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TOPOLOGIES RELATED TO SETS HAVING THE BAIRE PROPERTY

S. Scheinberg in [7] has constructed three topologies for the reals such that the Borel sets generated by these topologies are exactly the Lebesgue measurable sets. Here we shall do the same for sets having the Baire property. Our general construction of a topology will be based upon a lower density having some additional properties, this will be the content of the first part. In the second part we shall give three examples of lower density leading to three different topologies which can be considered as category analogues of topologies described in [7]. It should be mentioned that Scheinberg did not use the language of lower density, but his results can be easily translated. We shall examine basic properties of all three topologies, some of them, which depend only on the fact that the topology is described by lower density, are presented in part 1, the rest depending on the form of the lower density - in part 2.

In the sequel R will denote the real line, $\mathcal{B}$ - the family of subsets of R having the Baire property, $\mathcal{J}$ - the σ-ideal of sets of the first category, N - the set of natural numbers. We shall say that sets $A, B \in \mathcal{B}$ are equivalent ($A \sim B$) if and only if $A \Delta B \in \mathcal{J}$. Except where a topology $\mathcal{T}$ is specifically mentioned, all topological notions are with respect to the natural topology on R. So if $\mathcal{T}$ is some topology, then the notations $\mathcal{T}$-open, $\mathcal{T}$-Borel, $\mathcal{T}$-Int, $\mathcal{T}$-Cl and so on are self explaining. Recall that G is regular open if and only if $G = \mathrm{Int}(\mathrm{Cl}\, G)$.

1. We shall say (after [2] and [4]) that an operation $\Phi : \mathcal{B} \to 2^R$ is a lower density if and only if it fulfils the following conditions:

1. for each $A \in \mathcal{B}$, $\Phi(A) \sim A$,
2. if $A \sim B$, then $\Phi(A) = \Phi(B)$,
3. $\Phi(\emptyset) = \emptyset$, $\Phi(R) = R$,
4. $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$.

Observe that by virtue of 1. we have $\Phi(A) \in \mathcal{B}$ for $A \in \mathcal{B}$, so in fact $\Phi : \mathcal{B} \to \mathcal{B}$. Also in virtue of 4. if $A, B \in \mathcal{B}$ and $A \subset B$ then $\Phi(A) \subset \Phi(B)$. Moreover $\Phi(\Phi(A)) = \Phi(A)$ for $A \in \mathcal{B}$ by virtue of 1. and 2.

Suppose that Φ fulfils the following additional condition:

5. if $A = (G \triangle P)$, where G is regular open and $P \in \mathcal{J}$, then

$$G \subset \Phi(A) \subset Cl(G).$$

From 5. it immediately follows that if for some $A \in \mathcal{B}$, $\Phi(A)$ is nonempty, then it is of the second category. An operation Φ fulfilling 1. - 5. will be called a category lower density.

Let $\mathcal{T} = \{\Phi(A) - P : A \in \mathcal{B}, P \in \mathcal{J}\}$. (Compare [2], chapter V and [4], chapter 22).

T h e o r e m 1.1. $\mathcal{T}$ is a topology on the real line strictly stronger than the natural topology $\mathcal{T} \subset \mathcal{B}$.

P r o o f . To prove the fact that $\mathcal{T}$ is a topology $\mathcal{T} \subset \mathcal{B}$ it is sufficient to show that $\{A_s : s \in S\} \subset \mathcal{T}$ and $A = \bigcup_{s \in S} A_s \in \mathcal{T}$. If M denotes the Baire kernel of A, for each $s \in S$ there is $N_s \sim \emptyset$ such that $A_s \subset M \cup N_s$. Hence $\Phi(A_s) \subset \Phi(M)$ so that $M \subset A \subset \bigcup_{s \in S} \Phi(A_s) \subset \Phi(M)$. By 1., $M \sim \Phi(M)$ and $A \in \mathcal{B}$. Since, for each $s \in S$, $A_s \subset \Phi(A_s)$ we know that $A = \bigcup_{s \in S} A_s \subset \bigcup_{s \in S} \Phi(A_s) \subset \Phi(A)$. Therefore $A = \Phi(A) \setminus (\Phi(A) \setminus A)$, $\Phi(A) \setminus A \in \mathcal{J}$ and $A \in \mathcal{T}$.

