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SOME OSTROWSKI'S TYPE INEQUALITIES FOR FUNCTIONS WHOSE SECOND DERIVATIVES ARE s-CONVEX IN THE SECOND SENSE

Abstract. Some new inequalities of the Ostrowski type for twice differentiable mappings whose derivatives in absolute value are s-convex in the second sense are given.

1. Introduction

In 1938, Ostrowski proved the following integral inequality [12]:

Theorem 1. *Let $I \subseteq \mathbb{R}$, $f : I \to \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, the inequality holds:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\left\|f'\right\|_\infty,$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

For some applications of Ostrowski's inequality see ([1]–[4]) and for recent results and generalizations concerning Ostrowski's inequality see ([1]–[8]).

The class of s-convexity in the second sense is defined in the following way [9, 11]: a function $f : [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense if

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y),$$

for all $x, y \in [0,\infty)$, $t \in [0,1]$ and some fixed $s \in (0,1]$. This class is usually denoted by K_s^2.

In [10], Dragomir and Fitzpatrick proved the Hadamard inequality for s-convex functions in the second sense:

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Theorem 2. *Suppose that $f : [0, \infty) \to [0, \infty)$ is an s-convex function in the second sense, where $s \in (0, 1)$, and let $a, b \in [0, \infty)$, $a < b$. If $f \in L^1([a, b])$ then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{s+1}. \tag{1.1}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.1).

In [3], Cerone et al. proved the following inequalities of Ostrowski type and Hadamard type, respectively.

Theorem 3. *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping on (a, b) and $f'' : (a, b) \to \mathbb{R}$ is bounded, i.e. $\|f''\|_\infty = \sup_{t\in(a,b)} |f''(t)| < \infty$. Then we have the inequality:*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \left(x - \frac{a+b}{2}\right) f'(x)\right| \\ &\quad \le \left[\frac{1}{24}(b-a)^2 + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2\right] \|f''\|_\infty \le \frac{(b-a)^2}{6}\|f''\|_\infty , \end{aligned} \tag{1.2}$$

for all $x \in [a, b]$.

Corollary 1. *Under the above assumptions, we have the mid-point inequality:*

$$\left| f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{24}\|f''\|_\infty . \tag{1.3}$$

In this article, we establish new Ostrowski's type inequalities for s-convex functions in the second sense.

2. Main results

In order to establish our main results we need the following lemma.

Lemma 1. *Let $I \subseteq \mathbb{R}$, $f : I \to \mathbb{R}$ be a twice differentiable function on I° with $f'' \in L_1[a, b]$ where $a, b \in I$ with $a < b$. Then*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(u)du - f(x) + \left(x - \frac{a+b}{2}\right) f'(x) \\ &= \frac{(x-a)^3}{2(b-a)}\int_0^1 t^2 f''(tx + (1-t)a)dt + \frac{(b-x)^3}{2(b-a)}\int_0^1 t^2 f''(tx + (1-t)b)dt, \end{aligned} \tag{2.1}$$

for each $x \in [a, b]$.

Proof. By integration by parts, we have the following identity

$$\int_0^1 t^2 f''(tx+(1-t)a)dt \tag{2.2}$$

$$= \frac{t^2}{(x-a)} f'(tx+(1-t)a)\Big|_0^1 - \frac{2}{x-a}\int_0^1 t f'(tx+(1-t)a)dt$$

$$= \frac{f'(x)}{(x-a)} - \frac{2}{x-a}\left[\frac{t}{(x-a)} f(tx+(1-t)a)\Big|_0^1 - \frac{1}{x-a}\int_0^1 f(tx+(1-t)a)dt\right]$$

$$= \frac{f'(x)}{(x-a)} - \frac{2f(x)}{(x-a)^2} + \frac{2}{(x-a)^2}\int_0^1 f(tx+(1-t)a)dt.$$

