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METRIC VARIANTS OF THE BREZIS-BROWDER ORDERING PRINCIPLE

0. Introduction

Let X be a nonempty set and let $\leq$ be a quasi-ordering (i.e., a reflexive and transitive relation) over X. Given a function $\varphi: X \to R$, call the point $z \in X$, φ - maximal when $z \leq w$ implies $\varphi(z) = \varphi(w)$ (that is, φ is constant on $X(z,\leq) = \{w \in X;\ z \leq w\}$). To establish under which circumstances it is true that each element of X is bounded above by a φ-maximal one, a basic assumption must be made about our elements, namely

(H_1) each ascending sequence in X has an upper bound (the ambient space being termed N - inductive in such a case). Under this framework, the following 1976 Brézis-Browder answer [4] to the considered problem is the start point of our developments.

T h e o r e m 1. Let the function φ be decreasing and bounded from below. Then, to each $x \in X$ there corresponds a φ-maximal element $z \in X$ with $x \leq z$.

This principle, including the one of Ekeland [6] and having a number of interesting applications to convex as well as nonconvex analysis (as the above references show) has been generalized for the first time in 1982 by Altman [1] and the author [12]. A common extension of these contributions has been obtained in the same year by Galvin ([7]; cf. Section 1) and, respectively, two years later by Turinici [15], using different techniques. Finally, as a unifying effort in this direction we mention the 1985 Kang statement [8] incorporating all these results. The obtained contributions are, of course, interesting from a practical viewpoint, but we must emphasize that, in all concrete cases when a maximality principle of this type is to be applied, a substitution of it by the Brézis--Browder one is always possible. This fact raises the question of whether or not these generalizations of Theorem 1 are effective. It is the main aim of the present exposition to show that the answer is negative or, to be more precise, that the Kang maximality principle we already quoted is nothing but an equivalent formulation of the Brézis-Browder one. (As a completion of these facts, it is proved in Section 2 that the compactness type Zorn maximality result established by the author in his 1982 paper [illegible] is also reductible to Theorem 1 by suitably choosing the function φ). In the light of this conclusion, it is legitimate to ask whether extensions of this principle exist without being reductible to it. To solve this problem, two strategies are followed. The first one is essentially functional and consists in replacing the decreasing property of the function φ (basic to the above developments) by an asymptotic one; this will be done in Section 3. The second one is founded on a [illegible] version of (H_1) and is equivalent - as independent result - with Zorn's maximality principle, as we shall see in Section 4. Some further considerations about these problems including an appropriate vectorial treatment of the variational Ekeland's result (cf. the above reference) will be given in a forthcoming paper.

1. Results about d-maximal elements

Let $[0, \infty]$ stand for the extended set of positive numbers (ordered and metrized in the usual way). The following variant of Theorem 1 will be in effect for us.

T h e o r e m 2. Let the function $\varphi : X \to [0, \infty]$ be decreasing. If, in addition,

(C_1) for each $x \in X$, $\varepsilon > 0$, there exists $y = y(x, \varepsilon) \geq x$ with $\varphi(y) < \varepsilon$ then, conclusion of Theorem 1 holds with $\varphi(z) = 0$.

P r o o f . Letting $x \in X$ be given, choose by (C_1) an element $y \geq x$ with $\varphi(y) < \infty$ and apply Theorem 1 to the structure $(X(y,\leq),\leq)$ and the same function, to get a φ-maximal element $z \in X(y,\leq)$ (with, of course, $x \leq z$). Since, again by (C_1), $\varphi(z) > 0$ leads us to a contradiction, we are forced to accept that $\varphi(z) = 0$. This ends the argument, q.e.d.

So far, Theorem 2 is a consequence of Theorem 1. But, the converse is also valid as it will be the case from the considerations below. Let the notion of (real) pseudometric over X be used to designate any function $d:X^2 \rightarrow [0,\infty]$ satisfying $d(x,x) = 0$, $x \in X$. Having introduced such an element, call the point $z \in X$, d-maximal provided $z \leq u \leq v$ implies $d(u,v) = 0$. As in the preceding section, we shall be interested in establishing under which conditions any point in X is bounded above by a d-maximal one. In this direction, the following result is almost immediate.