To prove that $\mathcal{T}$ is stronger than the natural topology take an open set G and observe that $G = Int(Cl(G)) \setminus (Int(Cl(G)) \setminus G)$, $Int(Cl(G)) \setminus G \in \mathcal{J}$ and $Int(Cl(G))$ is regular open. Then by 5., $G \subset Int(Cl(G)) \subset \Phi(G)$, $\Phi(G) - Int(Cl(G)) \in \mathcal{J}$ (since $Cl(G) - G \in \mathcal{J}$) and

$G = \Phi(G)-(\Phi(G)-G) = \Phi(G)-((\Phi(G)-\mathrm{Int}(\mathrm{Cl}(G))) \cup (\mathrm{Int}(\mathrm{Cl}(G))-G))$.

Hence G is of the required form and $G \in \mathcal{T}$. The set of irrationals obviously belongs to topology $\mathcal{T}$, so $\mathcal{T}$ is strictly stronger than the natural topology.

T h e o r e m 1.2. $\mathcal{T} = \{A \in \mathcal{B} : A \subset \Phi(A)\}$.

P r o o f . The proof is essentially the same as the proof of Remark 2 in [5].

Here is another description of $\mathcal{T}$: We shall say that A is a $\mathcal{T}$-neighbourhood of x if and only if $x \in A$ and there exists a set $B \in \mathcal{B}$ such that $B \subset A$ and $x \in \Phi(B)$ (we do not assume that $A \in \mathcal{B}$).

T h e o r e m 1.3. $A \in \mathcal{T}$ if and only if A is a $\mathcal{T}$-neighbourhood of each of its points.

P r o o f . If $A \in \mathcal{T}$, then from Theorem 1.2 it immediately follows that A is a $\mathcal{T}$-neighbourhood of each of its points. Suppose now that A is a $\mathcal{T}$-neighbourhood of each of its points. It suffices to prove that A has the Baire property, since then obviously $A \subset \Phi(A)$. Let F and G be sets having the Baire property and such that $F \subset A \subset G$ and if $P_1, P_2 \in \mathcal{B}$, $P_1 \subset A - F$, $P_2 \subset G - A$, then $P_1, P_2 \in \mathcal{J}$ (such sets always exist; see for example [11]). We show that $G - F \in \mathcal{J}$. Take $E = (G - F) \cap \Phi(G - F)$. Then $\Phi(E) = \Phi(G - F) \cap \Phi(\Phi(G - F)) = \Phi(G - F)$ by 1., 2. and 4., so $E \subset \Phi(E)$. For each $x_0 \in A$, we have A is a $\mathcal{T}$-neighbourhood of x_0 and there exists a set $B \subset A$, $B \in \mathcal{B}$ such that $x_0 \in \Phi(B)$. Because $E \cap B \subset E \cap A \subset (G - F) \cap A = A - F$, we have $E \cap B$ is of the first category. Hence, $\Phi(E) \cap \Phi(B) = \Phi(E \cap B) = \emptyset$, so that $x_0 \notin \Phi(E)$. Hence, $\Phi(E) \cap A = \emptyset$, so that $E \cap A = \emptyset$. Then $E \subset G - A$, so that E is of the first category (and hence empty). Hence, $G - F \in \mathcal{J}$ and A has the Baire property.

T h e o r e m 1.4. The $\mathcal{T}$-Borel sets are precisely the sets having the Baire property.

