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THE DE RHAM COHOMOLOGY OF DIFFERENTIAL SPACES

Introduction

In [11] we have defined an exterior algebra $\hat{A}(M)$ on a differential space M in the sense of Sikorski [12], [13]. Introduced there the operator $\hat{d}$ satisfies the well-known axioms of the exterior derivation. So we may consider the de Rham complex in our case. Such complexes were considered on another differentiable spaces by Smith [15], Spallek [16], [17], Marshall [2], [7], Mostow [10] and Schwartz [18]. It seems that these complexes are of interest in global analysis. We investigate properties of the de Rham cohomology of the complex $(\hat{A}(M),\hat{d})$ on a Sikorski's differential space.

In Section 1 we recall some basic notions and notation. In Section 2 we describe some properties of the Cartesian product of differential spaces and consider smooth one-parameter families of differential forms. Next in Section 3 we define the homotopy operator L for $\hat{d}$, which let us easily give an axiomatic description of the considered de Rham cohomology.

1. Basic notions and notation

Let (M,C) be a differential space [13], [14]. By τ_C we denote the smallest topology on M such that all functions from C are continuous. Let C_0 be a set of real functions on M. The differential structure C is called generated by C_0 if C is the smallest differential structure containing C_0. We denote by M_p the space tangent to (M,C) at the point $p \in M$. Each element $v \in M_p$ is an R-linear mapping $v: C \to R$ satisfying the condition:

$$v(\alpha \cdot \beta) = \alpha(p)v(\beta) + \beta(p)v(\alpha) \quad \text{for any } \alpha, \beta \in C.$$

Let $TM := \bigcup_{p \in M} M_p$ be a disjoint sum of tangent spaces to (M,C). Let TC (see [5]) be the differential structure on TM generated by the set $\{\alpha \circ \pi : \alpha \in C\} \cup \{d\alpha : \alpha \in C\}$, where $\pi : TM \to M$ is the natural projection and $d\alpha: TM \to R$ is a function defined by the formula

$$(d\alpha)(v) = v(\alpha) \quad \text{for} \quad v \in TM.$$

Now, let us put

$$T^kM = \{(v_1, \dots, v_k) \in TM \times \dots \times TM : \pi(v_1) = \dots = \pi(v_k)\}$$

as well as

$$T^kC = (TC \times \dots \times TC)_{T^kM} \quad \text{for} \quad k = 1,2,\dots \text{ [5]}.$$

By $A^k(M)$, where $k = 1,2,\dots$, we denote the set of all smooth mappings $\omega : T^kM \to R$ such that the mapping $\omega|_{M_p \times \dots \times M_p}$ is skew-symmetric R-k-linear for each point $p \in M$ [5]. A direct sum $A(M) = \bigoplus_{k \geq 0} A^k(M)$, where $A^0(M) = C$, together with the canonical operations of addition and multiplication is a graded algebra over R.

Let $\mathfrak{M}^k(M)$ for $k \geq 1$ denote the set of all elements $\omega \in A^k(M)$ such that for each point $p \in M$ there exist an open neighbourhood $U \in \tau_C$ of p and a family of smooth functions $\alpha_{i_1}, \dots, \alpha_{i_{k-1}}, \alpha_{i_1 \dots i_{k-1}} \in C_U$ for $(i_1, \dots, i_{k-1}) \in I \subset N^{k-1}$, where I is a finite subset, such that

$$\omega|\pi^{-1}(U) = \sum_I d\alpha_{i_1 \dots i_{k-1}} \wedge d\alpha_{i_1} \wedge \dots \wedge d\alpha_{i_{k-1}}$$

as well as

$$\sum_I \alpha_{i_1 \dots i_{k-1}} \, d\alpha_{i_1} \wedge \dots \wedge d\alpha_{i_{k-1}} = 0.$$

Moreover for $k = 0$ we put $\mathfrak{M}^0(M) = \{0\}$ [11]. One can prove [11] that the direct sum $\mathfrak{M}(M) = \bigoplus_{k\geq 0} \mathfrak{M}^k(M)$ is a homogeneous ideal in the graded algebra $A(M)$.

Let $\mathcal{A}^k$ be a sheaf $U \mapsto A^k(U)$, $U \in \tau_C$. We denote by $\mathfrak{M}^k$ $(k = 0,1,\ldots)$ the sheaf $U \mapsto \mathfrak{M}^k(U)$, $U \in \tau_C$. Let $\mathcal{C}$ be the sheaf of all smooth functions on (M,C) [10]. Evidently for each $k \geq 1$ $\mathfrak{M}^k$ is a subsheaf of $\mathcal{C}$-modules of the sheaf $\mathcal{A}^k$ of $\mathcal{C}$-modules. Let $\mathcal{A} = \bigoplus_{k\geq 0} \mathcal{A}^k$ and $\mathfrak{M} = \bigoplus_{k\geq 0} \mathfrak{M}^k$ be direct sums of the correspondent sheaves. The both sheaves $\mathcal{A}$ and $\mathfrak{M}$ are evidently the sheaves of graded algebras. The sheaf $\mathfrak{M}$ is a subsheaf of homogeneous ideals in the sheaf of graded algebras $\mathcal{A}$ [11]. Let $\Lambda = \mathcal{A}/\mathfrak{M}$ be a quotient presheaf. Denote by $\hat{\mathcal{A}}$ the quotient sheaf associated with the quotient preasheaf Λ. If $\xi \in \Lambda^k(U)$, $U \in \tau_C$ then for each point $p \in U$ by ξ_p we will denote below a germ of ξ at the point p. In the set $\hat{\mathcal{A}}(U)$ of the cross-sections of the sheaf $\hat{\mathcal{A}}$ over $U \in \tau_C$ one can define in the natural way the operations of additions and exterior multiplication. Moreover for any $U \in \tau_C$'a direct sum $\hat{\mathcal{A}}(U) = \bigoplus_{k\geq 0} \hat{\mathcal{A}}^k(U)$ is a graded algebra over R. In the graded algebra $\hat{\mathcal{A}}(M)$ there exists exactly one operator $\hat{d}\colon \hat{\mathcal{A}}^k(M) \to \hat{\mathcal{A}}^{k+1}(M)$ for $k = 0,1,\ldots$ satisfying the well-known conditions of exterior derivative (see Th. 3.1 in [11]). Let $\omega \in \hat{\mathcal{A}}^k(M)$ be an arbitrary element, $k = 0,1,\ldots$. Recall that for $k = 0$ $\hat{\mathcal{A}}^0(M) = C$ and then $(\hat{d}\omega)(p) := [d\omega]_p$ for each point $p \in M$, where $[d\omega]_p$ is the germ of the equivalence class $[d\omega]$. Now, let $k \geq 1$ and p be an arbitrary point of M. There exist for ω an open neighbourhood $U \in \tau_C$ of p and an indexed family of smooth functions $\alpha_{i_1},\ldots,\alpha_{i_k}$, $\alpha_{i_1\ldots i_k} \in C_U = \mathcal{C}(U)$, $(i_1,\ldots,i_k) \in I \subset N^k$, such that

$$(1.1) \qquad \omega(q) = \Big[\sum_I \alpha_{i_1\ldots i_k}\, d\alpha_{i_1} \wedge \ldots \wedge d\alpha_{i_k}\Big]_q$$

for each point $q \in U$. Then we put

$$(1.2) \qquad (\hat{d}\omega)(p) := \left[\sum_I d\alpha_{i_1 \ldots i_k} \wedge d\alpha_{i_1} \wedge \ldots \wedge d\alpha_{i_k}\right]_p .$$