By using the change of the variable $u = tx + (1-t)a$ for $t \in [0,1]$ and multiplying the both sides of (2.2) by $\frac{(x-a)^3}{2(b-a)}$, we obtain

$$\frac{(x-a)^3}{2(b-a)}\int_0^1 t^2 f''(tx+(1-t)a)dt \tag{2.3}$$

$$= \frac{(x-a)^2 f'(x)}{2(b-a)} - \frac{(x-a)f(x)}{b-a} + \frac{1}{b-a}\int_a^x f(u)du.$$

Similarly, we observe that

$$\frac{(b-x)^3}{2(b-a)}\int_0^1 t^2 f''(tx+(1-t)b)dt \tag{2.4}$$

$$= -\frac{(b-x)^2 f'(x)}{2(b-a)} - \frac{(b-x)f(x)}{b-a} + \frac{1}{b-a}\int_x^b f(u)du.$$

Thus, adding (2.3) and (2.4) we get the required identity (2.1). ∎

The following result may be stated:

Theorem 4. *Let $I \subset [0,\infty)$, $f : I \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a,b]$ where $a, b \in I$ with $a < b$. If $|f''|$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f(x) + \left(x - \frac{a+b}{2}\right) f'(x)\right| \tag{2.5}$$

$$\leq \frac{1}{2(b-a)}\left\{\left[\frac{|f''(x)|}{s+3} + \frac{2\,|f''(a)|}{(s+1)(s+2)(s+3)}\right](x-a)^3 + \left[\frac{|f''(x)|}{s+3} + \frac{2\,|f''(b)|}{(s+1)(s+2)(s+3)}\right](b-x)^3\right\},$$

for each $x \in [a,b]$.

Proof. By Lemma 1 and by s-convex of $|f''|$, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(u)du - f(x) + \left(x-\frac{a+b}{2}\right)f'(x)\right| \\
&\le \frac{(x-a)^3}{2(b-a)}\int_0^1 t^2\left|f''(tx+(1-t)a)\right|dt \\
&\quad + \frac{(b-x)^3}{2(b-a)}\int_0^1 t^2\left|f''(tx+(1-t)b)\right|dt \\
&\le \frac{(x-a)^3}{2(b-a)}\int_0^1 t^2\left[t^s\left|f''(x)\right| + (1-t)^s\left|f''(a)\right|\right]dt \\
&\quad + \frac{(b-x)^3}{2(b-a)}\int_0^1 t^2\left[t^s\left|f''(x)\right| + (1-t)^s\left|f''(b)\right|\right]dt \\
&= \frac{(x-a)^3}{2(b-a)}\int_0^1 \left(t^{s+2}\left|f''(x)\right| + t^2(1-t)^s\left|f''(a)\right|\right)dt \\
&\quad + \frac{(b-x)^3}{2(b-a)}\int_0^1 \left(t^{s+2}\left|f''(x)\right| + t^2(1-t)^s\left|f''(b)\right|\right)dt \\
&= \frac{(x-a)^3}{2(b-a)}\left[\frac{|f''(x)|}{s+3} + \frac{2\,|f''(a)|}{(s+1)(s+2)(s+3)}\right] \\
&\quad + \frac{(b-x)^3}{2(b-a)}\left[\frac{|f''(x)|}{s+3} + \frac{2\,|f''(b)|}{(s+1)(s+2)(s+3)}\right] \\
&= \frac{1}{2(b-a)}\left\{\left[\frac{|f''(x)|}{s+3} + \frac{2\,|f''(a)|}{(s+1)(s+2)(s+3)}\right](x-a)^3\right. \\
&\qquad \left. + \left[\frac{|f''(x)|}{s+3} + \frac{2\,|f''(b)|}{(s+1)(s+2)(s+3)}\right](b-x)^3\right\},
\end{aligned}$$

where we have used the fact that

$$\int_0^1 t^{s+2}dt = \frac{1}{s+3} \quad \text{and} \quad \int_0^1 t^2(1-t)^s dt = \frac{2}{(s+1)(s+2)(s+3)}.$$