P r o o f . Since each $\mathcal{T}$-open set has the Baire property, all the $\mathcal{T}$-Borel sets have the Baire property. Conversely, if A has the Baire property, then $A = (G - P_1) \cup P_2$, where G is open,

$P_1, P_2 \in \mathcal{J}$. Observe that P_2 is $\mathcal{T}$-closed, since $R - P_2 = \Phi(R) - P_2 \in \mathcal{J}$ by 3. Simultaneously, $A = (G \cap (R - P_1)) \cup P_2 = (G \cap (\Phi(R) - P_1)) \cup P_2$ again by 3., so A is the union of a $\mathcal{T}$-open and a $\mathcal{T}$-closed set, hence it is $\mathcal{T}$-Borel.

Theorem 1.5. Each $\mathcal{T}$-Borel set is a $\mathcal{T}$-$F_{\sigma\delta}$ set. There exists a set $B \in \mathcal{B}$ which is not a $\mathcal{T}$-G_δ set.

Proof. If $A \in \mathcal{B}$, then $A = E \cup P$, where E is a G_δ set and $P \in \mathcal{J}$. Hence E is $F_{\sigma\delta}$. Because $\mathcal{T}$ is stronger than the natural topology, E is also a $\mathcal{T}$-$F_{\sigma\delta}$ set and P is $\mathcal{T}$-closed, so it is also $\mathcal{T}$-$F_{\sigma\delta}$. This ends the proof of the first part.

Now let B be an arbitrary countable dense set. Suppose that there exists a sequence $\{U_n\}_{n\in N}$ of $\mathcal{T}$-open sets such that $B = \bigcap_{n=1}^{\infty} U_n$. Then for each $n \in N$ we have $U_n = \Phi(A_n) - P_n$, where A_n is a set having the Baire property and $P_n \in \mathcal{J}$. Since $\Phi(A_n) \supset B$ for each n, then from 5. it follows that $\Phi(A_n)$ is a residual set on the real line. But then $\bigcap_{n=1}^{\infty} (\Phi(A_n - P_n))$ is still residual, since B is not - a contradiction.

So B is not $\mathcal{T}$-G_δ.

Recall that in the measure case (see again [7]) each Lebesgue measurable set is "$\mathcal{T}$-G_δ".

Theorem 1.6. $\mathcal{T}$ is not regular.

Proof. Let Q be the set of all rational numbers. Then $A = Q - \{0\}$ is $\mathcal{T}$-closed. Every $\mathcal{T}$-neighbourhood of A is residual, so 0 and A cannot be separated.

Theorem 1.7. A set $A \subset R$ is $\mathcal{T}$-nowhere dense ($\mathcal{T}$- of the first category) if and only if $A \in \mathcal{J}$. Every $\mathcal{T}$-nowhere dense set is $\mathcal{T}$-closed.

Proof. See the proof of th. 22.6 in [4]. (Compare also [11]).

Theorem 1.8. A set $A \subset R$ has $\mathcal{T}$-property of Baire if and only if $A \in \mathcal{B}$.

Proof. See the proof of th. 22.7 in [4]. (Compare again [11]).

Theorem 1.9. A set $G \in \mathcal{B}$ is $\mathcal{T}$-regular open if and only if $G = \Phi(A)$ for some $A \in \mathcal{B}$.

P r o o f . See the proof of th. 22.8 in [4].

T h e o r e m 1.10. A set $A \subset R$ is $\mathcal{J}$-closed and $\mathcal{J}$-discrete (i.e. a subspace topology in A is discrete) if and only if $A \in \mathcal{J}$.

P r o o f . If $A \in \mathcal{J}$, then A is $\mathcal{J}$-closed. If $B \subset A$, then $B \in \mathcal{J}$ so that B is also $\mathcal{J}$-closed. In other words, every subset of A is $\mathcal{J}$-closed. Equivalently, A is $\mathcal{J}$-closed and $\mathcal{J}$-discrete.