Denote by $\mathfrak{X}(M)$ the C-module of all smooth vector fields tangent to (M,C).

2. The Cartesian product of differential spaces. Smooth 1-parameter families of differential forms

Let (M,C) and (N,D) be differential spaces. For an arbitrary function $\alpha \in C$ we denote by $\bar{\alpha}$ the function $\bar{\alpha} : M\times N \to R$ given by

$$(2.1) \qquad \bar{\alpha} = \alpha \circ pr_1 ,$$

where $pr_1 : M\times N \to M$ is the projection of $M\times N$ onto M.

Analogously for any $\beta \in D$ let $\bar{\beta} : M\times N \to R$ be the function given by

$$(2.2) \qquad \bar{\beta} = \beta \circ pr_2 ,$$

where $pr_2 : M\times N \to N$ is the projection of $M\times N$ onto N.

Let $C\times D$ be the differential structure on $M\times N$ generated by the set of real functions $\{\bar{\alpha} : \alpha \in C\} \cup \{\bar{\beta} : \beta \in D\}$. The differential space $(M\times N, C\times D)$ is called the Cartesian product of differential spaces (M,C) and (N,D) [14]. If C is generated by a set C_0 and D is generated by a set D_0 then the differential structure $C\times D$ is generated by the set $\{\bar{\alpha} : \alpha \in C_0\} \cup \{\bar{\beta} : \beta \in D_0\}$.

For an arbitrary point $p_0 \in M$ let $j_{p_0} : N \to M\times N$ be the imbedding given by

$$(2.3) \qquad j_{p_0}(q) = (p_0, q) \quad \text{for} \quad q \in M.$$

For an arbitrary point $q_0 \in N$ let $j_{q_0} : N \to M\times N$ be the imbedding defined by

$$(2.4) \qquad j_{q_0}(p) = (p, q_0) \quad \text{for} \quad p \in M.$$

It is easy to verify the following equalities:

$$(2.5)\qquad pr_1 \circ j_{q_0} = id_M,$$

$$(2.6)\qquad pr_2 \circ j_{p_0} = id_N,$$

$$(2.7)\qquad (pr_1 \circ j_{p_0})(q) = p_0 \quad \text{for} \quad q \in N,$$

$$(2.8)\qquad (pr_2 \circ j_{q_0})(p) = q_0 \quad \text{for} \quad p \in M.$$

Let $(M\times N)_{(p,q)}$ be the tangent space to $(M\times N, C\times D)$ at a point (p,q). For any tangent vector $w \in (M\times N)_{(p,q)}$ we put

$$(2.9)\qquad w_M = (j_q \circ pr_1)_{*(p,q)} w,$$

$$(2.10)\qquad w_N = (j_p \circ pr_2)_{*(p,q)} w.$$

It is easy to see that $w = w_M + w_N$ and that the vectors w_M and w_N satisfy the following conditions:

$$(2.11)\qquad w_M(\bar{\beta}) = 0 \quad \text{for any} \quad \beta \in D,$$

$$(2.12)\qquad w_N(\bar{\alpha}) = 0 \quad \text{for any} \quad \alpha \in C.$$

D e f i n i t i o n 2.1. A vector $w \in (M\times N)_{(p,q)}$ is said to be parallel to (M,C) if $w(\bar{\beta}) = 0$ for any $\beta \in D$. A vector $w \in (M\times N)_{(p,q)}$ is said to be parallel to (N,D) if $w(\bar{\alpha}) = 0$ for any $\alpha \in C$.

Clearly for any $w \in (M\times N)_{(p,q)}$ the vector w_M is parallel to (M,C) and the vector w_N is parallel to (N,D). It is easy to see that the subspace $(j_q)_*(M_p)$ is the set of all vectors tangent to $(M\times N, C\times D)$ at (p,q) parallel to (M,C) and the subspace $(j_p)_{*q}(N_q)$ is the set of all vectors tangent to $(M\times N, C\times D)$ at (p,q) parallel to (N,D). One can prove [14] that the tangent space $(M\times N)_{(p,q)}$ is a direct sum of subspaces $(j_q)_{*p}(M_p)$ and $(j_p)_{*q}(N_q)$.

D e f i n i t i o n 2.2. A vector field $Z \in \mathfrak{X}(M\times N)$ is said to be parallel to (M,C) if $Z(\bar{\beta}) = 0$ for any $\beta \in D$. A vector field $Z \in \mathfrak{X}(M\times N)$ is said to be parallel to (N,D) if $Z(\bar{\alpha}) = 0$ for any $\alpha \in C$.

Now, let $Z \in \mathfrak{X}(M\times N)$ be an arbitrary vector field tangent to $(M\times N, C\times D)$. Let us put

$$Z_M(p,q) := (j_q \circ pr_1)_{*(p,q)} Z(p,q) \quad \text{for } (p,q) \in M\times N, \tag{2.13}$$

$$Z_N(p,q) := (j_p \circ pr_2)_{*(p,q)} Z(p,q) \quad \text{for } (p,q) \in M\times N. \tag{2.14}$$

One can prove the identities:

$$Z_M(\bar{\alpha}) = Z(\bar{\alpha}) \quad \text{for } \alpha \in C, \tag{2.15}$$

$$Z_M(\bar{\beta}) = 0 \quad \text{for } \beta \in D, \tag{2.16}$$

$$Z_N(\bar{\alpha}) = 0 \quad \text{for } \alpha \in C, \tag{2.17}$$

$$Z_N(\bar{\beta}) = Z(\bar{\beta}) \quad \text{for } \beta \in D. \tag{2.18}$$

So the vector fields Z_M and Z_N defined by (2.13)-(2.14) are smooth and parallel to (M,C) and (N,D) respectively. Moreover $Z = Z_M + Z_N$.