P r o p o s i t i o n 1. Let the conditions of Theorem 2 be fulfilled and, in addition,

(H_2) for each $y \in X$ with $\varphi(y) = 0$ there exists a d-maximal point $z \in X$ with $y \leq z$.

Then, the above property is necessarily holding.

To derive concrete circumstances under which (H_2) is to be fulfilled we have to introduce a number of appropriate notions as well as to indicate some basic facts about the ascending sequences of the ambient space. Firstly, the d-Cauchy property being defined as in the metrical case ($d(x_n,x_m) \rightarrow 0$ as $n,m \rightarrow \infty$, $n \leq m$), call the (ascending) sequence $(x_n)_{n\in N}$ in X, d-asymptotic when $\lim d(x_n,x_{n+1}) = 0$ as $n \rightarrow \infty$. Of course, each (ascending) d-Cauchy sequence is d-asymptotic too, the reverse implication being also valid when all such sequences are involved, making the global conditions:

(C_2) each ascending sequence is d-Cauchy,

(C_2') each ascending sequence is d-asymptotic,

be mutually equivalent; moreover, each of these conditions implies,

(C_3) $\begin{cases} d \text{ is } \leq\text{-asymptotic (to any } x \in X,\ \varepsilon > 0, \text{ there corresponds } y = y(x,\varepsilon) \geq x \text{ with } d(u,v) < \varepsilon \text{ for } y \leq u \leq v) \end{cases}$

as it can be readily verified. Secondly, call the pseudometric d, $\leq$ -triangular when, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $x \leq y \leq z$ and $d(x,y), d(x,z) < \delta$ imply $d(y,z) < \varepsilon$. We remark in this context that, under

(C_4) d is $\leq$ -triangular

the property of being d-Cauchy is equivalent, for an ascending sequence $(x_n)_{n\in N}$ with the (weaker in general) property of being d-semi-Cauchy (to any $\varepsilon > 0$ there corresponds $n = n(\varepsilon) \in N$ such that $p \geq n$ implies $d(x_n, x_p) < \varepsilon$). Accordingly - returning to the general case (modulo (C_4)) - term an (ascending) sequence $(x_n)_{n\in N}$ in X, d-semi-asymptotic when $\liminf d(x_n, x_{n+1}) = 0$ as $n \to \infty$; it will follow by the same way as before that the global conditions

(C_5) each ascending sequence is d-semi-Cauchy

(C_5') each ascending sequence is d-semi-asymptotic

are mutually equivalent and, moreover, that each of them implies

(C_6) $\begin{cases} d \text{ is } \leq\text{-semi-asymptotic (to any } x \in X,\ \varepsilon > 0, \text{ there corresponds } y = y(x,\varepsilon) \geq x \text{ with } d(y,z) < \varepsilon, \text{ for } z \geq y). \end{cases}$

Under these facts, the following particular version - due to Kang [8] - of the above proposition may be formulated.

T h e o r e m 3. Let the assumption (H_1) be valid and the pseudometric d on X be $\leq$-asymptotic. Then, for each $x \in X$ there exists a d-maximal element $z \in X$ with $x \leq z$.

P r o o f . Define a function $\varphi : X \to [0,\infty]$ by

$$\varphi(x) = \sup\{d(u,v);\ x \leq u \leq v\}, \quad x \in X.$$

Of course, hypotheses of Theorem 2 are fulfilled, as well as (H_2) (with $z = y$). Therefore, Proposition 1 is applicable. q.e.d.

As already precised, a sufficient condition for (C_3) to be valid is (C_2) or, equivalently, (C_2'). Concerning this fact, let (Y,τ) be a topological space and $0 \in Y$ a distinguished point. Given the mapping $d:X^2 \to Y$, call it a Y-pseudometric on X provided $d(x,x) = 0$, $x \in X$, and, in such a context, let us introduce the notions of d-maximal element and of d-asymptotic sequence by the same (formal) way as before, with $0 \in R$ substituted by $0 \in Y$ (the limit in the second case being taken in the sense of τ). As an application of Theorem 3, the following d-maximality result due to Galvin [7] will be (methodologically) useful in the sequel.

Theorem 4. Let the assumption (H_1) be fulfilled and the Y-pseudometric d over X be such that (C_2') plus

(C_7) $\{0\}$ is a G_δ-set

are being holding. Then, conclusion of Theorem 3 (in the form we just indicated) remains true.