Now suppose that $A \subset R$ is $\mathcal{J}$-closed and $\mathcal{J}$-discrete. That is, suppose that every subset of A is $\mathcal{J}$-closed. From the assumption it follows that R - A is $\mathcal{J}$-open, so by virtue of Theorem 1.2 $R - A \subset \Phi(R - A)$. Let $B = R - \Phi(R - A)$. Clearly $B \subset A$. We shall show that $B = \emptyset$. Assume otherwise that $B \neq \emptyset$ and choose $x_0 \in B$ and let $C = B - \{x_0\}$. Then $C \sim B$, so that $\Phi(R - C) = \Phi(R - B) = R - B$, the last equality follows from the definition of B. But $C \subset A$, hence C is $\mathcal{J}$-closed, so R - C is $\mathcal{J}$-open and $R - C \subset \Phi(R - C) = R - B$. Hence $B \subset C$ which gives a contradiction. Therefore $B = \emptyset$ and $\Phi(R - A) = R$. Then $R - A \sim \Phi(R - A) = R$, so that $A \in \mathcal{J}$.

T h e o r e m 1.11. $A \subset R$ is a regular subspace of $(R, \mathcal{J})$ if and only if A is of the first category.

P r o o f . Suppose that A is of the first category. Then A is a $\mathcal{J}$-discrete subspace, so all separation axioms are fulfilled.

Suppose now A is not of the first category. We shall show that A is not a regular subspace of $(R, \mathcal{J})$. From the fact that $A \notin \mathcal{J}$ it follows that there exists an interval (a,b) such that for each subinterval (c,d) the intersection $A \cap (c,d)$ is not of the first category. Take $x_0 \in A$ and a countable set $C \subset A$ such that $Cl(C) \supset (a,b)$, where (a,b) is the interval described above, and $x_0 \notin C$. C is $\mathcal{J}$-closed in R, so in A too. Let U_{x_0} and U_C be sets which are $\mathcal{J}$-open in A where $x_0 \in U_{x_0}$ and $C \subset U_C$. Then $U_{x_0} = A \cap \tilde{U}_{x_0}$ and $U_C = A \cap \tilde{U}_C$, where $\tilde{U}_{x_0}, \tilde{U}_C \in \mathcal{J}$. From the fact that $Cl(C) \supset (a,b)$ and from 5. it follows that $\tilde{U}_C$ is residual in (a,b). Also there exists $(c,d) \subset (a,b)$ in which $\tilde{U}_{x_0}$

is residual. Then $U_{x_0} \cap U_C = A \cap \tilde{U}_{x_0} \cap \tilde{U}_C \supset (c,d) \cap A \cap \tilde{U}_{x_0} \cap \tilde{U}_C \neq \emptyset$, because $\tilde{U}_{x_0} \cap \tilde{U}_C$ is residual in (c,d) and $A \cap (c,d)$ is not of the first category. Hence A is not regular.

T h e o r e m 1.12. $A \subset R$ is a normal subspace of $(R,\mathcal{T})$ if and only if A is of the first category.

P r o o f . Obviously follows from Theorem 1.11.

Comparing this result with [8] and [9] we see that the behaviour of our topology is different from that of the density topology, which is even completely regular and possesses a much wider variety of normal subspaces. Our topology, as it is stronger than the natural topology, is obviously Hausdorff.

Now let us turn for the moment to real functions which transform $(R,\mathcal{T})$ into R equipped with the natural topology.

T h e o r e m 1.13. A function $f : R \to R$ has the Baire property if and only if there exists a set $P \in \mathcal{J}$ such that f is continuous with respect to $\mathcal{T}$ on $R - P$.

P r o o f . See [2], Proposition 1, section 1, chapter V.

2. Now we shall present three different category lower densities.

If $A \in \mathcal{B}$, then (see for example [4], th. 4.6) there exists exactly one representation of A in the form $A = G \Delta P$, where G is regular open and P is of the first category. Put $\Phi_1(A) = G$.

T h e o r e m 2.1. Φ_1 is a category lower density.

P r o o f . 1. and 3. are straightforward, 2. follows from the uniqueness of representation mentioned above, 4. is a consequence of the fact that the intersection of two regular open sets is a regular open set (compare for example [4] th. 4.7) and 5. is obviously fulfilled.

The topology generated by Φ_1 will be labelled $\mathcal{T}_1$.

T h e o r e m 2.2. $\mathcal{T}_1 = \{G - P : G \text{ is open and } P \in \mathcal{J}\}$.

