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REMARK ON A PAPER OF TURINICI

This note contains three examples of discrete inequalities which can be proved simultaneously as their continuous analogies by the same theorem valid on an ordered linear space.

In his paper [8] M.Turinici proved some kind of comparison theorem which is valid on an ordered linear space. As an application of his theorem Turinici considered Gronwall-Bellman integral inequalities recently studied by Pachpatte [3]. The aim of this paper is to show that Turinici's result can be applied to Gronwall type discrete inequalities; so, it is a handy tool for obtaining estimations both in the continuous and discrete case. We give three examples of new discrete inequalities which can be proved using Turinici's theorem.

Following Turinici denote by X a linear space, X_+ a nonempty convex and pointed cone in X. The ordering $\leq$ on X induced by the cone X_+ is defined in the usual way $x \leq y$ iff $y-x \in X_+$. A mapping $T: X \to X$ is said to be monotone iff $x \leq y$ implies $Tx \leq Ty$. Also, $T: X \to X$ is called a normal mapping iff

a) T has a unique fixed point $z = z(T) \in X$,

b) if $x \in X$ is such that $x \leq Tx$ then $x \leq z$.

T h e o r e m (2.2 [8]). Let $T_i : X_+ \to X_+$, $i=1,\dots,t$ be monotone for all i, and $\sum_{i=1}^{t} T_i$ be a normal mapping. Then for every solution $x \in X_+$ of

$$(1) \qquad x \leq T_1[x + T_2[x + \dots + [x + T_t[x]]\dots]]$$

we have

(2) $x \leq T_1\left[(T_1 + T_2)\left[\ldots\left[(T_1 + \ldots + T_{t-1})[z]\right]\ldots\right]\right]$

where z is the unique fixed point of $\sum_{i=1}^{t} T_i$.

Denote by N - the set of positive integers, s - the linear space of all real sequences $x = \{x_n\}_{n=1}^{\infty}$, by s_+ - the cone of nonnegative sequences.

It is supposed, for $f: N \to R$

$$\sum_{j=n}^{n-1} f_j = 0 \quad \text{and} \quad \prod_{j=n}^{n-1} f_j = 1.$$

L e m m a 1. Let $a \in s_+$, x_0 be nonnegative constant. The operator $T: s_+ \to s_+$, defined by

(3) $$T[x] = \left\{ x_0 + \sum_{j=1}^{n-1} a_j x_j \right\}_{n=1}^{\infty}$$

is normal, with the unique fixed point

$$z = \left\{ x_0 \prod_{j=1}^{n-1} (1+a_j) \right\}_{n=1}^{\infty}.$$

P r o o f . We can easy check that z given above is a fixed point of T. Furthermore if $y \in s_+$ is such that $y \leq T[y]$ then $y \leq z$, so T is normal. Q.E.D.

Observe that T is monotone.

T h e o r e m 1. Let $a^k \in s_+$, $k=1,2,\ldots,t$ and x_0 be any nonnegative constant. If $x = \{x_n\}_{n=1}^{\infty} \in s_+$ is a solution of

(4) $$x_n \leq x_0 + \sum_{j_1=1}^{n-1} a^1_{j_1}\Bigg(x_{j_1} + \sum_{j_2=1}^{j_1-1} a^2_{j_2}\Big(x_{j_2} + \ldots + \sum_{j_t=1}^{j_{t-1}-1} a^t_{j_t} x_{j_t} \Big)\Bigg), \quad n \in N$$

then

(5) $$x_n \leq x_0 + \sum_{j_1=1}^{n-1} v^1_{j_1}\Big(x_0 + \sum_{j_2=1}^{j_1-1} v^2_{j_2}\Big(x_0 + \dots$$
$$\dots + \sum_{j_{t-1}=1}^{j_{t-2}-1} v^{t-1}_{j_{t-1}} x_0 \prod_{j_t=1}^{j_{t-1}-1}\left(1 + v^t_{j_t}\right)\dots\Big)\Big), \quad n \in N$$

where

(6) $$v^k_n = \sum_{j=1}^{k} a^j_n, \quad k = 1,2,\dots,t.$$

P r o o f . Let us take for $y = \{y_n\}_{n=1}^{\infty}$

(7) $$T_1[y] = \left\{x_0 + \sum_{j=1}^{n-1} a^1_j y_j\right\}_{n=1}^{\infty}$$

and

(8) $$T_k[y] = \left\{\sum_{j=1}^{n-1} a^k_j y_j\right\}_{n=1}^{\infty} \quad k = 2,\dots,t.$$

The operators (7) and (8) are monotone. By Lemma 1 the operator

$$\left(\sum_{k=1}^{t} T_k\right)[y] = \left\{x_0 + \sum_{j=1}^{n-1} v^t_j y_j\right\}_{n=1}^{\infty}$$

with v^t_n defined by (6) is a normal mapping of which the unique fixed point is

$$z = \left\{x_0 \prod_{j=1}^{n-1}\left(1 + v^t_j\right)\right\}_{n=1}^{\infty}.$$

Hence (5) follows from (2).

