right)$$

and

$$\begin{aligned}&\Delta_n\left(\mathbf{H}(f_1) - \mathbf{H}(f_2), [\mathbf{t}_1,\mathbf{t}_2]\right)\\ &\qquad\leq \Phi^{-1}\left(\frac{1}{Vol\,[\mathbf{t}_1,\mathbf{t}_2]}\right) Vol\,[\mathbf{t}_1,\mathbf{t}_2]\omega(\|f_1 - f_2\|_\Phi).\end{aligned} \tag{4.4}$$

To prove that h is continuous in the second variable, we proceed as follows: let y and $\widetilde{y}$ be two points in $\mathcal{C}$ and define

$$f_1(\mathbf{x}) := y \text{ and } f_2(\mathbf{x}) := \widetilde{y}, \qquad \mathbf{x} \in [\mathbf{a},\mathbf{b}].$$

Then, $f_1, f_2 \in \{f \in BRV_\Phi^n([\mathbf{a},\mathbf{b}]; M) : f([\mathbf{a},\mathbf{b}]) \subset \mathcal{C})\}$ and

$$\|f_1 - f_2\|_\Phi = d((f_1 - f_2)(\mathbf{a}), 0) + p_\Phi(f_1 - f_2) = d(y - \widetilde{y}, 0).$$

Set $T := \mathbf{H}(f_1) - \mathbf{H}(f_2)$. Then, for $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$, we have

$$(4.5) \quad d(h(\mathbf{x}, y) - h(\mathbf{x}, \widetilde{y}), 0) = d(\mathbf{H}f_1(\mathbf{x}) - \mathbf{H}f_2(\mathbf{x}), 0) = d(T(\mathbf{x}), 0) \\ \leq \Delta_n (T, [\mathbf{a}, \mathbf{x}]) + n \sum_{\mathbf{0}<\theta<\mathbf{1}} \Delta_{|\theta|} (T_\theta^{\mathbf{a}}, [\mathbf{a}, \mathbf{x}]\lfloor\theta) + \|T(\mathbf{a})\|.$$

Since (4.4) holds for any $\mathbf{t}_1, \mathbf{t}_2 \in [\mathbf{a}, \mathbf{b}]$, in particular it holds for $\mathbf{t}_1 = \mathbf{a}$ and $\mathbf{t}_2 = \mathbf{x}$, hence from (4.5), we get the inequality

$$(4.6) \quad d(h(\mathbf{x}, y) - h(\mathbf{x}, \widetilde{y}), 0)$$
$$\leq n\Delta_n (T, [\mathbf{a}, \mathbf{x}]) + n \sum_{\mathbf{0}<\theta<\mathbf{1}} \Delta_{|\theta|} (T_\theta^{\mathbf{a}}, [\mathbf{a}, \mathbf{x}]\lfloor\theta) + d(T(\mathbf{a}), 0)$$
$$\leq n \sum_{\mathbf{0}<\theta\leq\mathbf{1}} \omega(\|f_1 - f_2\|_\Phi)\Phi^{-1}\left(\frac{1}{Vol\,[\mathbf{a}, \mathbf{x}]\lfloor\theta}\right) Vol\,[\mathbf{a}, \mathbf{x}]\lfloor\theta + d((\mathbf{H}f_1 - \mathbf{H}f_2)(\mathbf{a}), 0)$$
$$\leq n \sum_{\mathbf{0}<\theta\leq\mathbf{1}} \omega(\|f_1 - f_2\|_\Phi)\Phi^{-1}\left(\frac{1}{Vol\,[\mathbf{a}, \mathbf{x}]\lfloor\theta}\right) Vol\,[\mathbf{a}, \mathbf{x}]\lfloor\theta + \|(\mathbf{H}f_1 - \mathbf{H}f_2)\|_\Psi$$
$$\leq n \sum_{\mathbf{0}<\theta\leq\mathbf{1}} \omega(\|f_1 - f_2\|_\Phi)\Phi^{-1}\left(\frac{1}{Vol\,[\mathbf{a}, \mathbf{x}]\lfloor\theta}\right) Vol\,[\mathbf{a}, \mathbf{x}]\lfloor\theta + \omega(\|f_1 - f_2\|_\Phi)$$
$$= \left\{ n \sum_{\mathbf{0}<\theta\leq\mathbf{1}} \Phi^{-1}\left(\frac{1}{Vol\,[\mathbf{a}, \mathbf{x}]\lfloor\theta}\right) Vol\,[\mathbf{a}, \mathbf{x}]\lfloor\theta + 1 \right\} \omega(\|f_1 - f_2\|_\Phi)$$
$$= \left\{ n \sum_{\mathbf{0}<\theta\leq\mathbf{1}} \Phi^{-1}\left(\frac{1}{Vol\,[\mathbf{a}, \mathbf{x}]\lfloor\theta}\right) Vol\,[\mathbf{a}, \mathbf{x}]\lfloor\theta + 1 \right\} \omega(\|y - \widetilde{y}\|_M).$$

