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Maciej Skwarczyński

SEMIGROUPS OF NONEXPANDING MAPPINGS

A classical theorem of E. Cartan, about isometries of a simply connected, complete riemannian manifold M of negative sectional curvature, says that a group G of such isometries has a common fixed point iff there exists $p \in M$, such that the orbit

$$\mathrm{Orb}_G(p) = \{f(p);\ f \in G\}$$

is bounded in M. A modern proof of Cartan's theorem (see [2] thm. 9.2) is based on the Haar integral (for the closure of G), and a notion of barycenter for M-valued mapping. A purpose of the present note is to obtain Cartan's theorem not by using the Haar integral, but as a corollary of an elementary result on semigroups of nonexpanding mappings. In fact we shall derive a necessary and sufficient condition in order that a semigroup F of nonexpanding self-mappings of M has a common fixed point.

1. The main result

Let M be as above. We shall denote by d the geodesic distance on M. By definition, a mapping $f: M \to M$ is nonexpanding if for every $p, q \in M$

$$d(f(p), f(q)) \leq d(p,q). \tag{1}$$

Let F be a semigroup (with identity) consisting of nonexpanding self-mappings of M. The orbit of $p \in M$ under F is the set

$$\mathrm{Orb}_F(p) = \{f(p);\ f \in F\}.$$

For every $p,q \in M$ the boundedness of $\mathrm{Orb}_F(p)$ together with (1) implies the boundedness of $\mathrm{Orb}_F(q)$. Therefore either all orbits under F are bounded, or all orbits under F are unbounded. We shall prove

T h e o r e m 1. Let M, F be as above. In order that all mappings belonging to F have a common fixed point it is necessary and sufficient, that there exists $p \in M$ with the following properties:

i. The orbit $P = \mathrm{Orb}_F(p)$ is bounded.

ii. For every $f_1, f_2 \in F$ there exists $g \in F$, such that

$$g(P) \subset f_1(P) \cap f_2(P). \tag{2}$$

Before giving the proof of this theorem we shall consider two examples:

E x a m p l e 1. Let us assume that F is a group of isometries with bounded orbits. Then every $p \in M$ satisfies conditions i. and ii. (In particular Cartan's theorem follows from Theorem 1). To see this note that i. is satisfied by assumption. Moreover, for a given $g \in F$ and every $f \in F$ we have $f = g(g^{-1}f)$ where $g^{-1}f \in F$. Therefore, for every $p \in M$

$$P = \mathrm{Orb}_F(p) = \{g((g^{-1}f)(p));\ f \in F\} \subset g(P).$$

Inclusion $g(P) \subset P$ is obvious, hence $g(P) = P$. The right side of (2) is also P (in view of $f_1(P) = P$ and $f_2(P) = P$). Therefore ii. holds.

E x a m p l e 2. Let us assume that F is a commutative semigroup with bounded orbits. Then every $p \in M$ satisfies conditions i. and ii. (In particular Theorem 1 shows that F has a common fixed point). To see this note that i. holds by assumption. As for ii. we shall show that for every $f_1, f_2 \in F$ inclusion (2) holds with $g = f_1 f_2 = f_2 f_1$. Indeed, (2) follows from obvious inclusions

$$g(P) = f_1(f_2(P)) \subset f_1(P)$$

$$g(P) = f_2(f_1(P)) \subset f_2(P)$$

and ii. is satisfied.

Let us state some properties of the manifold M which we shall need in the following. Since M is complete every pair of points $p,q \in M$ can be joined by a (minimal) geodesic segment. Moreover, by the theorem of Hopf-Rinov, a subset of M is compact iff its is closed and bounded. Next, M satisfies assumptions of Hadamard's theorem (cf. [1], thm. 13.3), hence the exponential mapping $Exp_p : TM_p \to M$ is a diffeomorphism for every $p \in M$; it follows that every geodesic segment in M is uniquely determined by its endpoints. It is also known that Exp_p^{-1} in nonexpanding for every $p \in M$ (cf. [1], thm. 13.1).