This completes the proof. ∎

Corollary 2. *If we put* $M = \sup_{x\in[a,b]}|f''|$ *in Theorem* 4, *then we get*

$$\begin{aligned}
(2.6)\qquad &\left|\frac{1}{b-a}\int_a^b f(u)du - f(x) + \left(x-\frac{a+b}{2}\right)f'(x)\right| \\
&\le 3M\left(\frac{s^2+3s+4}{(s+1)(s+2)(s+3)}\right)\left[\frac{1}{24}(b-a)^2 + \frac{1}{2}\left(x-\frac{a+b}{2}\right)^2\right] \\
&\le M\frac{(b-a)^2}{2}\left(\frac{s^2+3s+4}{(s+1)(s+2)(s+3)}\right).
\end{aligned}$$

Here, simple computation shows that

$$(x-a)^3+(b-x)^3=(b-a)\left[\frac{(b-a)^2}{4}+3\left(x-\frac{a+b}{2}\right)^2\right].$$

Remark 1. If in Corollary 2 we choose $s=1$, then we recapture the inequality (1.2) for functions f with convex $|f''|$.

Corollary 3. *If in Corollary 2 we choose* $x=\frac{a+b}{2}$, *then we get the mid-point inequality*

$$\left|\frac{1}{b-a}\int_a^b f(u)du-f(\frac{a+b}{2})\right|\le M\frac{(b-a)^2}{2}\left(\frac{s^2+3s+4}{(s+1)(s+2)(s+3)}\right).$$

Theorem 5. *Let* $I\subset[0,\infty)$, $f:I\to\mathbb{R}$ *be a twice differentiable function on* I° *such that* $f''\in L_1[a,b]$ *where* $a,b\in I$ *with* $a<b$. *If* $|f''|^q$ *is s-convex in the second sense on* $[a,b]$, *for some fixed* $s\in(0,1]$, $p,q>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$, *then the following inequality holds:*

$$\begin{aligned}(2.7)\qquad &\left|\frac{1}{b-a}\int_a^b f(u)du-f(x)+\left(x-\frac{a+b}{2}\right)f'(x)\right|\\ &\le\frac{(x-a)^3}{2(b-a)}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{|f''(x)|^q+|f''(a)|^q}{s+1}\right)^{\frac{1}{q}}\\ &+\frac{(b-x)^3}{2(b-a)}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{|f''(x)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}},\end{aligned}$$

for each $x\in[a,b]$.

Proof. Suppose that $p>1$. From Lemma 1 and by the Hölder inequality, we have

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(u)du-f(x)+\left(x-\frac{a+b}{2}\right)f'(x)\right|\\ &\le\frac{(x-a)^3}{2(b-a)}\int_0^1 t^2\left|f''(tx+(1-t)a)\right|dt+\frac{(b-x)^3}{2(b-a)}\int_0^1 t^2\left|f''(tx+(1-t)b)\right|dt\\ &\le\frac{(x-a)^3}{2(b-a)}\left(\int_0^1 t^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\\ &+\frac{(b-x)^3}{2(b-a)}\left(\int_0^1 t^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

Since $|f''|^q$ is s-convex in the second sense, we have

$$\int_0^1 \left|f''(tx+(1-)a)\right|^q dt \leq \int_0^1 \left[t^s \left|f''(x)\right|^q + (1-t)^s \left|f''(a)\right|^q\right] dt$$
$$= \frac{|f''(x)|^q + |f''(a)|^q}{s+1}$$

and

$$\int_0^1 \left|f''(tx+(1-t)b)\right|^q dt \leq \int_0^1 \left[t^s \left|f''(x)\right|^q + (1-t)^s \left|f''(b)\right|^q\right] dt$$
$$= \frac{|f''(x)|^q + |f''(b)|^q}{s+1}.$$

Therefore, we have

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f(x) + \left(x - \frac{a+b}{2}\right) f'(x)\right|$$
$$\leq \frac{(x-a)^3}{2(b-a)}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{|f''(x)|^q + |f''(a)|^q}{s+1}\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^3}{2(b-a)}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{|f''(x)|^q + |f''(b)|^q}{s+1}\right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$, which is required. ■

Corollary 4. *Under the above assumptions, we have the following inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f(x) + \left(x - \frac{a+b}{2}\right) f'(x)\right| \tag{2.8}$$
$$\leq \frac{3M}{(2p+1)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\frac{(b-a)^2}{24} + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2\right].$$

This follows by Theorem 5 with $M = \sup_{x\in[a,b]} |f''|$.