P r o o f . Let $\mathcal{T}^* = \{G - P : G \text{ is open and } P \in \mathcal{J}\}$. Obviously $\mathcal{T}_1 \subset \mathcal{T}^*$, since each regular open set is open. Let now $B \in \mathcal{T}^*$; i.e. $B = A - P$, where A is open and $P \in \mathcal{J}$. Then

$B = Int(Cl\ A) - ((Int(Cl\ A) - A) \cup P) = \Phi_1(A) - ((\Phi_1(A) - A) \cup P)$. Here $\Phi_1(A) - A \in \mathcal{J}$ from 1., so $(\Phi_1(A) - A) \cup P \in \mathcal{J}$ and B is of the required form. The equality is proved.

The topology $\mathcal{T}^*$ is called a Hashimoto topology (see [1]).

Theorem 2.3. The topology $\mathcal{T}_1$ is connected.

Proof. If A is $\mathcal{T}_1$-open and $\mathcal{T}_1$-closed, then, according to theorem 2.2, $A = G - P_1 = F \cup P_2$, where G is open, F is closed and $P_1, P_2 \in \mathcal{J}$. We can (and shall) assume that $P_1 \subset G$. Hence $G = F \cup P_1 \cup P_2$ and $G - F \subset P_1 \cup P_2$. Since $G - F$ is open and $P_1 \cup P_2$ is of the first category, we have $G - F = \emptyset$, so $G \subset F$. But from the equality $G - P_1 = F \cup P_2$ it follows that $G \supset F$, so finally $G = F$. Since the natural topology is connected, we have either $G = F = \emptyset$ or $G = F = R$. In the first case $A = \emptyset - P_1 = \emptyset$, in the second case $A = R \cup P_2 = R$, so $\mathcal{T}_1$ is connected.

Theorem 2.4. A set $A \subset R$ is $\mathcal{T}_1$-regular open if and only if A is regular open.

Proof. This is an immediate consequence of Theorem 1.9 and the definition of Φ_1.

Theorem 2.5. A function $f : R \to R$ is $\mathcal{T}_1$-continuous if and only if it is continuous.

Proof. See [10], th. 34.1.

Observe that Theorem 2.3 is a consequence of Theorem 2.5, but we have given a complete proof because of its simplicity.

To describe the next category lower density, which was presented for the first time in [5] and [6] let us introduce the following denotations and definitions: If $A \subset R$ and $x_0 \in R$, then $x_0 \cdot A = \{x_0 \cdot x : x \in A\}$ and $A - x_0 = \{x - x_0 : x \in A\}$, where χ_A will mean the characteristic function of the set A. We shall say that 0 is an $\mathcal{J}$-density point of a set $A \in \mathcal{B}$ if and only if the sequence $\{\chi_{(n \cdot A) \cap [-1,1]}\}_{n \in N}$ converges to 1 with respect to the σ-ideal $\mathcal{J}$ (it means that for every increasing sequence $\{n_m\}_{m \in N}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p \in N}$ such that $\{\chi_{(n_{m_p} \cdot A) \cap [-1,1]}\}_{p \in N}$ converges to 1 except on a set belonging to $\mathcal{J}$. Further, x_0 is an $\mathcal{J}$-density point of

$A \in \mathcal{B}$ if and only if 0 is an $\mathcal{J}$-density point of the set $A - x_0$. If x_0 is an $\mathcal{J}$-density point of $R - A$ we say that x_0 is an $\mathcal{J}$-dispersion point of A. Let $\Phi_2(A) = \{x \in R : x$ is an $\mathcal{J}$-density point of $A\}$ for $A \in \mathcal{B}$.

Theorem 2.6. Φ_2 is a category lower density.

Proof. See [5] Theorem 2 for 1.-4. The condition 5. follows from the definition of Φ_2.

The topology generated by Φ_2 will be called the $\mathcal{J}$-density topology and labelled $\mathcal{T}_2$ (in [5] and [6] - $\mathcal{T}_{\mathcal{J}}$).