Proof. By (C_7), $\{0\} = \cap \{G_n;\ n \in N\}$ where G_n is open in Y for each $n \in N$. Let $\psi : Y \to [0,\infty]$ be defined, for each $y \in Y$, by

$$\psi(y) = \infty, \quad \text{when } \{n \in N;\ y \in G_n\} \text{ is empty}$$
$$= \inf \{2^{-n};\ y \in G_n\}, \text{ when the same set is not empty}$$

and put

$$e(u,v) = \psi(d(u,v)), \quad u,v \in X.$$

Clearly, $d(u,v) = 0 \in Y$ if and only if $e(u,v) = 0 \in R$ (which, among others, shows that e is a (real) pseudometric over X). On the other hand, (C_2') will be surely fulfilled with d replaced by e. This, along with the preceding statement, ends the proof. q.e.d.

In particular, when $Y = R$, and the pseudometric d on X is, in addition $\leqslant$-triangular, Theorem 4 reduces to author's maximality result [15] which, as precised there, contains Altman's extended variant [1] of Theorem 1. So, Theorem 3 may be viewed as a common extension of all these results (hence, in particular, an extension of Theorem 1); but, as a consequence of its proof, the converse of this last assertion is also valid, making all these statements be reductible to each another. This fact has, of course, a theoretical impact on the above maximality principles but, in general, not a practical one since, in many concrete cases, one or another of them can be more directly handled than the original Brézis-Browder one; as a good illustration of this assertion we mention the solvability result (subsumed to Theorem 3) in the above quoted author's paper.

2. Metrical Zorn type results

Let $(X,\leq)$ be a quasi-ordered set. In the following, we shall be interested in determining circumstances under which a point x of X be bounded above by a $\leq$-maximal one, $z \in X$ (introduced as: $z \leq w$ implies $w \leq z$). This is in fact a standard maximality problem of Zorn type and the natural way to solve it is, e.g., that an upper boundedness condition be imposed on every chain C (or, equivalently, on every well ordered part W) of $(X,\leq)$. However, when an additional pseudometric structure is being added, this (transfinite) method seems to be somehow difficult to be followed (as a typical example of this kind being the author's maximality result in [14]) and the legitimate question arises whether a denumerable upper boundedness condition of the form (H_1) cannot replace it. To get an appropriate answer, the following consequence of Theorem 2 will be useful for us.

P r o p o s i t i o n 2. Let the conditions of Theorem 2 be fulfilled and, in addition assume

(H_3) $\begin{cases} \text{for each } y \in X \text{ with } \varphi(y) = 0 \text{ there exists a } \leq\text{-maximal point } z \in X \text{ with } y \leq z. \end{cases}$

Then, the Zorn property we formulated before is necessarily holding.

As already precised, the setting in which (H_3) is to be discussed is the pseudometric one; so, let $d:X^2 \to [0,\infty]$ be such an object. For each $x \in X$ and each subset Y of X, let $d(x,Y)$ denote, as usually, the infimum of all distances $d(x,y)$, $y \in Y$; under this convention, call the considered pseudometric d, weakly $\leq$-asymptotic when, to any $x \in X$, $\varepsilon > 0$, there corresponds $y = y(x,\varepsilon) \geq x$ with $d(u,X(v,\leq)) < \varepsilon$ for $y \leq u \leq v$. At the same time, letting $\leftarrow$ denote the dual convergence structure over X (defined as: $x \leftarrow x_n$ whenever $d(x,x_n) \to 0$ as $n \to \infty$) call the considered quasi-ordering $\leq$, weakly d-self-closed when for each $x \in X$ there exists $y \geq x$ with: $(x_n)_{n\in N}$ ascending in $X(y,\leq)$ and $x \leftarrow x_n$ imply $x_n \leq y$, for some $n \in N$.

Under these preliminaries, the following Zorn type result may be formulated.

T h e o r e m 5. Let the quasi-ordered space $(X,\leq)$ fulfiling (H_1) be such that, a pseudometric d over X may be determined with

(C_8) d is weakly $\leq$-asymptotic

(C_9) $\leq$ is weakly d-self-closed.

Then, for any $x \in X$, there is a $\leq$-maximal element $z \in X$ with $x \leq z$.