2. P r o o f of Theorem 1

Necessity is obvious, since a common fixed point p satisfies i. and ii. It remains to prove sufficiency. Let us assume that $p \in M$ satisfies i. and ii. With $P := Orb_F(p)$, we define for every $q \in M$ and $f \in F$

$$r_f(q) := \inf\{R > 0;\ f(P) \subset B(q,R)\}.$$

Obviously $f(P) \subset P$, hence $f(P) \subset B(q,R)$ implies that $B(q,R)$ intersects P and (as a consequence) $dist(q,P) \leq R$. It follows dhat $dist(q,P) \leq r_f(q)$. Since $f \in F$ was arbitrary we see that

$$r(q) := \inf\{r_f(q);\ f \in F\}$$

satisfies $dist(q,P) \leq r(q)$. Let us consider $M \cup \{\infty\}$, the Alexandrov compactification of M. Since P is bounded (by i.) it follows that $\lim_{q \to \infty} dist(q,P) = \infty$; this implies that $\lim_{q \to \infty} r(q) = \infty$.

Denote $r^* := \inf\{r(q);\ q \in M\}$. Let us chose a sequence $q_n \in M$, $n = 1,2,\ldots$ such that

$$\lim_{n \to \infty} r(q_n) = r^*.$$

Since $r(q)$ becomes infinite at infinity, the sequence q_n must be bounded, and we may assume with no loss of generality that q_n converges to some $q_0 \in M$. We shall show that q_0 is a common fixed point for F, by proving that

a) $r(q_0) = r^*$,

b) $r(f(q_0)) = r^*$, for every $f \in F$,

c) $r(q_1) = r^* = r(q_2)$ implies that $q_1 = q_2$.

(Indeed, (a), (b) and (c) imply that $f(q_0) = q_0$).

Ad a). Take any $\epsilon > 0$. If n is sufficiently large then $r(q_n) < r^*+\epsilon$, and by definition of $r(q_n)$, there exists $f_n \in F$, such that

(3) $$f_n(P) \subset B(q_n, r^*+\epsilon).$$

We may chose n so large, that $d(q_0,q_n) < \epsilon$. This inequality and (3) imply

(4) $$f_n(P) \subset B(q_0, r^*+2\epsilon).$$

From (4) follows that $r_{f_n}(q_0) \leqslant r^*+2\epsilon$, hence $r(q_0) \leqslant r^*+2\epsilon$.

Since ϵ was arbitrary we have $r(q_0) \leqslant r^*$. This obviously implies $r(q_0) = r^*$, as claimed.

Ad b). It suffices to show that $r(f(q_0)) \leqslant r^*$. Take any $\epsilon > 0$. Since $r(q_0) = r^*$ (by (a)), there exists $h \in F$, such that $h(P) \subset B(q_0,r^*+\epsilon)$. By assumption $f \in F$ is nonexpanding, therefore

$$(fh)(P) \subset f(B(q_0,r^*+\epsilon) \subset B(f(q_0),r^*+\epsilon).$$

This shows that $r_{fh}(f(q_0)) \leqslant r^*+\epsilon$, hence $r(f(q_0)) \leqslant r^*+\epsilon$. Since ϵ was arbitrary it follows that $r(f(q_0)) \leqslant r^*$, as claimed.

Ad c). Let us consider first the case when $r^* = 0$. Let us assume (by contradiction), that there are points $q_1 \neq q_2$ in M, such that $r(q_1) = 0$, $r(q_2) = 0$. Chose $\epsilon > 0$ so small that the balls $B(q_1,\epsilon)$ and $B(q_2,\epsilon)$ are disjoint. Since $r(q_i) = 0$ $(i = 1,2)$ there exists $f_i \in F$, such that $f_i(P) \subset B(q_i,\epsilon)$. Let $g \in F$ be such that (2) holds. Since

$$g(P) \subset B(q_1,\epsilon) \cap B(q_2,\epsilon) = \emptyset$$

it follows that $g(P) = \emptyset$, a contradiction.