R e m a r k 1. By Remark 2.2 [8] we may obtain a weaker estimation of x, namely

$$x_n \leq x_0 \prod_{j=1}^{n-1} (1 + v_j^t), \quad n \in N.$$

Inequality (4) for the case t = 2 was considered in [1] and [2].

As the second example we consider discrete analogs of the Gronwall-Wendroff type inequalities.

Denote by s^2 the linear space of functions $x = \{x_{n,m}\}_{n=1}^{\infty}{}_{m=1}^{\infty} : N^2 \to R$ and by s_+^2 the subset of s^2 with $x : N^2 \to [0, \infty)$. Furthermore, put $\overline{1,k} = \{1,2,\ldots,k\}$. For any $a \in s^2$ we define

$$(9) \qquad \sum_{\substack{p\in\overline{1,n} \\ q\in\overline{1,m}}}{}_{r}^{2} a_{p,q} = \sum_{\substack{n\geq p_1>p_2>\ldots>p_r\geq 1 \\ m\geq q_1>q_2>\ldots>q_r\geq 1}} \prod_{k=1}^{r} a_{p_k,q_k}$$

for $1 \leq r \leq \min(n,m)$

and

$$\sum_{\substack{p\in\overline{1,n} \\ q\in\overline{1,m}}}{}_{r}^{2} a_{p,q} = 0 \quad \text{for the cases } r > \min(n,m) \text{ or } n < 1 \text{ or } m < 1.$$

For the basic properties of the above summation operator see [4].

L e m m a 2. Let $a \in s_+^2$, and x_0 be a nonnegative constant. The operator $T : s_+^2 \to s_+^2$ defined by

$$(10) \qquad T[x] = \left\{ x_0 + \sum_{j=1}^{n-1} \sum_{i=1}^{m-1} a_{j,i} x_{j,i} \right\}_{n=1}^{\infty}{}_{m=1}^{\infty}$$

is normal with the unique fixed point

$$(11)\qquad z = \left\{ x_0 \left(1 + \sum_{r=1}^{m-1} \sum_{\substack{p\in\overline{1,n-1} \\ q\in\overline{1,m-1}}}{}^{2}_{r}\, a_{p,q}\right)\right\}_{n=1}^{\infty}{}_{m=1}^{\infty}.$$

P r o o f . It is easy, by using properties of (9) to check that z defined by (11) is the solution of the equation

$$z_{n,m} = x_0 + \sum_{j=1}^{n-1}\sum_{i=1}^{m-1} a_{j,i} z_{j,i} \qquad (n,m)\in N^2.$$

Furthermore by Theorem 1 in [4] if $y \le T[y]$ then $y \le z$. Hence T is normal. Q.E.D.

Note that T is a monotone mapping on s_+^2.

T h e o r e m 2. Let $a^k \in s_+^2$, $k = 1,\dots,t$, x_0 be non-negative constant. If $x = \{x_{n,m}\}_{n=1}^{\infty}{}_{m=1}^{\infty} \in s_+^2$ is a solution of

$$(12)\qquad x_{n,m} \le x_0 + \sum_{j_1=1}^{n-1}\sum_{i_1=1}^{m-1} a^1_{j_1,i_1}\Big(x_{j_1,i_1} + \sum_{j_2=1}^{j_1-1}\sum_{i_2=1}^{i_1-1} a^2_{j_2,i_2}\Big(x_{j_2,i_2} + \dots + \sum_{j_t=1}^{j_{t-1}-1}\sum_{i_t=1}^{i_{t-1}-1} a^t_{j_t,i_t} x_{j_t,i_t}\Big)\Big), \qquad (n,m)\in N^2$$

then

$$(13)\quad x_{n,m} \leq x_0 + \sum_{j_1=1}^{n-1} \sum_{i_1=1}^{m-1} v^1_{j_1,i_1} \Big(x_0 + \sum_{j_2=1}^{j_1-1} \sum_{i_2=1}^{i_1-1} v^2_{j_2,i_2} \Big(x_0 + \dots$$

$$\dots + \sum_{j_{t-1}=1}^{j_{t-2}-1} \sum_{i_{t-1}=1}^{i_{t-2}-1} v^{t-1}_{j_{t-1},i_{t-1}} z_{j_{t-1},i_{t-1}} \dots \Big)\Big), \quad (n,m) \in N^2,$$

where

$$(14)\qquad z_{n,m} = x_0 \Big(1 + \sum_{r=1}^{m-1} \sum_{\substack{p \in \overline{1,n-1} \\ q \in \overline{1,m-1}}}{}^{2}_{r} \Big(\sum_{k=1}^{t} a^k_{p,q} \Big)\Big)$$

$$v^k_{n,m} = \sum_{j=1}^{k} a^j_{n,m} \qquad k = 1,\dots,t-1, \qquad (n,m) \in N^2.$$

Proof of this theorem follows by Lemma 2 in a similar way to that of Theorem 1.