Theorem 2.7. The topology $\mathcal{T}_2$ is connected.

Proof. See [5], Corollary 2.

Theorem 2.8. If a set $A \subset R$ is $\mathcal{T}_2$-regular open, then $A \in F_{\sigma\delta}$.

Proof. From 1.9 it follows that $A = \Phi_2(A)$. We shall prove that for each $A \in \mathcal{B}$, $\Phi_2(A) \in F_{\sigma\delta}$. Since $A = F \Delta P$, where F is closed and $P \in \mathcal{J}$, we have $\Phi_2(A) = \Phi_2(F)$. In [3] it is proved that $x \in \Phi_2(F)$ if and only if for each $n \in N$ there exist $p \in N$ and $k \in N$ such that for each $h \in \left(0, \frac{1}{p}\right)$ and for each $i \in \{1,\dots,n\}$ there exist two natural numbers $j_r, j_l \in \{1,\dots,k\}$ such that the intervals

$$d^+_{j_r}(x) = \left[x + \frac{(i-1)\cdot k + j_r - 1}{n\cdot k}\cdot h,\ x + \frac{(i-1)\cdot k + j_r}{n\cdot k}\cdot h\right] \subset F$$

and

$$d^-_{j_l}(x) = \left[x - \frac{(i-1)\cdot k + j_l}{n\cdot k}\cdot h,\ x - \frac{(i-1)\cdot k + j_l - 1}{n\cdot k}\cdot h\right] \subset F.$$

Let

$$A^+_{n,k,h,i,j} = \{x : d^+_j(x) \subset F\},$$

$$A^-_{n,k,h,i,j} = \{x : d^-_j(x) \subset F\}.$$

For each n,k,h,i and j both sets $A^+_{n,k,h,i,j}$ and $A^-_{n,k,h,i,j}$ are closed, since F is closed. But

$$\Phi(F) = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcap_{h \in \left(0,\frac{1}{p}\right)} \bigcap_{i=1}^{n} \bigcup_{\substack{j_r=1\\ j_l=1}}^{k} \left(A^+_{n,k,h,i,j_r} \cap A^-_{n,k,h,i,j_l}\right)$$

and

$$\Phi(F) \in F_{\sigma\delta}.$$

T h e o r e m 2.9. If a function $f : R \to R$ is $\mathcal{J}_2$-continuous, then f is Baire one and has the Darboux property.

P r o o f . See [5], Theorem 8.

Now, we proceed to the third category lower density. Let $\mathcal{A}$ be the collection of all sets $A \in \mathcal{B}$ such that 0 is an $\mathcal{J}$-density point of A. $\mathcal{A}$ is a filter in the family $\mathcal{B}$. Extend A to an ultrafilter $\mathcal{U}$ in $\mathcal{B}$. Then $\mathcal{U}$ has the following properties:

(0) $\emptyset \notin \mathcal{U}$.

(1) if $A \in \mathcal{B}$, then either A or $R - A$ belongs to $\mathcal{U}$.

(2) if A and B belong to $\mathcal{U}$, then $A \cap B \in \mathcal{U}$.

(3) if $A \in \mathcal{U}$ and $B \in \mathcal{B}$ and $B \supset A$, then $B \in \mathcal{U}$.

(4) if $A \in \mathcal{B}$ and 0 is an $\mathcal{J}$-density point of A, then $A \in \mathcal{U}$; if 0 is an $\mathcal{J}$-dispersion point of A, then $A \notin \mathcal{U}$.

(5) if $A \in \mathcal{B}$ and A meets every member of $\mathcal{U}$, then $A \in \mathcal{U}$.

(6) if $A \in \mathcal{U}$ and $B \sim A$, then $B \in \mathcal{U}$.

