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Mihai Cristea

WHEN CLOSED MAPS ARE PROPER OR CONTINUOUS

It is frequently of interest in applications to know that a given proper map is closed, or that a closed map is proper. It is known (see [1], [5]) that if E and F are Hausdorff spaces, F a first countable space or a locally compact space and $f:E \to F$ is continuous and proper, then f is closed. Also it is known (see [1]) that if E and F are Hausdorff spaces, $f:E \to F$ is closed and for every $y \in F$, $f^{-1}(y)$ is compact, then f is proper. We shall use this theorem to find conditions for a closed map to become a proper map.

We shall also find conditions for a closed map $f:E \to F$, where E is of the second Baire category, to become continuous on E, with the possible exception of a set of the first Baire category in E.

Throughout this paper, E and F will be Hausdorff spaces.

T h e o r e m 1. Let E be a normal space such that every sequentially compact set from E is compact, F be a first countable space or a locally compact space, $f:E \to F$ closed such that for every $x \in E$, $V \in V(f(x))$ and $Q \in V(x)$, $f(Q) \cap V \neq \{f(x)\}$. Then f is proper.

P r o o f . Suppose that there exists $y \in F$ such that $f^{-1}(y)$ is not compact. Then we can find x_i such that $x_i \neq x_j$ for $i \neq j$, $f(x_i) = y$; $i \in N$, and $\{x_i\}'_{i\in N} = \emptyset$. We take $V_n \in V(y)$, $V_{n+1} \subset V_n$ a countable base of y if F is a first countable space and $V_n = W$, where $W \in V(y)$ is relatively compact if F is a locally compact space. Choose $Q_i \in V(x_i)$ such that $\bar{Q}_i \cap \bar{Q}_j = \emptyset$

for $i \neq j$. Denote $K = \{x_n\}_{n\in N}$. Let $H = Fr \bigcup_{i=1}^{\infty} Q_i$. Since K is closed, H is closed and $K \cap H = \emptyset$, we can find U, V open such that $K \subset U$, $H \subset V$ and $U \cap V = \emptyset$. Then for every sequence $y_n \in U \cap Q_n$, $\{y_n\}'_{n\in N} = \emptyset$. We can find $z_n \in U \cap Q_n$ such that $f(z_n) \in V_n$, $f(z_i) \neq f(z_j)$ for $i \neq j$ and $f(z_n) \neq y$ for $n \in N$. Denote $A = \{z_n\}_{n\in N}$ and remark that A is closed. If F is a first countable space, $f(z_n) \to y$, hence $y \in \overline{f(A)} \setminus f(A)$. If F is locally compact, then $f(z_n) \in \overline{W}$, hence we can find $z \in \overline{W}$ and z_{n_k} a subsequence of z_n such that $f(z_{n_k}) \to z$ and $f(z_{n_k}) \neq z$ for every $k \in N$. If we denote $B = \{z_{n_k}\}_{k\in N}$, then B is closed and $z \in \overline{f(B)} \setminus f(B)$. We obtained a contradiction, hence f is a proper map, q.e.d.

C o r o l l a r y 1. Let E be a metric space, F a first countable space, or a locally compact space, $f:E \to F$ a closed, continuous map such that for every open set $Q \subset E$, card $f(Q) \geq 2$. Then f is proper.

C o r o l l a r y 2. Let E be a locally connected, metric space, F a first countable regular space or a locally compact space, $f:E \to F$ closed such that f carries connected sets into connected sets and such that for every open set $Q \subset E$, card$f(Q) \geq 2$. Then f is proper.

P r o o f . Let $x \in E$, $Q \in V(x)$ connected and $V \in V(f(x))$. Then either $f(Q) \subset V$ or $f(Q) \cap FrV \neq \emptyset$.

C o r o l l a r y 3. Let E and F be n dimensional manifolds, E a metric space, $f:E \to F$ continuous, closed, open or light. Then f is proper.

T h e o r e m 2. Let E be a metrisable, connected, infinite dimensional Fréchet manifold, $f:E \to F$ continuous and closed. Then f is either a proper map or is the constant map.

P r o o f . Suppose that f is not constant and let $y \in F$ and $A = f^{-1}(y) \neq \emptyset$. Since E is connected, $FrA \neq \emptyset$. Then FrA is compact, since, by the same argument as in Theorem 1, we cannot find $x_n \in FrA$, $x_i \neq x_j$ for $i \neq j$ and $\{x_n\}'_{n\in N} = \emptyset$. If

$\operatorname{int}A = \emptyset$, it results that $f^{-1}(y)$ is compact. If $\operatorname{int}A \neq \emptyset$, we see that $E \setminus \operatorname{Fr}A$ is nowhere disconnecting (see [4]), hence $E \setminus \operatorname{Fr}A$ is connected and $\operatorname{Int}\operatorname{Fr}A = \emptyset$. Then f is constant on $E \setminus \operatorname{Fr}A$, hence on E, which represents a contradiction, and the theorem is proved.