Corollary 5. *With the assumptions in Corollary 4, one has the mid-point inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{8\,(2p+1)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}} M.$$

This follows by Corollary 4, choosing $x = \frac{a+b}{2}$.

Corollary 6. *With the assumptions in Corollary 4, one has the following perturbed trapezoid like inequality:*

$$\left|\int_a^b f(u)du - \frac{(b-a)}{2}\left[f(a)+f(b)\right] + \frac{(b-a)^2}{4}\left(f'(b)-f'(a)\right)\right| \leq \frac{(b-a)^3}{2(2p+1)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}} M.$$

This follows using Corollary 4 with $x = a$, $x = b$, adding the results and using the triangle inequality for the modulus.

Theorem 6. *Let $I \subset [0,\infty)$, $f : I \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a,b]$ where $a,b \in I$ with $a < b$. If $|f''|^q$ is s-convex in the second sense on $[a,b]$, for some fixed $s \in (0,1]$ and $q \geq 1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f(x) + \left(x-\frac{a+b}{2}\right)f'(x)\right| \tag{2.9}$$
$$\leq \frac{(x-a)^3}{2(b-a)}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{|f''(x)|^q}{s+3} + \frac{2|f''(a)|^q}{(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^3}{2(b-a)}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{|f''(x)|^q}{s+3} + \frac{2|f''(b)|^q}{(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}},$$

for each $x \in [a,b]$.

Proof. Suppose that $q \geq 1$. From Lemma 1 and by the well known power mean inequality, we have

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f(x) + \left(x-\frac{a+b}{2}\right)f'(x)\right|$$
$$\leq \frac{(x-a)^3}{2(b-a)}\int_0^1 t^2\left|f''(tx+(1-t)a)\right|dt + \frac{(b-x)^3}{2(b-a)}\int_0^1 t^2\left|f''(tx+(1-t)b)\right|dt$$
$$\leq \frac{(x-a)^3}{2(b-a)}\left(\int_0^1 t^2 dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2\left|f''(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^3}{2(b-a)}\left(\int_0^1 t^2 dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2\left|f''(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f''|^q$ is s-convex in the second sense, we have

$$\int_0^1 t^2 \left|f''(tx+(1-t)a)\right|^q dt \le \int_0^1 \left[t^{s+2}\left|f''(x)\right|^q + t^2(1-t)^s\left|f''(a)\right|^q\right]dt$$
$$= \frac{|f''(x)|^q}{s+3} + \frac{2\,|f''(a)|^q}{(s+1)(s+2)(s+3)}$$

and

$$\int_0^1 t^2 \left|f''(tx+(1-t)b)\right|^q dt \le \int_0^1 \left[t^{s+2}\left|f''(x)\right|^q + t^2(1-t)^s\left|f''(b)\right|^q\right]dt$$
$$= \frac{|f''(x)|^q}{s+3} + \frac{2\,|f''(b)|^q}{(s+1)(s+2)(s+3)}.$$

Therefore, we have

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f(x) + \left(x-\frac{a+b}{2}\right)f'(x)\right|$$
$$\le \frac{(x-a)^3}{2(b-a)}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{|f''(x)|^q}{s+3} + \frac{2\,|f''(a)|^q}{(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^3}{2(b-a)}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{|f''(x)|^q}{s+3} + \frac{2\,|f''(b)|^q}{(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}}. \blacksquare$$

Corollary 7. *Under the above assumptions we have the following inequality*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f(x) + \left(x-\frac{a+b}{2}\right)f'(x)\right|$$
$$\le M\left(\frac{3\left(s^2+3s+4\right)}{(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}}\left[\frac{(b-a)^2}{24} + \frac{1}{2}\left(x-\frac{a+b}{2}\right)^2\right].$$

This follows by Theorem 6 *with* $M = \sup_{x\in[a,b]}|f''|$.