Now, let $X \in \mathfrak{X}(M)$ be an arbitrary smooth vector field tangent to (M,C). Let $\bar{X}\colon M\times N \to T(M\times N)$ be the mapping given by

$$\bar{X}(p,q) := (j_q)_{*p} X(p) \quad \text{for } (p,q) \in M\times N. \tag{2.19}$$

It is easy to verify that $\bar{X}$ is a smooth vector field tangent to $(M\times N, C\times D)$, parallel to (M,C) and satisfies the following condition:

$$\bar{X}(\bar{\alpha}) = \overline{X(\alpha)} \quad \text{for any } \alpha \in C. \tag{2.20}$$

Analogously for any $Y \in \mathfrak{X}(N)$ we can define the vector field $\bar{Y} \in \mathfrak{X}(M\times N)$ parallel to (N,D) by the following formula

$$\bar{Y}(p,q) := (j_p)_{*q}\, Y(q) \quad \text{for } (p,q) \in M\times N. \tag{2.21}$$

One can verify the following identities:

$$[\bar{X},\bar{Y}] = 0 \quad \text{for any } X \in \mathfrak{X}(M),\ Y \in \mathfrak{X}(N), \tag{2.22}$$

(2.23) $$[\bar{X}_1,\bar{X}_2] = \overline{[X_1,X_2]} \quad \text{for any} \quad X_1,X_2 \in \mathfrak{X}(M),$$

(2.24) $$[\bar{Y}_1,\bar{Y}_2] = \overline{[Y_1,Y_2]} \quad \text{for any} \quad Y_1,Y_2 \in \mathfrak{X}(N).$$

Now, let $\varphi \in C\times D$ be an arbitrary real function. Denote by $\pi : T(M\times N) \to M\times N$ the canonical projection [5]. Let π_1 and π_2 be the coordinates of π, i.e. $\pi = (\pi_1,\pi_2)$. Let us put

(2.25) $$(d_M\varphi)(v) = v(\varphi(\cdot,\pi_2(v))) \quad \text{for} \quad v \in T(M\times N),$$

(2.26) $$(d_N\varphi)(v) = v(\varphi(\pi_1(v),\cdot)) \quad \text{for} \quad v \in T(M\times N).$$

One can verify that $d_M\varphi$ and $d_N\varphi$ are smooth 1-forms on $M\times N$ and

(2.27) $$d\varphi = d_M\varphi + d_N\varphi.$$

1-form $d_M\varphi$ is called the partial differential of φ with respect to M and 1-form $d_N\varphi$ is called the partial differential of φ with respect to N. It is easy to check the following identities:

(2.28) $$j_p^*(d\varphi) = j_p^*(d_N\varphi) \quad \text{for any} \quad p \in M,$$

(2.29) $$j_q^*(d\varphi) = j_q^*(d_M\varphi) \quad \text{for any} \quad q \in N.$$

Now we consider the Cartesian product of $(R,\mathcal{E})$ and (M,C), where $\mathcal{E}$ is the natural differential structure on R generated by the function $\theta = id_R$. If the differential structure C is generated by C_o then the differential structure $\mathcal{E}\times C$ is generated by the set $\{\bar{\theta}\} \cup \{\bar{\alpha} : \alpha \in C_o\}$. Let $T = \overline{\frac{d}{dt}} \in \mathfrak{X}(R\times M)$ be the vector field defined by (2.13), where $\frac{d}{dt}$ is the basis vector field tangent to $(R,\mathcal{E})$.

For any $\omega \in A^k(R\times M)$, $k \in N$, let $\imath_T\omega$ be the $(k-1)$-form defined by

(2.30) $$(\imath_T\omega)(v_1,\dots,v_{k-1}) := \omega(T(t,p),v_1,\dots,v_{k-1})$$

for $v_1,\dots,v_{k-1} \in (R\times M)_{(t,p)}$, $(t,p) \in R\times M$.

Moreover for $\omega \in A^0(R\times M)$ we put $\imath_T \omega := 0$.

It is easy to see that the operator $\imath_T : A(R\times M) \to A(R\times M)$, $\omega \mapsto \imath_T\omega$, is $\mathcal{E}\times C$-linear and satisfies the following condition:

$$(2.31) \qquad \imath_T(\omega_1 \wedge \omega_2) = \imath_T\omega_1 \wedge \omega_2 + (-1)^{|\omega_1|}\, \omega_1 \wedge \imath_T\omega_2$$

for $\omega_1, \omega_2 \in A(R\times M)$. Of course $\imath_T \circ \imath_T = 0$.

We shall prove

L e m m a 2.1. Every k-form $\omega \in A^k(R\times M)$ may be uniquely presented in the form

$$(2.32) \qquad \omega = d\bar{\theta} \wedge \omega_1 + \omega_2,$$

where $\omega_1 \in A^{k-1}(R\times M)$ and $\omega_2 \in A^k(R\times M)$ are such forms that $\imath_T\omega_1 = 0$ and $\imath_T\omega_2 = 0$.

P r o o f . Put $\omega_1 := \imath_T\omega$ and $\omega_2 := \omega - d\bar{\theta} \wedge \imath_T\omega$. Of course $\imath_T\omega_1 = 0$ and $\imath_T\omega_2 = 0$. It remains to prove the uniqueness of the decomposition (2.32). In fact, if $\omega = d\bar{\theta} \wedge \omega_1' + \omega_2'$, where $\imath_T\omega_1' = 0$ and $\imath_T\omega_2' = 0$ then

$$\imath_T\omega = \imath_T(d\bar{\theta} \wedge \omega_1') = \imath_T(d\bar{\theta} \wedge \omega_1).$$

Hence $\omega_1' - d\bar{\theta} \wedge \imath_T\omega_1' = \omega_1 - d\bar{\theta} \wedge \imath_T\omega_1$. Consequently $\omega_1' = \omega_1$ and

$$\omega_2 = \omega - d\bar{\theta} \wedge \omega_1 = \omega - d\bar{\theta} \wedge \omega_1' = \omega_2'.$$

Denote by $A^{0,k}(R\times M)$ the $\mathcal{E}\times C$-submodule of k-forms $\omega \in A^k(R\times M)$ satisfying the condition $\imath_T\omega = 0$ and by $A^{1,k-1}(R\times M)$ the submodule of the $\mathcal{E}\times C$-module $A^k(R\times M)$ of all forms of the form $d\bar{\theta} \wedge \omega_1$, where $\omega_1 \in A^{k-1}(R\times M)$ and $\imath_T\omega_1 = 0$. From Lemma 2.1 it follows

C o r o l l a r y 2.2. The $\mathcal{E}\times C$-module $A^k(R\times M)$ is the direct sum of the $\mathcal{E}\times C$-modules $A^{1,k-1}(R\times M)$ and $A^{0,k}(R\times M)$.