Consequently, as $y \to \widetilde{y}$ the limit of the right hand side of (4.6) is zero which proves the continuity of h in the second variable.

Now we will show that h satisfies the *Jensen equation* in the second variable.

Indeed, let $\mathbf{t}_1 = (t_1^{(i)})_{i=1}^n$ and $\mathbf{t}_2 = (t_2^{(i)})_{i=1}^n \in [\mathbf{a}, \mathbf{b}]$, suppose further that $\mathbf{t}_1 \leq \mathbf{t}_2$, and define the functions

$$\eta_i(t) := \begin{cases} 0, & \text{if } a_i \leq t \leq t_1^{(i)}, \\ \dfrac{t - t_1^{(i)}}{t_2^{(i)} - t_1^{(i)}}, & \text{if } t_1^{(i)} \leq t \leq t_2^{(i)}, \\ 1, & \text{if } t_2^{(i)} \leq t \leq b_i. \end{cases}$$

Next, consider $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{C}$, $\mathbf{y}_1 \neq \mathbf{y}_2$ and define

$$(4.7) \qquad f_j(\mathbf{x}) := \frac{1}{2}\Big[\prod_{i=1}^n \eta_i(x_i)(\mathbf{y}_1 - \mathbf{y}_2) + \mathbf{y}_j + \mathbf{y}_2\Big],$$

for $j = 1, 2$, where $\mathbf{x} := (x_1, x_2, \ldots, x_n)$.

Notice that

$$
\begin{aligned}
&f_1(\mathbf{x}) - f_2(\mathbf{x}) \\
&\quad = \frac{1}{2}\Big[\prod_{i=1}^{n} \eta(x_i)(\mathbf{y}_1 - \mathbf{y}_2) + \mathbf{y}_1 + \mathbf{y}_2 - \prod_{i=1}^{n} \eta(x_i)(\mathbf{y}_1 - \mathbf{y}_2) - \mathbf{y}_2 - \mathbf{y}_2\Big] \\
&\quad = \frac{\mathbf{y}_1 - \mathbf{y}_2}{2}.
\end{aligned}
$$

Hence $f_1 - f_2$ has zero Φ-variation and

$$
\begin{aligned}
\|f_1 - f_2\|_{(\Phi,M)} &= d((f_1 - f_2)(\mathbf{a}), 0) + p_\varphi(f_1 - f_2) \\
&= d((f_1 - f_2)(\mathbf{a}), 0) = d\left(\frac{\mathbf{y}_1 - \mathbf{y}_2}{2}, 0\right) = d\Big(\frac{\mathbf{y}_1}{2}, \frac{\mathbf{y}_2}{2}\Big) > 0.
\end{aligned}
$$

Notice then that

- If $\mathbf{x} = \mathbf{t}_\alpha$ where $\alpha_i = 2$, for $i = 1, 2, \ldots, n$ then

$$
\prod_{i=1}^{n} \eta(t_{\alpha_i}^{(i)}) = \prod_{i=1}^{n} \frac{t_{\alpha_i}^{(i)} - t_1^{(i)}}{t_2^{(i)} - t_1^{(i)}} = 1.
$$

- If $\mathbf{x} = \mathbf{t}_\alpha$ with $\alpha_i \neq 2$, for some $1 \leq i \leq n$ then

$$
\prod_{i=1}^{n} \eta(t_{\alpha_i}^{(i)}) = \prod_{i=1}^{n} \frac{t_{\alpha_i}^{(i)} - t_1^{(i)}}{t_2^{(i)} - t_1^{(i)}} = 0.
$$

Thus, by (4.7)