Corollary 8. *With the assuptions in Corollary* 7, *one has the mid-point inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right)\right| \le M\left(\frac{3\left(s^2+3s+4\right)}{(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}}\frac{(b-a)^2}{24}.$$

This follows by Corollary 7, *choosing* $x=\frac{a+b}{2}$.

Remark 2. If in Corollary 8 we choose $s = 1$ and $q = 1$, then we have the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right)\right| \leq M\frac{(b-a)^2}{24},$$

which is the inequality (1.3) for functions f with convex $|f''|$.

Corollary 9. *With the assumptions in Corollary 7, one has the following perturbed trapezoid like inequality:*

$$\left|\int_a^b f(u)du - \frac{(b-a)}{2}\left[f(a)+f(b)\right] + \frac{(b-a)^2}{4}\left(f'(b)-f'(a)\right)\right| \leq \frac{(b-a)^3}{6}\left(\frac{3\left(s^2+3s+4\right)}{(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}} M.$$

This follows by using Corollary 7 with $x = a$, $x = b$, adding the results and using the triangle inequality for the modulus.

Remark 3. All of the above inequalities hold for functions f with convex $|f''|$. Simply choose $s = 1$ in each of those results to get desired formulas.

The following result holds in the s-concave case.

Theorem 7. *Let $I \subset [0,\infty)$, $f : I \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a,b]$ where $a, b \in I$ with $a < b$. If $|f''|^q$ is s-concave in the second sense on $[a,b]$, for some fixed $s \in (0,1]$, $p, q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f(x) + \left(x - \frac{a+b}{2}\right)f'(x)\right| \leq \frac{2^{(s-1)/q}}{(2p+1)^{1/p}(b-a)}\left(\frac{(x-a)^3\left|f''(\frac{x+a}{2})\right| + (b-x)^3\left|f''(\frac{b+x}{2})\right|}{2}\right), \tag{2.10}$$

for each $x \in [a,b]$.

Proof. Suppose that $q > 1$. From Lemma 1 and by the Hölder inequality, we have

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f(x) + \left(x - \frac{a+b}{2}\right) f'(x)\right|$$
$$\leq \frac{(x-a)^3}{2(b-a)}\int_0^1 t^2 \left|f''(tx+(1-t)a)\right| dt + \frac{(b-x)^3}{2(b-a)}\int_0^1 t^2 \left|f''(tx+(1-t)b)\right| dt$$
$$\leq \frac{(x-a)^3}{2(b-a)}\left(\int_0^1 t^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^3}{2(b-a)}\left(\int_0^1 t^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f''|^q$ is s-concave in the second sense, using (1.1) we obtain

$$\int_0^1 \left|f''(tx+(1-t)a)\right|^q dt \leq 2^{s-1}\left|f''(\frac{x+a}{2})\right|^q \tag{2.11}$$

and

$$\int_0^1 \left|f''(tx+(1-t)b)\right|^q dt \leq 2^{s-1}\left|f''(\frac{b+x}{2})\right|^q. \tag{2.12}$$

A combination of (2.11) and (2.12) gives

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f(x) + \left(x - \frac{a+b}{2}\right) f'(x)\right|$$
$$\leq \frac{2^{(s-1)/q}}{(2p+1)^{1/p}(b-a)}\left(\frac{(x-a)^3\left|f''(\frac{x+a}{2})\right| + (b-x)^3\left|f''(\frac{b+x}{2})\right|}{2}\right).$$

This completes the proof. ■

Corollary 10. *If in* (2.10), *we choose* $x = \frac{a+b}{2}$, *then we have*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right)\right| \leq \frac{2^{(s-1)/q}(b-a)^2}{16\,(2p+1)^{1/p}}\left[\left|f''(\frac{3a+b}{4})\right| + \left|f''(\frac{a+3b}{4})\right|\right]. \tag{2.13}$$

For instance, if $s = 1$, *then we have*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{16\,(2p+1)^{1/p}}\left[\left|f''(\frac{3a+b}{4})\right| + \left|f''(\frac{a+3b}{4})\right|\right].$$

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