(0) - (4) are obvious. (5) follows from the fact that if $A \notin \mathcal{U}$, then by (1) $R - A \in \mathcal{U}$, but A cannot meet $R - A$. To prove (6) suppose that $B \notin \mathcal{U}$. Then $R - B \in \mathcal{U}$ by (1) and $A \cap (R - B) \in \mathcal{U}$ by (2). But $A \cap (R - B) \in \mathcal{J}$, so 0 is an $\mathcal{J}$-dispersion point of $A \cap (R - B)$ and $A \cap (R - B) \in \mathcal{U}$ by (4) - a contradiction.

Let us call $\mathcal{U} = \mathcal{U}_0$. Then $\mathcal{U}_x$ is obtained by translation of $\mathcal{U}_0$ to x ($\mathcal{U}_x = \mathcal{U}_0 + x = \{A + x : A \in \mathcal{U}_0\}$). Let $\Phi_3(A) = \{x \in R : A \in \mathcal{U}_x\}$ for $A \in \mathcal{B}$.

T h e o r e m 2.10. Φ_3 is a category lower density.

P r o o f . 1. and 5. If $A \in \mathcal{B}$, then there exist regular open set G and $P_1, P_2 \in \mathcal{J}$ such that $A = (G - P_1) \cup P_2$. Observe first that $G \subset \Phi_3(A)$. Indeed, if $x \in G$, then $G \in \mathcal{U}_x$, so from (6), $A \in \mathcal{U}_x$ and $x \in \Phi_3(A)$. Now we shall prove that $\Phi_3(A) \subset Cl(G)$. Indeed, if $x \notin Cl(G)$, then there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \cap A = (x - \varepsilon, x + \varepsilon) \cap P_2 \in \mathcal{J}$ and $(x - \varepsilon, x + \varepsilon) \cap A \notin \mathcal{U}_x$ by (4). Hence $x \notin \Phi_3(A)$. So $\Phi_3(A) \sim G \sim A$.

2. If $x \in \Phi_3(A)$, then $A \in \mathcal{U}_x$ by definition. Hence $B \in \mathcal{U}_x$ by (6), so $x \in \Phi_3(B)$ and $\Phi_3(A) \subset \Phi_3(B)$. The inverse inclusion can be proved in the same way.

3. Obvious.

4. Suppose that $x \in \Phi_3(A \cap B)$. Then $A \cap B \in \mathcal{U}_x$. By (3) $A \in \mathcal{U}_x$ and $B \in \mathcal{U}_x$, so $x \in \Phi_3(A) \cap \Phi_3(B)$. Suppose now that $x \in \Phi_3(A) \cap \Phi_3(B)$. Then $A \in \mathcal{U}_x$ and $B \in \mathcal{U}_x$, hence $A \cap B \in \mathcal{U}_x$ by (2) and $x \in \Phi_3(A \cap B)$. Observe also that $\Phi_3(A \cup B) = \Phi_3(A) \cup \Phi_3(B)$. Indeed, from 4. it follows that $\Phi_3(A) \subset \Phi_3(A \cup B)$ and $\Phi_3(B) \subset \Phi_3(A \cup B)$, hence $\Phi_3(A) \cup \Phi_3(B) \subset \Phi_3(A \cup B)$. Suppose now that $x \in \Phi_3(A \cup B)$. Then by the definition $A \cup B \in \mathcal{U}_{x_0}$. If $R - A \in \mathcal{U}_{x_0}$ and $R - B \in \mathcal{U}_{x_0}$, then $R - (A \cup B) = (R - A) \cap (R - B) \in \mathcal{U}_{x_0}$ - a contradiction. Hence $A \in \mathcal{U}_{x_0}$ or $B \in \mathcal{U}_{x_0}$, which means that $x_0 \in \Phi_3(A) \cup \Phi_3(B)$. So Φ_3 is a lifting. For the comments concerning the existence of Borel lifting see [2], p.131.

We have defined $\mathcal{U}_x$ as $\mathcal{U}_0 + x$. But in the above proof we have never used this property. So it is also possible to construct an ultrafilter separately for each x and to obtain another lower density which is not invariant with respect to translations. It is worth mentioning that all lower densities Φ_1, Φ_2, Φ_3 are invariant, i.e. $\Phi_i(A + x) = \Phi_i(A) + x$.