- If $\alpha_i = 2$, for $i = 1, 2, \ldots, n$ then

$$
\begin{aligned}
f_1(\mathbf{t}_\alpha) &:= \frac{1}{2}\Big[\prod_{i=1}^{n} \eta(t_{\alpha_i}^{(i)})(\mathbf{y}_1 - \mathbf{y}_2) + \mathbf{y}_1 + \mathbf{y}_2\Big] = \mathbf{y}_1, \text{ and} \\
f_2(\mathbf{t}_\alpha) &:= \frac{1}{2}\Big[\prod_{i=1}^{n} \eta(t_{\alpha_i}^{(i)})(\mathbf{y}_1 - \mathbf{y}_2) + \mathbf{y}_2 + \mathbf{y}_2\Big]\frac{\mathbf{y}_1 + \mathbf{y}_2}{2}.
\end{aligned}
$$

- If $\alpha_k \neq 2$, for some $1 \leq k \leq n$ then

$$
\begin{aligned}
f_1(\mathbf{t}_\alpha) &:= \frac{1}{2}\left[\mathbf{y}_1 + \mathbf{y}_2\right] = \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}, \\
f_2(\mathbf{t}_\alpha) &:= \mathbf{y}_2.
\end{aligned}
$$

Thus, by the definition of $\mathbf{H}$, we have

$$\begin{aligned}
\mathbf{H}f_1(\mathbf{t}_2) &= h(\mathbf{t}_2, f_1(\mathbf{t}_2)) = h(\mathbf{t}_2, \mathbf{y}_1),\\
\mathbf{H}f_2(\mathbf{t}_2) &= h(\mathbf{t}_2, f_2(\mathbf{t}_2)) = h\Big(\mathbf{t}_2, \frac{\mathbf{y}_1+\mathbf{y}_2}{2}\Big),\\
\mathbf{H}f_1(\mathbf{t}_1) &= h(\mathbf{t}_1, f_1(\mathbf{t}_1)) = h\Big(\mathbf{t}_1, \frac{\mathbf{y}_1+\mathbf{y}_2}{2}\Big),\\
\mathbf{H}f_2(\mathbf{t}_1) &= h(\mathbf{t}_1, f_2(\mathbf{t}_1)) = h(\mathbf{t}_1, \mathbf{y}_2),
\end{aligned}$$

and, if θ is a non-zero multi-index different from $\mathbf{1}$

$$\begin{aligned}
\mathbf{H}f_1(\theta\,\mathbf{t}_1 + (1-\theta)\mathbf{t}_2) &= h\Big(\theta\,\mathbf{t}_1 + (1-\theta)\mathbf{t}_2, \frac{\mathbf{y}_1+\mathbf{y}_2}{2}\Big),\\
\mathbf{H}f_2(\theta\,\mathbf{t}_1 + (1-\theta)\mathbf{t}_2) &= h(\theta\mathbf{t}_1 + (1-\theta)\mathbf{t}_2, \mathbf{y}_2).
\end{aligned}$$

Letting $\mathbf{t}_2 \to \mathbf{t}_1$ on the left hand side of (4.4), we get

$$\lim_{\mathbf{t}_2\to\mathbf{t}_1} d\Big(\sum_{\theta\le\mathbf{1}} (-1)^{|\theta|}\,(\mathbf{H}(f_1)-\mathbf{H}(f_2))(\theta\,\mathbf{t}_1 + (\mathbf{1}-\theta)\mathbf{t}_2), 0\Big) \tag{4.8}$$

$$\begin{aligned}
&= d\Big(h(\mathbf{t}_1,\mathbf{y}_1) - h\Big(\mathbf{t}_1, \frac{\mathbf{y}_1+\mathbf{y}_2}{2}\Big)\\
&\quad + \lim_{\mathbf{t}_2\to\mathbf{t}_1} \sum_{\substack{\theta\le\mathbf{1}\\ \theta\ne\mathbf{1}}} (-1)^{|\theta|}\,(\mathbf{H}(f_1)-\mathbf{H}(f_2))(\theta\,\mathbf{t}_1 + (\mathbf{1}-\theta)\mathbf{t}_2), 0\Big)\\
&= d\Big(h(\mathbf{t}_1,\mathbf{y}_1) - h\Big(\mathbf{t}_1, \frac{\mathbf{y}_1+\mathbf{y}_2}{2}\Big)\\
&\quad + \lim_{\mathbf{t}_2\to\mathbf{t}_1} \sum_{\substack{\theta\le\mathbf{1}\\ \theta\ne\mathbf{1}}} (-1)^{|\theta|}\Big[h\Big(\theta\,\mathbf{t}_1 + (\mathbf{1}-\theta)\mathbf{t}_2, \frac{\mathbf{y}_1+\mathbf{y}_2}{2}\Big) - h(\theta\,\mathbf{t}_1 + (\mathbf{1}-\theta)\mathbf{t}_2, \mathbf{y}_2)\Big], 0\Big)
\end{aligned}$$