Now let $\varphi \in \mathcal{E}\times C$ be an arbitrary smooth function on $R\times M$. Let $\frac{\partial\varphi}{\partial t} : R\times M \to R$ be the function defined by

$$(2.33) \qquad \frac{\partial\varphi}{\partial t}(s,p) = \frac{d}{dt}\Big|_s (t \mapsto \varphi(t,p)).$$

It is easy to see that $\frac{\partial\varphi}{\partial t}$ is smooth. In fact, for an arbitrary point $(t,p) \in R\times M$ there exist an open interval $(a,b) \ni t$, an open neighbourhood $V \in \tau_C$ of p and smooth functions $\alpha_1,\ldots,\alpha_n \in C$, $\gamma \in C^\infty(R^{n+1})$ such that

$$\varphi|(a,b)\times V = \gamma(\bar\theta, \bar\alpha_1,\ldots,\bar\alpha_n)|(a,b)\times V.$$

Then

$$(2.34) \qquad \frac{\partial\varphi}{\partial t}|(a,b)\times V = \gamma'_{|1} \circ (\bar\theta, \bar\alpha_1,\ldots,\bar\alpha_n)|(a,b)\times V.$$

From (2.26), (2.27) and (2.34) it is easy to observe the equality:

$$(2.35) \qquad d\varphi = \frac{\partial\varphi}{\partial t}\, d\bar\theta + d_M\varphi.$$

Of course $d_M\varphi \in A^{0,1}(R\times M)$ and $\frac{\partial\varphi}{\partial t}\, d\bar\theta = d_R\varphi \in A^{1,0}(R\times M)$. Moreover $\imath_T(d\varphi) = \frac{\partial\varphi}{\partial t}$.

Now let $(\omega_t)_{t\in R}$, $\omega_t \in A^k(M)$, be a 1-parameter family of differential k-forms on (M,C).

D e f i n i t i o n 2.3. 1-parameter family $(\omega_t)_{t\in R}$ is called smooth if the function $\tilde\omega : R\times T^kM \to R$ defined by the formula

$$(2.36) \qquad \tilde\omega(t,v_1,\ldots,v_k) = \omega_t(v_1,\ldots,v_k) \text{ for } (t,v_1,\ldots,v_k)\in R\times T^kM$$

is smooth.

D e f i n i t i o n 2.4. Let $(\omega_t)_{t\in R}$ be a smooth 1-parameter family of differential k-forms. Then the k-form $\int_a^b \omega_t\, dt$ defined by

$$(2.37) \qquad \left(\int_a^b \omega_t\, dt\right)(v_1,\ldots,v_k) := \int_a^b \omega_t(v_1,\ldots,v_k)\, dt$$

for $(v_1,\ldots,v_k) \in T^kM$, is said to be the defined integral from $(\omega_t)_{t\in R}$.

Example 2.1. Let $(\omega_t)_{t\in R}$ be a smooth 1-parameter family of k-forms on (M,C) of the form

$$\omega_t = \sum_I \varphi^{i_1\ldots i_k}(t,\cdot)\, d\alpha_{i_1}\wedge\ldots\wedge d\alpha_{i_k},$$

where $\varphi^{i_1\ldots i_k}\in\mathcal{E}\times C$, $\alpha_{i_1},\ldots,\alpha_{i_k}\in C$ for $(i_1,\ldots,i_k)\in I\subset N^k$ and I is a finite set of indices. Then one can check that

$$\int_a^b \omega_t\, dt = \sum_I \left(\int_a^b \varphi^{i_1\ldots i_k}(t,\cdot)dt\right) d\alpha_1\wedge\ldots\wedge d\alpha_k.$$

Lemma 2.2. If $(\alpha_t)_{t\in R}$ is a smooth 1-parameter family of 0-forms on a differential space (M,C) then

$$d\left(\int_a^b \alpha_t\, dt\right) = \int_a^b d\,\alpha_t\, dt. \tag{2.38}$$

Proof. The smoothness of 1-parameter family $(\alpha_t)_{t\in R}$ of 0-forms means that the function $\tilde{\alpha}: R\times M \to R$ defined by

$$\tilde{\alpha}(t,p) = \alpha_t(p) \quad \text{for} \quad (t,p)\in R\times M$$

is smooth. Thus for any point $p\in M$ there exist an open neighbourhood $V\in\tau_C$ of p, a sequence of real numbers $c_0,c_1,\ldots,c_n$ such that $a = c_0 < c_1 < \ldots < c_n = b$ and functions $\gamma_1,\ldots,\gamma_n \in \mathcal{E}_{k+1}$, $\beta_1,\ldots,\beta_k\in C$, $k\in N$, such that

$$\tilde{\alpha}\,|\,(c_{i-1},c_i)\times V = \gamma_i(\bar{\theta},\bar{\beta}_1,\ldots,\bar{\beta}_k)\,|\,(c_{i-1},c_i)\times V \quad \text{for} \quad i = 1,\ldots,n.$$

Then we have

$$\left(\int_a^b \alpha_t\, dt\right)\Big|V = \left(\sum_{i=1}^{n}\int_{c_{i-1}}^{c_i} \gamma_i(t,\bar{\beta}_1,\ldots,\bar{\beta}_k)dt\right)\Big|V \tag{2.39}$$

and

$$(2.40)\qquad \Big(\int_a^b d\alpha_t\ dt\Big)\Big|V = \Big(\sum_{i=1}^{n}\int_{c_{i-1}}^{c_i} d\gamma_i(t,\bar\beta_1,\dots,\bar\beta_k)dt\Big)\Big|V.$$

One can easily verify the equality

$$(2.41)\qquad d\Big(\int_{c_{i-1}}^{c_i}\gamma_i(t,\bar\beta_1,\dots,\bar\beta_k)dt\Big) = \int_{c_{i-1}}^{c_i} d\gamma_i(t,\bar\beta_1,\dots,\bar\beta_k)dt.$$

From (2.39)-(2.41) it follows that

$$d\Big(\int_a^b \alpha_t\ dt\Big)\Big|V = \Big(\int_a^b d\alpha_t\ dt\Big)\Big|V.$$

This finishes the proof.