P r o o f . Let the function $\varphi : X \to [0,\infty]$ be defined as

$$\varphi(x) = \sup\{d(u,X(v,\leq));\ x \leq u \leq v\},\quad x \in X.$$

By (C_8), conditions of Theorem 2 are verified. Moreover, let $y \in X$ with $\varphi(y) = 0$ and let $z \geq y$ be that introduced by (C_9). We claim z is $\leq$-maximal (and this will complete the proof, by Proposition 2). Indeed, letting $w \in X(z,\leq)$, it will follow, in view of

$$d(y,X(v,\leq)) = 0,\quad v \in X(w,\leq),$$

that an ascending sequence $(v_n)_{n\in N}$ in $X(w,\leq)$ may be determined with $d(y,v_n) < 2^{-n}$, $n \in N$ (that is, $y \leftarrow v_n$); this, along with the choice of z gives $v_n \leq z$, for some $n \in N$ (hence $w \leq z$), and the result follows. q.e.d.

Let us call the (ascending) sequence $(x_n)_{n\in N}$ in X, weakly d-asymptotic when for each $n \in N$, $\varepsilon > 0$, one has $d(x_p,x_q) < \varepsilon$ for some $p,q \geq n$, $p < q$; then, a sufficient condition for (C_8) is

(C_{10}) each ascending sequence is weakly d-asymptotic

as it can be readily seen; at the same time, it is clear that

(C_{11}) $\left\{\begin{array}{l}\leqslant \text{ is d-self-closed } ((x_n)_{n\in N} \text{ ascending and } x \leftarrow x_n \\ \text{imply } x_n \leqslant x,\ n \in N)\end{array}\right.$

is a sufficient condition for (C_9). In particular, with d being a metric over X, the corresponding variant of Theorem 5 reduces to the compactness type author's maximality result [13] proved by a direct Zorn argument and including the variational Ekeland's principle [6]. Of course, Brondsted's contribution [5] in this direction cannot be covered by this result but, as proved in [12], it might be viewed, however, as a particular case of Theorem 3.

3. Maximality principles of sequential type

The developments we performed in Section 1 raise the question of whether or not extensions of Theorem 1 (or its variants) exist without being reductible to it. This seems to be a very delicate question; to give an appropriate answer - of a functional type - we start from the remark that, as the decreasing property of the ambient function φ is basic in all those treatments, a natural way of extending such statements is to replace it by another conditions, among which the most interesting one is (C_1). This will require, however, a sequential treatment of the maximality problem. To be more precise, let $(X,\leqslant)$ be a quasi-ordered set. Denoting by ubd (x_n) the set of all upper bounds of the sequence $(x_n)_{n\in N}$ in X, let $\varphi: X \to [0,\infty]$ be a given function and call the point z in X, sequentially φ-maximal provided $X(z,\leqslant) \subset$ ubd (x_n) for some ascending sequence $(x_n)_{n\in N}$ in X with $\varphi(x_n) \to 0$. The following statement is almost immediate.

P r o p o s i t i o n 3. Let the assumptions of Theorem 2 (minus the decreasing property for the function φ) be fulfilled. Then, to each $x \in X$ there corresponds a sequentially φ-maximal point $z \in X$ with $x \leqslant z$.

In particular, letting $d: X^2 \to [0,\infty]$ be a pseudometric over X, a good choice for this function is

$$\varphi(x) = \sup\{d(x,y);\ x \leq y\}, \quad x \in X,$$

in which case, condition (C_1) becomes a $\leq$-semi-asymptotic property of the considered pseudometric. Furthermore, letting $\longrightarrow$ denote the (primal) convergence structure on X (introduced as: $x_n \rightarrow x$ when $d(x_n,x) \rightarrow 0$ as $n \longrightarrow \infty$) put $\lim (x_n) = \{x \in X;\ x_n \rightarrow x\}$ and $\text{olim}\,(x_n) = \lim (x_n) \cap \text{ubd}\,(x_n)$, for each (ascending) sequence $(x_n)_{n \in N}$ in X. The corresponding version of Proposition 3 may now be stated as follows.

T h e o r e m 3′. Let the assumption (H_1) be fulfilled and the pseudometric d over X be $\leq$-semi-asymptotic. Then, conclusion of Proposition 3 is valid in the sense $X(z,\leq) \subset \subset \text{olim}\,(x_n)$.