which, by continuity of h in the first variable, is equal to

$$\begin{aligned}
&= d\Big(h(\mathbf{t}_1,\mathbf{y}_1) - h\Big(\mathbf{t}_1, \frac{\mathbf{y}_1+\mathbf{y}_2}{2}\Big)\\
&\quad + \sum_{\substack{\theta\le\mathbf{1}\\ \theta\ne\mathbf{1}}} (-1)^{|\theta|}\Big[h\Big(\theta\mathbf{t}_1 + (\mathbf{1}-\theta)\mathbf{t}_1, \frac{\mathbf{y}_1+\mathbf{y}_2}{2}\Big) - h(\theta\mathbf{t}_1 + (\mathbf{1}-\theta)\mathbf{t}_1, \mathbf{y}_2)\Big], 0\Big)\\
&= d\Big(h(\mathbf{t}_1,\mathbf{y}_1) - h\Big(\mathbf{t}_1, \frac{\mathbf{y}_1+\mathbf{y}_2}{2}\Big) + \sum_{\substack{\theta\le\mathbf{1}\\ \theta\ne\mathbf{1}}} (-1)^{|\theta|}\Big[h\Big(\mathbf{t}_1, \frac{\mathbf{y}_1+\mathbf{y}_2}{2}\Big) - h(\mathbf{t}_1, \mathbf{y}_2)\Big], 0\Big).
\end{aligned}$$

Now, for $1 \le k \le n$, the number of n-tuples with k entries equal to 1, is equal to $\binom{n}{k} = \dfrac{n!}{(n-k)!\,k!}$, thus

$$
\begin{aligned}
&\sum_{\substack{\theta\le\mathbf{1}\\ \theta\neq\mathbf{0}}}(-1)^{|\theta|}\left[h\left(\mathbf{t}_1,\frac{\mathbf{y}_1+\mathbf{y}_2}{2}\right)-h(\mathbf{t}_1,\mathbf{y}_2)\right]\\
&\qquad=\left[h\left(\mathbf{t}_1,\frac{\mathbf{y}_1+\mathbf{y}_2}{2}\right)-h(\mathbf{t}_1,\mathbf{y}_2)\right]\sum_{k=1}^{n}(-1)^k\binom{n}{k}\\
&\qquad=\left[h\left(\mathbf{t}_1,\frac{\mathbf{y}_1+\mathbf{y}_2}{2}\right)-h(\mathbf{t}_1,\mathbf{y}_2)\right]\left\{\sum_{k=0}^{n}(-1)^k\binom{n}{k}-\binom{n}{0}\right\}\\
&\qquad=\left[h\left(\mathbf{t}_1,\frac{\mathbf{y}_1+\mathbf{y}_2}{2}\right)-h(\mathbf{t}_1,\mathbf{y}_2)\right]\left\{(-1+1)^n-\binom{n}{0}\right\}\\
&\qquad=\left[h\left(\mathbf{t}_1,\frac{\mathbf{y}_1+\mathbf{y}_2}{2}\right)-h(\mathbf{t}_1,\mathbf{y}_2)\right]\{-1\}\,.
\end{aligned}
$$

Hence, from this identity and (4.8), we get

$$
\begin{aligned}
(4.9)\qquad &\lim_{\mathbf{t}_2\to\mathbf{t}_1} d\Big(\sum_{\theta\le\mathbf{1}}(-1)^{|\theta|}\,(\mathbf{H}(f_1)-\mathbf{H}(f_2))(\theta\,\mathbf{t}_1+(\mathbf{1}-\theta)\mathbf{t}_2),0\Big)\\
&=d\Bigg(h(\mathbf{t}_1,\mathbf{y}_1)-h\left(\mathbf{t}_1,\frac{\mathbf{y}_1+\mathbf{y}_2}{2}\right)\\
&\quad+\sum_{\substack{\theta\le\mathbf{1}\\ \theta\neq 0}}(-1)^{|\theta|}\left[h\left(\mathbf{t}_1,\frac{\mathbf{y}_1+\mathbf{y}_2}{2}\right)-h(\mathbf{t}_1,\mathbf{y}_2)\right],0\Bigg)\\
&=d\left(h(\mathbf{t}_1,\mathbf{y}_1)-h\left(\mathbf{t}_1,\frac{\mathbf{y}_1+\mathbf{y}_2}{2}\right)-h\left(\mathbf{t}_1,\frac{\mathbf{y}_1+\mathbf{y}_2}{2}\right)+h(\mathbf{t}_1,\mathbf{y}_2),0\right).
\end{aligned}
$$