One can prove

Lemma 2.3. If

$$\sum_I \varphi^{i_1\dots i_k}\, d\bar\alpha_{i_1}\wedge\dots\wedge d\bar\alpha_{i_k} = 0,$$

for $\varphi^{i_1\dots i_k}\in\mathcal{E}\times C$, $\alpha_{i_1},\dots,\alpha_{i_k}\in C$, $(i_1,\dots,i_k)\in I\subset N^k$, where I is a finite set, then

$$\sum_I \frac{\partial\varphi^{i_1\dots i_k}}{\partial t}\, d\bar\alpha_{i_1}\wedge\dots\wedge d\bar\alpha_{i_k} = 0.$$

Lemma 2.4. If $\mu\in\mathfrak{M}^{k+1}(R\times M)$ then $\imath_T\mu\in\mathfrak{M}^k(R\times M)$.

Proof. Let $\mu\in\mathfrak{M}^{k+1}(R\times M)$. For each point $(t,p)\in R\times M$ there exist an open interval $(a,b)\ni t$, an open neighbourhood V of p and smooth functions $\varphi^{i_1\dots i_k}\in(\mathcal{E}\times C)_{(a,b)\times V}$, $\alpha_{i_1},\dots,\alpha_{i_k}\in C_V$, $(i_1,\dots,i_k)\in I\subset N^k$, $\psi^{j_1\dots j_{k-1}}\in(\mathcal{E}\times C)_{(a,b)\times V}$, $\beta_{j_1},\dots,\beta_{j_{k-1}}\in C_V$, $(j_1,\dots,j_{k-1})\in J\subset N^{k-1}$, where I and J are finite subsets of indices, such that

$$\mu|\pi^{-1}((a,b)\times V) = \sum_I d\varphi^{i_1\ldots i_k} \wedge d\bar{\alpha}_{i_1} \wedge \ldots \wedge d\bar{\alpha}_{i_k} + \tag{2.42}$$

$$+ \sum_I d\psi^{j_1\ldots j_{k-1}} \wedge d\bar{\theta} \wedge d\bar{\beta}_{j_1} \wedge \ldots \wedge d\bar{\beta}_{j_k}$$

as well as

$$\sum_I \varphi^{i_1\ldots i_k}\, d\bar{\alpha}_{i_1} \wedge \ldots \wedge d\bar{\alpha}_{i_k} + \tag{2.43}$$

$$+ \sum_J d\psi^{j_1\ldots j_{k-1}} \wedge d\bar{\theta} \wedge d\bar{\beta}_{j_1} \wedge \ldots \wedge d\bar{\beta}_{j_{k-1}} = 0.$$

Using i_T to (2.42) and (2.43) from Lemma 2.3 and (2.35) it follows that

$$i_T\mu|\pi^{-1}((a,b)\times V) = -\sum_I d\psi^{j_1\ldots j_{k-1}} \wedge d\bar{\beta}_{j_1} \wedge \ldots \wedge d\bar{\beta}_{j_{k-1}}$$

and

$$\sum_J \psi^{j_1\ldots j_{k-1}}\, d\bar{\beta}_{j_1} \wedge \ldots \wedge d\bar{\beta}_{j_{k-1}} = 0.$$

Hence $i_T\mu \in \mathfrak{M}^k(R\times M)$.

C o r o l l a r y 2.5. Let $\mu \in \mathfrak{M}^k(R\times M)$ and μ_1, μ_2 be elements from the decomposition (2.32) such that $\mu = d\bar{\theta}\wedge\mu_1 + \mu_2$. Then $\mu_1 \in \mathfrak{M}^{k-1}(R\times M)$ and $\mu_2 \in \mathfrak{M}^k(R\times M)$.

P r o o f . Since $\mu = d\bar{\theta}\wedge\mu_1 + \mu_2$ and $\mu \in \mathfrak{M}^k(R\times M)$ by Lemma 2.4 $i_T\mu = \mu_1 \in \mathfrak{M}^{k-1}(R\times M)$. Because $\mathfrak{M}^k(R\times M)$ is an ideal then also $d\bar{\theta}\wedge\mu_1 \in \mathfrak{M}^k(R\times M)$. But $\mu_2 = \mu - d\bar{\theta}\wedge\mu_1$. Thus $\mu_2 \in \mathfrak{M}^k(R\times M)$.

Now we prove

L e m m a 2.6. Let $\mu \in \mathfrak{M}^k(R\times M) \cap A^{0,k}(R\times M)$ be an arbitrary element and $j_t\colon M \to R\times M$ be the imbedding defined by (2.3).

Then $(j_t^*(\mu))_{t\in R}$ is a smooth 1-parameter family of k-forms on (M,C) from the ideal $\mathfrak{M}^k(M)$ and $\int_a^b j_t^*(\mu)dt \in \mathfrak{M}^k(M)$.

P r o o f . For an arbitrary point $p\in M$ there exist an open neighbourhood $V\in\tau_C$ of the point $p\in M$, a sequence of real numbers $a = c_0 < c_1 < \dots < c_n = b$, a family of smooth functions $\alpha_{j i_1},\dots,\alpha_{j i_{k-1}} \in C_V$, $\overset{i_1\dots i_{k-1}}{\underset{j}{\alpha}} \in \mathcal{E}(c_{j-1},c_j)\times C_V$, where $(i_1,\dots,i_{k-1}) \in I \subset N^{k-1}$, $j = 1,\dots,n$ and I is a finite subset, such that

$$\mu\,|\,(c_{j-1},c_j)\times V = \sum_I d\overset{i_1\dots i_k}{\underset{j}{\alpha}} \wedge d\bar{\alpha}_{j i_1}\wedge\dots\wedge d\bar{\alpha}_{j i_{k-1}}$$

and

$$\sum_I \overset{i_1\dots i_{k-1}}{\underset{j}{\alpha}}\, d\bar{\alpha}_{j i_1}\wedge\dots\wedge d\bar{\alpha}_{j i_k} = 0 \quad \text{for} \quad j = 1,\dots,n.$$

Hence by simple calculation one can check:

$$\Big(\int_a^b j_t^*(\mu)dt\Big)\Big|V = \Big(\sum_{j=1}^n \int_{c_{j-1}}^{c_j} j_t^*(\mu)dt\Big)\Big|V =$$

$$= \sum_{j=1}^n \sum_I \Big(\int_{c_{j-1}}^{c_j} d\Big(j_t^*\, \overset{i_1\dots i_{k-1}}{\underset{j}{\alpha}}\Big)dt\Big) \wedge d\alpha_{j i_1}\wedge\dots\wedge d\alpha_{j i_{k-1}}$$

and

$$\sum_{j=1}^n \sum_I \Big(\int_{c_{j-1}}^{c_j} j_t^*\, \overset{i_1\dots i_{k-1}}{\underset{j}{\alpha}}\, dt\Big)\, d\alpha_{j i_1}\wedge\dots\wedge d\alpha_{j i_{k-1}} = 0.$$

Hence it follows that $\int_a^b j_t^*(\mu)dt|V \in \mathfrak{M}^k(V)$ and consequently $\int_a^b j_t^*(\mu)dt \in \mathfrak{M}^k(M)$.