By convention, a point $z \in X$ with the above property will be referred to as a sequentially d-maximal one. Of course, under an extra condition like (C_4), Theorem 3′ is identical with Theorem 3 or, equivalently, with Brézis-Browder′s ordering principle subsumed to Theorem 1. The question of whether this property continues to hold in the absence of (C_4) remains open; we conjecture that the answer is negative.

Finally, as another useful choice of our function, let us consider the case

$$\varphi(x) = \sup\{d(x,X(y,\leq));\ x \leq y\}, \quad x \in X,$$

when (C_1) becomes a weakly $\leq$-semi-asymptotic property for the considered pseudometric, expressed as

$$(C_8') \begin{cases} \text{to each } x \in X,\ \varepsilon > o \text{ there corresponds } y = y(x,\varepsilon) \geq x \\ \text{with } d(y,X(u,\ )) < \varepsilon,\ y \leq u. \end{cases}$$

Under the same lines as before, a corresponding version of Proposition 3 may be formulated; in the sequel, we shall be interested, however, in exploiting this vérsion to get a sequentially type counterpart of Theorem 5 which, in a convenient form, can be given as

T h e o r e m 5′. Let the quasi-ordered space $(X,\leq)$ fulfiling (H_1) be such that, for some pseudometric d over X, conditions (C_8') as well as

(C_9') to any ascending sequence $(x_n)_{n\in N}$ in X and $x \in$ ubd (x_n) there corresponds $y = y((x_n),x) \geq x$ with: $(y_n)_{n\in N}$ ascending in $X(x,\leq)$ and $d(x_n,y_n) \to 0$ as $n \to \infty$ give $y_n \leq y$ for some $n \in N$,

hold. Then, conclusion of Theorem 5 remains valid.

P r o o f . Letting $x \in X$ be arbitrary fixed we get, by (C_8') in conjunction with Proposition 3 an ascending sequence $(x_n)_{n\in N}$ in X with $x \leq x_n$, $n \in N$, and

$$d(x_n,X(u,\leq)) < 2^{-n}, \quad n \in N, \quad x_n \leq u.$$

Let $y \in$ ubd (x_n) be the point indicated by (H_1); we claim each element $z = z((x_n),y) \geq y$ taken in accordance with (C_9') is a $\leq$-maximal one (and this will complete the argument). Indeed, given $w \geq z$, it follows, by the above property of (x_n) that an ascending sequence $(y_n) \subset X(w,\leq)$ may be determined with $d(x_n,y_n) \to 0$ as $n \to \infty$; this, again with (C_9'), gives $y_n \leq z$, for some $n \in N$, (hence $w \leq z$), which was to be proved. q.e.d.

By convention, a sequence (x_n) satisfying the requirement of (C_9') will be termed strongly bounded above. In particular, when d is a metric over X, the corresponding variant of Theorem 5′ with (C_{10}) in place of (C_8) reduces to Theorem 1 in [14], proved by a Zorn technique. A direct (logical) connection between this result and these of Section 1 cannot be (generally) established; some further considerations about this problem will be given elsewhere.

4. Transfinite variants of Theorem 3

As we had already occasion to say (cf. the introductory part), the second way of extending Brézis-Browder's ordering principle (or, equivalently, its metric variants given in Section 1) is the ordinality one. To be more precise, let $(X,\leq)$ be a quasi-ordered set and (B^A denoting the class of all functions $f:A \to B$) let $D = (d_i)_{i\in M}$ be $[0,\infty]^M$-pseudometric over X where, by the Zermelo principle ([9], Ch.2, Sect.2), the index set M can be viewed as a well-ordered one. Calling an element $z \in X$, D-maximal provided $D(u,v) = 0$ for all u, $v \in X$, $z \leq u \leq v$, the point is (cf. Section 1) to determine conditions under which any element of X is bounded above by a D-maximal one. To begin with, let us note the fact that (calling the subset P of M an initial segment whenever $M(m,\geq) \subset P$ for all $m \in P$) condition (H_1) (basic to the considerations we performed until now) is no more appropriate in this extended setting, unless M is order isomorphic with an initial segment of N (ordered in the standard way). So, without any loss, assume N is order isomorphic with a segment of M; then (the notion of (ascending) net in X being used to designate each (increasing) mapping $p \mapsto x_p$ from the well-ordered set $(P,\leq)$ to $(X,\leq)$) the following (transfinite) counterpart of (H_1) must be taken into consideration