R e m a r k 2. A weaker estimation of x, by Remark 2.2 [8] is $x_{n,m} \leq z_{n,m}$, $(n,m) \in N^2$ where $z_{n,m}$ is given by (14).

The result of Theorem 2 generalizes some cases of inequalities considered in [6] and [7].

In the third example instead operators of summation we use operators of the form

$$(15)\qquad T[x] = \Big\{ \prod_{j=1}^{n-1} a_j x_j \Big\}_{n=1}^{\infty}$$

and

$$(16)\qquad T[x] = \Big\{ a_n \prod_{j=1}^{n-1} x_j \Big\}_{n=1}^{\infty}$$

acting from s_+ into s_+ for $a \in s_+$. Observe that these operators are monotone.

L e m m a 3. The operator (16) is normal with the unique fixed point

$$z = \left\{ a_n \prod_{j=1}^{n-1} (a_j)^{2^{n-1-j}} \right\}_{n=1}^{\infty} . \tag{17}$$

P r o o f . We prove that z given by (17) is the unique fixed point of (16). Indeed

$$T[z] = \left\{ a_n \prod_{j=1}^{n-1} z_j \right\}_{n=1}^{\infty} = \left\{ a_n \prod_{j=1}^{n-1} a_j \prod_{i=1}^{j-1} (a_i)^{2^{j-1-i}} \right\}_{n=1}^{\infty} =$$

$$= \left\{ a_n \prod_{j=1}^{n-1} (a_j)^{\left(1 + \sum_{i=0}^{n-2-j} 2^i\right)} \right\}_{n=1}^{\infty} = \left\{ a_n \prod_{j=1}^{n-1} (a_j)^{2^{n-1-j}} \right\}_{n=1}^{\infty} = z.$$

Let $y = T[y]$ and $y \neq z$. We get

$$y_1 = a_1 \prod_{j=1}^{0} y_j = a_1 = z_1.$$

Suppose $y_k = z_k$ for $k = 1,2,\dots,m$. Hence

$$y_{m+1} = a_{m+1} \prod_{j=1}^{m} y_j = a_{m+1} \prod_{j=1}^{m} z_j = z_{m+1}.$$

So we get $y_m = z_m$ for all $m \in N$. Therefore z is the unique fixed point of T. Let now $y \in s_+$ be such that $y \leq T[y]$. Hence

$$y_1 \leq a_1 \prod_{j=1}^{0} y_j = a_1 = z_1.$$

Supposing $y_k \leq z_k$ for $k = 1,2,\dots,m$ in a similar way we obtain $y_{m+1} \leq z_{m+1}$. From this it follows $y \leq z$. Q.E.D.

Theorem 3. Let $a^k \in s_+$, $k = 1,2,\dots,t$. If $x \in s_+$ is a solution of

$$x_n \leq \prod_{j_1=1}^{n-1} a^1_{j_1}\Big(x_{j_1} + \prod_{j_2=1}^{j_1-1} a^2_{j_2}\Big(x_{j_2} + \dots + \prod_{j_t=1}^{j_{t-1}-1} a^t_{j_t} x_{j_t} \dots\Big)\Big), \quad n \in N \tag{18}$$

then

$$x_n \leq \prod_{j_1=1}^{n-1} a^1_{j_1}\Big(v^2_{j_1} \prod_{j_2=1}^{j_1-1} v^3_{j_2}\Big(\dots \dots v^t_{j_{t-1}} \prod_{j_t=1}^{j_{t-1}-1} \left(v^t_{j_t}\right)^{2^{-1-j_t+j_{t-1}}} \dots\Big)\Big), \quad n \in N \tag{19}$$

where

$$v^k_n = \sum_{s=1}^{k} \prod_{j=1}^{n-1} a^s_j \qquad k = 2,\dots,t.$$

Proof. Let us take

$$T_k[x] = \left\{\prod_{j=1}^{n-1} a^k_j x_j\right\}_{n=1}^{\infty}$$

then

$$\Big(\sum_{s=1}^{k} T_s\Big)[x] = \left\{v^k_n \prod_{j=1}^{n-1} x_j\right\}_{n=1}^{\infty}.$$

The operator

$$\Big(\sum_{s=1}^{t} T_s\Big) \text{ is of the form (16).}$$

Hence by Lemma 3 and (2) we obtain (19).

R e m a r k 3. For the case $t = 2$, (19) is

$$x_n \leqslant \prod_{j_1=1}^{n-1} a^1_{j_1} \left(v^2_{j_1} \prod_{j_2=1}^{j_1-1} \left(v^2_{j_2}\right)^{2^{-1-j_2+j_1}} \right), \quad n \in N.$$

By Remark 2.2 [8] we may get another estimation of x, namely

$$x_n \leqslant v^t_n \prod_{j=1}^{n-1} \left(v^t_j\right)^{n-1-j}, \quad n \in N.$$

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