3. The homotopy operator on a differential space

Now, using the above lemmas, we shall construct the homotopy operator for the complex $(\hat{A}(M),\hat{d})$.

For an arbitrary $\omega \in \hat{A}^k(R\times M)$, $k \geq 1$, denote by $\imath_T\omega$ the element of $\hat{A}^{k-1}(R\times M)$ given by

$$(3.1) \qquad (\imath_T\omega)(t,p) := [\imath_T\xi]_{(t,p)} \quad \text{for} \quad (t,p) \in R\times M,$$

where $\xi \in A^k(V)$ is a k-form on an open neighbourhood $V \in \tau_{\mathcal{E}\times C}$ of p such that $\omega(t,p) = [\xi]_{(t,p)}$.

The correctness of (3.1) follows from Lemma 2.4.

Let $\imath_T: \hat{A}^k(R\times M) \to \hat{A}^{k-1}(R\times M)$, $k \geq 1$, be the mapping defined by

$$\omega \mapsto \imath_T\omega.$$

Denote by $\hat{A}^{0,k}(R\times M)$, for $k = 1,2,\dots$, the $\mathcal{E}\times C$-module of elements $\omega \in \hat{A}^k(R\times M)$ such that $\imath_T\omega = 0$ and by $\hat{A}^{1,k-1}(R\times M)$ the submodule of the $\mathcal{E}\times C$-module $\hat{A}^k(R\times M)$ of all elements of the form $\hat{d}\bar{\theta}\wedge\omega_1$, where $\omega_1 \in \hat{A}^{k-1}(R\times M)$ and $\imath_T\omega_1 = 0$. From Lemma 2.1 it follows that $\hat{A}^k(R\times M)$ is the direct sum of the modules $\hat{A}^{0,k}(R\times M)$ and $\hat{A}^{1,k-1}(R\times M)$.

Now let $\omega \in \hat{A}^{0,k-1}(R\times M)$, $k \geq 1$, be an arbitrary element. For any point $p \in M$ there exist an open neighbourhood $U \in \tau_C$ of p and a sequence of real numbers $0 = c_0 < c_1 < \dots < c_n = 1$, a family of (k-1)-forms $\omega_i \in A^{0,k-1}((c_{i-1},c_i)\times U)$, $i = 1,\dots,n$, such that

$$\omega(t,q) = [\omega_i]_{(t,q)} \quad \text{for} \quad (t,q) \in (c_{i-1},c_i)\times U, \quad i=1,\dots,n.$$

Of course $\int_{c_{i-1}}^{c_i} j_t^*(\omega_i)\,dt \in A^{k-1}(U)$ for $i = 1,\dots,n$.

Let us put

$$(3.2) \qquad (I_0^1\omega)(p) := \left[\sum_{i=1}^{n} \int_{c_{i-1}}^{c_i} j_t^*(\omega_i)dt\right]_p \quad \text{for} \quad p \in M.$$

Formula (3.2) defines an element $I_0^1\omega \in \hat{A}^{k-1}(M)$ called the definite integral of $\omega \in A^{0,k-1}(R\times M)$ from 0 to 1. The correctness of (3.2) follows from Lemma 2.5.

The mapping $I_0^1 : \hat{A}^{0,k-1}(R\times M) \to \hat{A}^{k-1}(M)$, $\omega \mapsto I_0^1\omega$, is called the integral operator on (M,C).

The composition $L = I_0^1 \circ \partial_T : \hat{A}^k(R\times M) \to \hat{A}^{k-1}(M)$, $k = 1,2,\dots$, is said to be the homotopy operator on (M,C). If $k = 0$, define $L = 0$.

P r o p o s i t i o n 3.1. The homotopy operator L is C-linear and satisfies

$$\hat{d}\circ L + L\circ\hat{d} = j_1^* - j_0^*. \tag{3.3}$$

P r o o f . It suffices to verify (3.3) for $\omega \in \hat{A}^{0,k}(R\times M)$ and for $\omega \in \hat{A}^{1,k-1}(R\times M)$.

Let $\omega \in \hat{A}^{0,k}(R\times M)$. For any point $p\in M$ there exist an open neighbourhood $U \in \tau_C$ of p and a sequence of real numbers $0 = c_0 < c_1 < \dots < c_n = 1$, $\varepsilon > 0$, a finite family of functions $\varphi_l^{i_1\dots i_k} \in (\mathcal{E}\times C)_{(c_{l-1}-\varepsilon,c_l+\varepsilon)\times U}$, $\alpha_{i_1},\dots,\alpha_{i_k} \in C_U$, $(i_1,\dots,i_k)\in I$, such that

$$\omega(t,q) = \left[\sum_I \varphi_l^{i_1\dots i_k}\, d\bar{\alpha}_{i_1}\wedge\dots\wedge d\bar{\alpha}_{i_k}\right]_{(t,q)}$$

for $(t,q) \in (c_{l-1}-\varepsilon,c_l+\varepsilon)\times U$, $l = 1,\dots,n$.

Then by (3.2) we have

$$L(\hat{d}\omega)(p) = I_0^1\big(\partial_T(\hat{d}\omega)\big)(p) =$$

$$= \left[\sum_{l=1}^{n}\int_{c_{l-1}}^{c_l}\left(\sum_I j_t^*\left(\frac{\partial\varphi_l^{i_1\dots i_k}}{\partial t}\right)d\alpha_{i_1}\wedge\dots\wedge d\alpha_{i_k}\right)dt\right]_p =$$

$$= \left[\sum_{l=1}^{n} \sum_{I} \left(\underset{l}{\varphi}^{i_1 \dots i_k}(c_l, \cdot) - \underset{l}{\varphi}^{i_1 \dots i_k}(c_{l-1}, \cdot) \right) d\alpha_{i_1} \wedge \dots \wedge d\alpha_{i_k} \right]_p =$$

$$= \sum_{l=1}^{n} \left(j^*_{c_l}(\omega) - j^*_{c_{l-1}}(\omega) \right)(p) = j^*_1(\omega)(p) - j^*_0(\omega)(p).$$

It is obvious that $\partial_T \omega = 0$ for $\omega \in \hat{A}^{0,k}(R\ M)$. Thus $(\hat{d} \circ L)(\omega) = 0$. Therefore (3.3) is true for $\omega \in \hat{A}^{0,k}(R \times M)$.