(H_4) $\begin{cases} \text{each ascending net } (x_p)_{p\in P} \text{ in } X, \text{ where } P \text{ is an} \\ \text{initial segment of } M, \text{ has an upper bound} \end{cases}$

(the ambient space being termed M-inductive in such a case). Under these preliminaries, an useful answer to the above posed problem is contained in

T h e o r e m 6. Let the assumption (H_4) be valid and, in addition, each component of D be $\leq$-asymptotic. Then, to each $x \in X$ there corresponds a D-maximal element $z \in X$ with $x \leq z$.

P r o o f . Letting $x \in X$ be arbitrary fixed, for each $i \in M$ denote by $X_i(x)$ the subset of all d_i-maximal elements $y \in X(x,\leq)$ (not empty, by Theorem 3). Now, $\mathcal{F}(x)$ being the family of all couples (K,f) where K is an initial segment of M and $f:K \to X$ is increasing and K-selective (in the sense $f(p) \in X_p(x)$, for all $p \in K$). Let us introduce an ordering structure on it by the convention

$$(K,f) \leq (H,g) \quad \text{whenever} \quad K \subset H \quad \text{and} \quad f = g/K.$$

Suppose $\mathcal{L} = (K_q,f_q)_{q\in Q}$ is an arbitrary chain in $\mathcal{F}(x)$; then, putting $K = \bigcup\{K_q;\ q \in Q\}$ and defining $f:K \to X$ by $f(p) = f_q(p)$ when $p \in K_q$, it is transparent that $(K,f) \in \mathcal{F}(x)$ and $(K_q,f_q) \leq$ $\leq (K,f)$, $q \in Q$, which tells us $\mathcal{L}$ is bounded above in $\mathcal{F}(x)$.

By the Zorn maximality principle ([2], §2, Sect.4) we have that, given $(K,f) \in \mathcal{F}(x)$, a maximal (modulo $\leq$) element $(H,g) \in \mathcal{F}(x)$ exists with $(K,f) \leq (H,g)$. Assume H is distinct from M and let r be the first element of $M \setminus H$. Treating the function g as an increasing net $(x_p)_{p\in H}$, there exists, by (H_4), a point $x_0 \in X$ with $x_p \leq x_0$, $p \in H$ (hence $x \leq x_0$); furthermore, given $x_0 \in X$, $r \in M$, there exists, again by Theorem 3, a point $x_0^* \in X_r(x_0)$ with $x_0 \leq x_0^*$. Putting $H^* = H \cup \{r\}$, define $g^*: H^* \to X$ by $g^*/H = g$ and $g^*(r) = x_0^*$; then, clearly, (H^*, g^*) is an element of $\mathcal{F}(x)$ which majorizes strictly (H,g), in contradiction to its maximality. So, $H = M$ and, in such a case, any upper bound, z, of $(x_p)_{p\in M}$ (existing by (H_4) and satisfying also $x \leq z$) is our desired element, as it can be directly seen. q.e.d.

As already precised, when $M = N$, the corresponding variant of Theorem 6 is reductible to Kang's maximality result subsumed to Theorem 3. Returning to the general case, the following (pseudo) uniform variant of the above result will be (methodologically) useful for us. Denote by Δ the diagonal of X (the subset of all (x,x) with $x \in X$) and let $\mathcal{U}$ be a family of subsets of X^2 containing Δ (referred to in the sequel as a pseudo-uniformity over X under a terminology similar to that of Nachbin ([10], Ch.2, Sect.2). The point $z \in X$ will be said to be $\mathcal{U}$-maximal, provided $(u,v) \in \cap\,\mathcal{U}$, for each $u,v \in X$, $z \leq u \leq v$; of course, this property does not change when $\mathcal{U}$ is replaced by a basis (= cofinal part) of it (characterized as: for each $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ (the considered basis index) with $U \supset V$). To get appropriate conditions under which any point of X is majorized by a $\mathcal{U}$-maximal one, call the (ascending) net $(x_p)_{p\in P}$ in X, $\mathcal{U}$-Cauchy when to any $U \in \mathcal{U}$ there corresponds $p = p(U) \in P$ such that $(x_q, x_r) \in U$ for all $q, r \in P(p,\leq)$, $q \leq r$; note that, for the case of an (ascending) sequence, this property is stronger than that of being $\mathcal{U}$-asymptotic (for each $U \in \mathcal{U}$ there exists $n = n(U) \in N$ with $(x_m, x_{m+1}) \in U$ for all $m \in N(n,\leq)$) but, when all such sequences are involved, the reverse implication is also valid, in the sense that the global conditions

(C_{12}) each ascending net is $\mathcal{U}$-Cauchy

(C_{12}') each ascending sequence is $\mathcal{U}$-asymptotic

are mutually equivalent, as it can be directly seen. Under these facts, the following answer to the above problem can be given.