It remains to consider the case when $r^* > 0$. Take any $\epsilon > 0$; there exist $f_1, f_2 \in F$, such that $f_i(P) \subset B(q_i, r^*+\epsilon)$. Take $g \in F$ such that (2) holds; it follows that

$$(5) \qquad g(P) \subset B(q_1,\ r^*+\epsilon) \cap B(q_2,\ r^*+\epsilon).$$

Denote by q_* the midpoint of the (minimal) geodesic segment which joins q_1 and q_2; the length of this segment will be denoted by $2m$. Let q be an arbitrary point in the right side of (5). Diffeomorphism $\mathrm{Exp}_{q_*}^{-1} : M \longrightarrow TM_{q_*}$ maps points q_*, q_1, q_2, q in M respectively onto points 0, v_1, v_2, v in TM_{q_*}, see Fig.1.

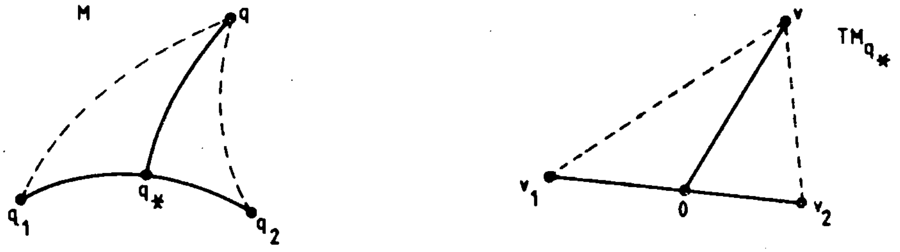

Fig.1

On an arbitrary Riemannian manifold the exponential mapping maps the segments of rays emanating from the origin onto geodesic segments of the same length. Therefore

$$(6) \qquad \|v_1\| = m = d(q\ ,q_1), \quad \|v_2\| = m = d(q\ ,q_2)$$

and

$$(7) \qquad \|v\| = d(q_*,q).$$

Since $\mathrm{Exp}_{q_*}^{-1} : M \longrightarrow TM_{q_*}$ is nonexpanding it follows that

$$(8) \qquad \|v - v_1\| \leq d(q,q_1) \quad \|v - v_2\| \leq d(q,q_2).$$

In view of (6), (7), (8) an elementary inequality

(9) $$\|v\|^2 \leqslant \max^2(\|v - v_1\|, \|v - v_2\|) - m^2$$

yields

(10) $$d(q_*, q)^2 \leqslant \max^2(d(q, q_1), d(q, q_2)) - m^2.$$

By the definition of q the right side is not greater then $(r^*+\epsilon)^2 - m^2$. Therefore $q \in B(q_*, \sqrt{(r^*+\epsilon)^2 - m^2})$ for all q in the right side of (5). Hence (5) yields

$$g(P) \subset B(q_*, \sqrt{(r^*+\epsilon)^2 - m^2}).$$

Consequently $r_g(q_*) \leqslant \sqrt{(r^*+\epsilon)^2 - m^2}$, hence

$$r(q_*) \leqslant \sqrt{(r^*+\epsilon)^2 - m^2}.$$

For sufficiently small ϵ, the right side is smaller then r^*, contradicting the definition of r^*. Therefore (c) holds and the proof of Theorem 1 is complete.

3. Additional remarks

1. Let q_1, q_2 be two different points of M. Let f_1, f_2 be self-mappings of M such that $f_i(M) = \{q_i\}$, $i = 1,2$. Then $F = \{id, f_1, f_2\}$ is a semigroup of nonexpanding mappings, which has no common fixed point.

2. In general a common fixed point of F is not unique. (Consider for example a group of rotations around a fixed axis in R^3).

3. In the case when F is generated by a single non-expanding mapping f Theorem 1 was obtained by S.Wereński [4]. In his proof the problem is reduced to the Brouwer fixed point theorem. Let us remark, that the case when f is lipschitzian with a constant $L < 1$ is covered by a general theorem on contractions in metric spaces (cf. [3] p.48); in this case there is the unique fixed point of f.

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Author's address: Smoleńskiego 27a/14, 01-698 WARSZAWA
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