Now, let $\omega \in \hat{A}^{1,k-1}(R \times M)$. For an arbitrary point $p \in M$ there exist an open neighbourhood $U \in \tau_C$ of p and a sequence of real numbers $0 = c_0 < c_1 < \dots < c_m = 1$, $\varepsilon > 0$, a finite family of functions $\underset{l}{\varphi}^{i_1 \dots i_{k-1}} \in (\mathcal{E} \times C)_{(c_{l-1}-\varepsilon, c_l+\varepsilon) \times U}, {}^{\alpha_{i_1}, \dots, \alpha_{i_{k-1}}} \in$ $\in C_U$, $(i_1, \dots, i_{k-1}) \in I'$, $l = 1, 2, \dots, m$, such that

$$\omega(t,q) = \left[\sum_{I'} \underset{l}{\varphi}^{i_1 \dots i_{k-1}} \, d\bar{\theta} \wedge d\bar{\alpha}_{i_1} \wedge \dots \wedge d\bar{\alpha}_{i_{k-1}} \right](t,q)$$

for $(t,q) \in (c_{l-1}-\varepsilon, c_l+\varepsilon) \times U$, $l = 1, \dots, m$.

Hence using (2.35) we obtain

$$\hat{d}\omega(t,q) =$$

$$= \left[\sum_{I'} \left(\frac{\partial \underset{l}{\varphi}^{i_1 \dots i_{k-1}}}{\partial t} d\bar{\theta} + d_M \underset{l}{\varphi}^{i_1 \dots i_{k-1}} \right) \wedge d\bar{\theta} \wedge d\bar{\alpha}_{i_1} \wedge \dots \wedge d\bar{\alpha}_{i_{k-1}} \right]_{(t,q)} =$$

$$= \left[\sum_{I'} d_M \underset{l}{\varphi}^{i_1 \dots i_{k-1}} \wedge d\bar{\theta} \wedge d\bar{\alpha}_{i_1} \wedge \dots \wedge d\bar{\alpha}_{i_{k-1}} \right]_{(t,q)},$$

for $(t,q) \in (c_{l-1}-\varepsilon, c_l+\varepsilon) \times U$, $l = 1, \dots, m$.

Thus from (3.1) it follows that

$$\partial_T(\hat{d}\omega)(t,q) = \left[-\sum_{I'} d_M \varphi_l^{i_1\ldots i_{k-1}} \wedge d\bar{\alpha}_{i_1} \wedge \ldots \wedge d\bar{\alpha}_{i_{k-1}}\right](t,q)$$

for $(t,q) \in (c_{l-1}-\varepsilon, c_l+\varepsilon)\times U$, $l = 1,\ldots,m$.

Therefore by (3.2) we have

$$(3.5) \qquad L(\hat{d}\omega)(p) = I_0^1(\partial_T(\hat{d}\omega))(p) =$$

$$= \left[-\sum_{l=1}^{m}\sum_{I'}\int_{c_{l-1}}^{c_l} d\, \varphi_l^{i_1\ldots i_{k-1}}(t,\cdot)\, d\alpha_{i_1} \wedge \ldots\, d\alpha_{i_{k-1}}\right]_p .$$

From the other hand we have

$$(3.6) \qquad L(\omega)(p) = I_0^1(\partial_T(\omega))(p) =$$

$$= \left[\sum_{l=1}^{m}\sum_{I'}\int_{c_{l-1}}^{c_l} \varphi_l^{i_1\ldots i_{k-1}}(t,\cdot)\, d\alpha_{i_1} \wedge \ldots \wedge d\alpha_{i_{k-1}}\right]_p .$$

From (3.5) and (3.6) it follows that

$$L(\hat{d}\omega)(p) + \hat{d}(L(\omega))(p) = 0$$

for any $p \in M$ and $\omega \in \hat{A}^{1,k-1}(R\times M)$.

Thus

$$(3.7) \qquad (L\circ\hat{d} + \hat{d}\circ L)(\omega) = 0 \quad \text{for any} \quad \omega \in \hat{A}^{1,k-1}(R\times M).$$

It is easy to see that

$$j_t^*(\omega) = 0 \quad \text{for any} \quad \omega \in \hat{A}^{1,k-1}(R\times M) \quad \text{and} \quad t \in R.$$

In particular $j_1^*(\omega) = j_0^*(\omega) = 0$. Hence and from (3.7) it follows that

$$L(\hat{d}\omega) + \hat{d}(L(\omega)) = j_1^*\omega - j_0^*\omega$$

for any $\omega \in \hat{A}^{1,k-1}(R\times M)$. This finishes the proof of (3.3).

The following lemma is an analogue of Poincare lemma in manifolds.

L e m m a 3.2. Let (M,C) be a differential space which is locally smoothly contractible [9]. Then the sequence

$$(3.8)\qquad 0 \longrightarrow \ker \hat{\mathcal{A}}^k \xrightarrow{i_k} \hat{\mathcal{A}}^k \xrightarrow{\hat{d}_k} \ker \hat{\mathcal{A}}^{k+1} \longrightarrow 0$$

of sheaf homomorphism, where i_k is the injection, is locally exact.

P r o o f . This follows immediately from (3.3).

The de Rham group of degree $k \geq 0$ of (M,C) is the group

$$H^k_{dR}(M) = \ker \hat{d}_k / \hat{d}_{k-1}(\hat{A}^{k-1}(M)).$$

One has $H^0_{dR}(M) = \ker \hat{d}_0$.

It is easy to prove that for paracompact differential spaces for all $k \geq 0$ the sheaf $\mathcal{A}^k$ is fine [4]. This is a simple consequence of the existence of smooth partition of unity in paracompact differential spaces [5].

Let us denote by $\check{H}^k(M,R)$ the k-th Čech cohomology group of (M,C) with coefficients in the real constant sheaf. Similarly to Theorem 4.3 in [2] one can prove

T h e o r e m 3.3. If (M,C) is locally smoothly contractible and paracompact differential space, then

$$(3.9)\qquad H^k_{dR}(M) = \check{H}^k(M,R) \quad \text{for} \quad k = 0,1,2,\dots .$$

In the sequal let $\mathfrak{A}$ be the category whose objects are pairs (M,N) of differential spaces admitting smooth partition of unity to any open cover with N as a closed differential subspace of M, and whose morphism are smooth mapping of these pairs (see [2]). The category $\mathfrak{A}$ is an admissible category for a cohomology theory in the sense of Eilenberg-Steenrod [3].

For each pair (M,N) in $\mathfrak{A}$ with imbedding $i : N \longrightarrow M$ we put $\hat{A}^k(M,N) = \ker i^*$, where $i^* : \hat{A}^k(M) \longrightarrow \hat{A}^k(N)$. Since $\hat{d}_k \circ i^* = i^* \circ \hat{d}_k$, it follows that $\hat{d}_k$ induces a homomorphism

(3.10) $$\tilde{d}_k : \hat{A}^k(M,N) \to \hat{A}^{k+1}(M,N).$$

Let us put $D^k(M,N) = \ker \tilde{d}_k$.