T h e o r e m 7. Let the pseudo-uniformity $\mathcal{U}$ on X fulfiling (C_{12}) (or (C_{12}')) be such that, for some basis $\mathcal{V} = (V_i)_{i \in M}$ of it, condition (H_4) is valid. Then, to each $x \in X$ there corresponds a $\mathcal{U}$-maximal point $z \in X$ with $x \leq z$.

P r o o f . Denote, for each $i \in M$

$$d_i(x,y) = 0, \quad \text{if} \quad (x,y) \in V_i$$
$$\infty, \quad \text{otherwise}$$

and observe that conditions of Theorem 6 are fulfilled for the $[0,\infty]^M$-pseudometric $D = (d_i)_{i \in M}$ over X. It follows by the conclusion of the above quoted statement, that, given $x \in X$, a point $z \in X$ exists with (a) $x \leq z$, (b) $z \leq u \leq v$ implies $D(u,v) = 0$ (hence $(u,v) \in V_i$, for all $i \in M$). This completes the proof in view of the fact fhat $\mathcal{V}$ is cofinal in $\mathcal{U}$. q.e.d.

Summing up these developments we found that the above theorems - hence, a fortiori, their "universal" variants based on a general counterpart of (H_4) like

(H_4') each ascending net $(x_p)_{p \in P}$ where P is a well ordered set, has an upper bound

as well as on a general index set M - are technically reductible to Zorn s theorem. But the converse of this fact is also valid; indeed, letting $\leq$ be an ordering (that is, an antisymmetric quasi-ordering) on X with respect to which any well-ordered part is bounded above, denote by $\mathcal{S}(X)$ the class of all ascending sequences $\Sigma = (x_n)_{n \in N}$ in X. Under the convention

$$(\Sigma ; n) = \{ x_n, x_{n+1}), (x_{n+1}, x_{n+2}), \dots \}, \quad \Sigma \in \mathcal{S}(X), \quad n \in N,$$

let us put

$$U_f = \bigcup\{(\Sigma; f(\Sigma)); \Sigma \in \mathcal{S}(X)\}, \quad f \in N^{\mathcal{S}(X)}.$$

It is now clear that the family $\mathcal{U} = \{U_f;\ f \in N^{\mathcal{S}(X)}\}$ is a pseudouniformity over X for which (C_{12}') is being satisfies do, by Theorem 7 (under its "universal" form) we have that, each $x \in X$ is bounded above by a $\mathcal{U}$-maximal element $z \in X$. We want to show z is $\leq$-maximal (that is, $z \leq w$ implies $z = w$). Suppose not; then, for the function $f: \mathcal{S}(X) \to N$ defined as

$$f((x_n)) = \text{arbitrary, when } \{n \in N;\ x_n = w\} \text{ is empty}$$
$$= \min\{n \in N;\ x_n = w\}, \text{ in the opposite case}$$

we cannot have $(z,w) \in U_f$, contradicting the $\mathcal{U}$-maximality property claimed for z, and ending the argument. In other words, the "universal" versions of Theorems 6 and 7 are (when taken as independent results) equivalent formulations of Zorn's theorem.

Returning to the initial statements it is clear that, when $\mathcal{U}$ is a uniformity on X in the sense of Bourbaki ([3], Ch.2, § 2, Sect.1) the corresponding variant of Theorem 7 is easily shown to contain the maximality Valyi's result [16] proved by a direct Zorn argument. Also, we mention that the statement in Theorem 6 seems to be the most adequate one in deriving a further extension of the Nemeth's variational principle [11] (appearing as a (normed) vectorial variant of the one due to Ekeland [6]); we shall give the necessary details in a forthcoming paper.

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