On the other hand, property (3.3) implies that the limit as $\mathbf{t}_2\to\mathbf{t}_1$ on the right side of (4.4) is zero, therefore

$$
d\left(h(\mathbf{t}_1,\mathbf{y}_1)-h\left(\mathbf{t}_1,\frac{\mathbf{y}_1+\mathbf{y}_2}{2}\right)-h\left(\mathbf{t}_1,\frac{\mathbf{y}_1+\mathbf{y}_2}{2}\right)+h(\mathbf{t}_1,\mathbf{y}_2),0\right)=0
$$

or equivalently

$$
\frac{h(\mathbf{t}_1,\mathbf{y}_1)+h(\mathbf{t}_1,\mathbf{y}_2)}{2}=h\left(\mathbf{t}_1,\frac{\mathbf{y}_1+\mathbf{y}_2}{2}\right).
$$

Thus $h(\mathbf{t}_1,\cdot)$ is a solution for the Jensen equation in $\mathcal{C}$ for $\mathbf{t}_1\in[\mathbf{a},\mathbf{b}]$.

Adapting a classical standard argument (c.f Kuczma [7], see also [9]), we conclude that there exist $A(\mathbf{t}_1)\in\mathcal{L}(M,N)$ and $B\in N^{[\mathbf{a}.\mathbf{b}]}$ such that

$$
(4.10)\qquad h(\mathbf{t}_1,\mathbf{y})=A(\mathbf{t}_1)\mathbf{y}+B(\mathbf{t}_1)\ \ \mathbf{y}\in\mathcal{C}.
$$

Finally, notice that if $0\in\mathcal{C}$ then taking $y=0$ in (4.10), we have $h(\mathbf{t},\mathbf{0})=B(\mathbf{t})$, for $\mathbf{t}\in[\mathbf{a},\mathbf{b}]$, which implies that $B\in BRV^n_\Psi([\mathbf{a},\mathbf{b}];N)$. ∎

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References

[1] J. Appell, P. P. Zabrejko, *Nonlinear Superposition Operator*, Cambridge University Press, New York, 1990.
[2] V. V. Chistyakov, *Functions of several variables of finite variation and superposition operators*, in: Real Analysis Exchange 26th Summer Symposium, Lexington, VA, USA, 2002, pp. 61–66.
[3] V. V. Chistyakov, *A selection principle for mappings of bounded variation of several variables*, in: Real Analysis Exchange 27th Summer Symposium, Opava, Czech Republic, 2003, pp. 217–222.
[4] V. V. Chistyakov, Y. Tretyachenko, *Maps of several variables of finite total variation and Helly-type selection principles*, J. Math. Anal. Appl. 370(2) (2010), 672–686.
[5] J. A. Clarkson, C. R. Adams, *On definitions of bounded variation for functions of two variables*, Trans. Amer. Math. Soc. 35 (1933), 824–854.
[6] T. H. Hildebrandt, *Introduction to the Theory of Integration*, Academic Press, New York and London, 1963.
[7] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Polish Scientific Editors and Silesian University, Warszawa, Kraków, Katowice, 1985.
[8] K. Lichawski, J. Matkowski, J. Miŝ, *Locally defined operators in the space of differentiable functions*, Bull. Polish Acad. Sci. Math. 37 (1989), 315–125.
[9] J. Matkowski, *Uniformly continuous superposition operators in the space of bounded variation functions*, Math. Nachr. 283(7) (2010), 1060–1064.
[10] F. A. Talalyan, *A multidimensional analogue of a theorem of F. Riesz*, Sbornik: Mathematics 186(9) (1995), 1363–1374.
[11] G. Vitali, *Sui gruppi di punti e sulle funzioni di variabili reali*, Atti Accad. Sci. Torino 43 (1908), 75–92.
[12] M. Wróbel, *Representation theorem for local operators in the space of continuous and monotone functions*, J. Math. Anal. Appl. 372 (2010), 45–54.
[13] M. Wróbel, *Locally defined operators in the Hölder spaces*, Nonlinear Anal. 74 (2011), 317–323.

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