In this way we obtain a complex [2]

(3.11) $$\ldots \to \hat{A}^{k-1}(M,N) \xrightarrow{\tilde{d}_{k-1}} \hat{A}^k(M,N) \xrightarrow{\tilde{d}_k} \hat{A}^{k+1}(M,N) \to \ldots .$$

The de Rham group of degree $k \geq 0$ of the pair (M,N) is the group

$$H^k_{dR}(M,N) = D^k(M,N)/d_{k-1}(A^{k-1}(M,N)).$$

Observe that $H^0_{dR}(M,N) = D^0(M,N)$.

If $f: (M,N) \to (M',N')$ is a morphism, then the equality $\hat{d}\circ f^* = f^*\circ\hat{d}$ implies that there is a canonically associated homomorphism

(3.12) $$H^k_{dR}(f): H^k_{dR}(M',N') \to H^k_{dR}(M,N).$$

$H_{dR} = (H^k_{dR})_{k=0,1,2,\ldots}$ is a contravariant functor from the category $\mathcal{O}\!\iota$ into the category of graded abelian groups and homomorphisms of degree 0. Applying H^k_{dR} to the sequence $(N,\emptyset) \xrightarrow{i} (M,\emptyset) \xrightarrow{j} (M,N)$ one obtains

(3.13) $$H^k_{dR}(M,N) \xrightarrow{j^*} H^k_{dR}(M) \xrightarrow{i^*} H^k_{dR}(N), \quad k = 0,1,2,\ldots .$$

One can prove

L e m m a 3.4. Let (M,C) be a differential space which admits smooth partition of unity subordinate to any open cover. Then for any closed differential subspace (N,C_N) the mapping $i^* : \hat{A}^k(M) \to \hat{A}^k(N)$, $k = 0,1,2,\ldots$, is surjective.

From Lemma 3.4 it follows that the sequences

(3.14) $$0 \to \hat{A}^k(M,N) \to \hat{A}^k(M) \xrightarrow{i^*} \hat{A}^k(N) \to 0, \quad k=0,1,2,\ldots$$

are exact.

Let $\delta^k: H^k_{dR}(N) \to H^{k+1}_{dR}(M,N)$ be a connecting homomorphism for $k = 0,1,2,\dots$ [4], [2]. The collection $\delta = (\delta)^k_{k=0,1,2,\dots}$ is a natural transformation of degree 1 from the functor H_{dR} on (M,N) to the functor H_{dR} on N.

Now one can prove (see [2])

T h e o r e m 3.5. The pair (H_{dR}, δ) is a cohomology theory of $\mathcal{O}\!\mathcal{l}$. (H_{dR}, δ) satisfies the following axioms:

1. Homotopy Axiom. If $f,g:(M,N) \to (M,N)$ are smoothly homotopic, then

$$H_{dR}(f) = H_{dR}(g).$$

2. Exactness Axiom. For any pair (M,N) of $\mathcal{O}\!\mathcal{l}$ the sequence $0 \to H^o_{dR}(M,N) \to \dots \to H^k_{dR}(M,N) \xrightarrow{j^*} H^k_{dR}(M) \xrightarrow{i^*} H^k_{dR}(N) \xrightarrow{\delta^k} H^{k+1}_{dR}(M,N) \to$ obtained from (3.13) and δ^k by composition is exact.

3. Excision Axiom. Let (M,N) be a pair of $\mathcal{O}\!\mathcal{l}$ and U be an open subset of whose closure U is contained in the interior of N. If $(M-U,N-U)$ with induced differential structures is an object of $\mathcal{O}\!\mathcal{l}$ and the inclusion map $\tilde{j}: (M-U,N-U) \to (M,N)$ is a morphism, then

$$H_{dR}(\tilde{j}): H_{dR}(M,N) \to H_{dR}(M-U,N-U)$$

is an isomorphism.

4. Dimension Axiom. If P is a one-point differential space, then

$$H^o_{dR}(P) = R \quad \text{and} \quad H^k_{dR}(P) = 0 \quad \text{for} \quad k = 1,2,\dots .$$

REFERENCES

[1] G. B r e d o n : Sheaf theory, McGraw-Hill, New York, 1967.

[2] M. B r e u e r , Ch.D. M a r s h a l l : Banachian differentiable spaces, Math. Ann. 237 (1978), 105-120.

[3] S. E i l e n b e r g , N. S t e e n r o d : Foundations of algebraic topology, Princeton University Press, Princeton, 1952.

[4] R. G o d e m e n t : Theorie des faisceaux, Hermann, Paris, 1958.

[5] A. K o w a l c z y k : Tangent differential spaces and smooth forms, Demonstratio Math. 13 (1980), 893-905.

[6] A. K o w a l c z y k : On partition of unity in differential spaces, Bull. Acad. Polon. Math. 28 (1980), 391-395.

[7] Ch.D. M a r s h a l l : The de Rham cohomology of subcartesian spaces, J. Differential Geometry 10 (1975), 575-588.

[8] Ch.D. M a r s h a l l : Calculus on subcartesian spaces, J. Differential Geometry 10 (1975), 551-574.

[9] M.A. M o s t o w : The differentiable space structure of Milnor classifying space, simplicial complexes and geometric realizations, J. Differential Geometry 14 (1979), 255-293.

[10] W. S a s i n , Z. Ż e k a n o w s k i : On some sheaves over a differential space, Arch. Math. 4, Scripta Fac. Sci. Nat. UJEP Brunensis 18 (1982), 193-200.

[11] W. S a s i n : On some exterior algebra of the differential forms over a differential space, Demonstratio Math. 19 (1986), 1063-1075.

[12] R. S i k o r s k i : Abstract covariant derivative, Colloq.Math. 18 (1967), 251-272.

[13] R. S i k o r s k i : Differential modules, Colloq. Math. 24 (1971), 45-70.

[14] R. S i k o r s k i : Wstęp do geometrii różniczkowej, PWN, Warszawa, 1972.

[15] J.W. S m i t h : The de Rham theorem for general spaces, Tâhoku Math. Journ. 18 (1966), 115-137.

[16] K. S p a l l e k : Differential forms on differentiable spaces, Rend. Mat. 4 (1971) 231-258.

[17] K. S p a l l e k : Differential forms on differentiable spaces, II, Rend. Mat. 5 (1972), 375-389.
[18] J. S c h w a r t z : De Rham's theorem for arbitrary spaces, Amer. J. Math. 77 (1955), 29-44.

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