T h e o r e m 3. Let $n \geq 2$ and $f: R^n \to R^n$ be continuous and closed. Then either f is a proper map, or there exists $r > 0$ such that f s constant on $CB(0,r)$.

P r o o f . Let $y \in R^n$ and $A = f^{-1}(y) \neq \emptyset$. Then $\operatorname{Fr}A$ is compact, hence there exists $r>0$ such that $\operatorname{Fr}A \subset \bar{B}(0,r)$. If $A \subset \bar{B}(0,r)$, then A is compact. If not, there exists $z \in \operatorname{Int}A \cap C\bar{B}(0,r)$. Let $b \in C\bar{B}(0,r)$. We can join z and b by an arc in $C\bar{B}(0,r)$ and since $\operatorname{Fr}A \subset \bar{B}(0,r)$, it results that $b \in \operatorname{int}A$, hence $A \supset C\bar{B}(0,r)$.

T h e o r e m 4. Let E be of the second Baire category, F a regular space with a countable base B, $f: E \to F$ such that for every $S_i \in B$, $f^{-1}(\bar{S}_i)$ is closed. Then f is continuous on E with the possible exception of a set of the first Baire category in E.

P r o o f . Let $H_i = \operatorname{Fr}f^{-1}(\bar{S}_i)$. Since $f^{-1}(\bar{S}_i)$ is closed, $\operatorname{int}H_i = \emptyset$. Let $H = \bigcup_{i=1}^{\infty} H_i$. Let now $x \in E \setminus H$ be fixed and $m, n \in N$ such that $\bar{S}_m \subset S_n$ and $S_m, S_n \in V(f(x))$. Since $x \in \operatorname{int}f^{-1}(\bar{S}_m)$, we can find $U \in V(x)$ such that $f(U) \subset \bar{S}_m \subset S_n$, and since S_i is a base, f is continuous at x.

C o r o l l a r y 4. Let E be of the second Baire category, F a locally compact space with a countable base and $f: E \to F$ proper. Then f is continuous on E with the possible exception of a set of the first Baire category in E.

C o r o l l a r y 5. Let E be of the second Baire category and a first countable space, F a locally compact space with a countable base and $f: E \to F$. Suppose that f satisfies one of the following conditions:

a) The graph of f is closed,

b) f carries compact sets into closed sets and $f^{-1}(y)$ is closed for every $y \in F$.

c) f carries compact sets into closed sets and for every $x \in E$, $Q \in V(x)$ and $V \in V(f(x))$, $V \cap f(Q) \neq \{f(x)\}$.

Then f is continuous on E with the possible exception of a set of the first Baire category in E.

P r o o f . Let S_i be a base in F such that $\bar{S}_i$ is compact for every $i \in N$ and $x_n \in f^{-1}(\bar{S}_i)$ such that $x_n \to x$. We shall prove that $f^{-1}(\bar{S}_i)$ is closed for every $i \in N$. In the first case, $f(x_n) \in \bar{S}_i$, hence we can find $y \in \overline{\{f(x_n)\}}_{n\in N}$, hence $y = f(x)$ and $x \in f^{-1}(\bar{S}_i)$. In the second case, suppose that $f(x_p) \neq f(x_j)$ for $p \neq j$. Then either $f(x) = f(x_k)$ for some $k \in N$, or $f(x) \in \{f(x_n)\}'_{n\in N}$. Indeed, $\{x_n\}_{n\in N} \cup \{x\}$ is compact, hence $\{f(x_n)\}_{n\in N} \cup \{f(x)\}$ is closed. Since there exists $y \in \{f(x_n)\}'_{n\in N}$, and we can suppose that $y \neq f(x_n)$ for every $n \in N$, it results that $y = f(x)$, hence $x \in f^{-1}(\bar{S}_i)$. Suppose now that $f(x_p) = f(x_j) = z$ for $p \neq j$. Since $f^{-1}(z)$ is closed, it results that $x \in f^{-1}(z) \subset f^{-1}(\bar{S}_i)$. For the third case suppose that $f(x_p) = f(x_j) = z \neq f(x)$ for $p \neq j$. We can find $z_n \in V(x_n)$ such that $z_n \to x$ and $f(z_p) \neq f(z_j)$ for $p \neq j$ and $\overline{f(z_n)} \in \bar{S}_i$ for every $n \in N$. As before, we find that $f(x) \in \overline{\{f(z_n)\}}_{n\in N}$, hence $x \in f^{-1}(\bar{S}_i)$.

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INSTITUTUL DE MATEMATICA, R-70109 BUCURESTI, ROMANIA
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