Let $\mathcal{T}_3$ be a topology determined by the lower density Φ_3.

T h e o r e m 2.11. $\mathcal{T}_3$ is totally disconnected.

P r o o f . From (5) it follows that if x is a $\mathcal{T}_3$-cluster point of $A \in \mathcal{B}$, then $x \in \mathcal{T}_3\text{-Int}(\{x\} \cup A)$. Therefore $\mathcal{T}_3$-closure of any $\mathcal{T}_3$-open set is again $\mathcal{T}_3$-open.

T h e o r e m 2.12. If f is a function having the Baire property then there is a unique function $\bar{f}$ such that $f = \bar{f}$ $\mathcal{J}$- a.e. (it is $\{x : \bar{f}(x) \neq f(x)\} \in \mathcal{J}$) and $\bar{f}$ is continunuous as a function from $(R, \mathcal{T}_3)$ to R equipped with a natural topology.

P r o o f . Uniqueness. Since $\mathcal{T}_3$ consists of sets having the Baire property, every continuous function from $(R, \mathcal{T}_3)$ to (R,natural) has the Baire property. Let g and h be two such functions. If $g - h = 0$ $\mathcal{J}$-a.e., then $g - h \equiv 0$, since the complement of the set of the first category is everywhere $\mathcal{T}_3$-dense.

E x i s t e n c e . From the fact that Φ_3 is a lifting it follows immediately that $\Phi_3(R \setminus A) = R \setminus \Phi_3(A)$ for each $A \in \mathcal{B}$.

Now, let $\bar{f}(x) = \sup \{r : x \notin \Phi_3([f^{-1}((-\infty, r))])\}$ for each $x \in R$, where f is a function having the Baire property and $[A]$ denotes the element $\mathcal{B}/\mathcal{J}$ corresponding to $A \in \mathcal{B}$. It is easy to show that

$$\bar{f}^{-1}((-\infty, r)) = \bigcup_{\substack{s<r \\ s\text{-rational}}} \Phi_3([f^{-1}((-\infty, s))])$$

and

$$\bar{f}^{-1}((r, +\infty)) = \bigcup_{\substack{s<r \\ s\text{-rational}}} R \setminus \Phi_3([f^{-1}((-\infty, s]))]).$$

Then, by the first observation $\bar{f}^{-1}((-\infty, r)) \in \mathcal{T}_3$ and $\bar{f}((-r, \infty)) \in \mathcal{T}_3$ for each $r \in R$. Since $\bar{f}^{-1}((-\infty, r)) \setminus f^{-1}((-\infty, r)) \in \mathcal{J}$ and $f^{-1}((-\infty, r)) \setminus \bar{f}^{-1}((-\infty, r)) \in \mathcal{J}$ thus f and $\bar{f}$ differ only on a set belonging to $\mathcal{J}$.

T h e o r e m 2.13. $\mathcal{T}_1 \subsetneq \mathcal{T}_2 \subsetneq \mathcal{T}_3$.

P r o o f . Since for each $A \in \mathcal{B}$, $\Phi_1(A) \subset \Phi_2(A) \subset \Phi_3(A)$, then from theorem 1.2 it follows immediately that $\mathcal{T}_1 \subset \mathcal{T}_2 \subset \mathcal{T}_3$. To show that $\mathcal{T}_1 \neq \mathcal{T}_2$ observe that if the sequence of intervals $\{(a_n, b_n)\}_{n \in N}$ is such that $\{b_n\}_{n \in N}$ tends decreasingly to 0, $a_{n+1} < b_{n+1} < a_n$ for each $n \in N$ and 0 is an $\mathcal{J}$-density point of the set $A = (-\infty, 0] \cup \bigcup_{n=1}^{\infty} (b_{n+1}, a_n)$ (for the existence of such a sequence see [6], Theorem 1), then $A \in \mathcal{T}_2 - \mathcal{T}_1$. Then $\mathcal{T}_2 \neq \mathcal{T}_3$ because $\mathcal{T}_2$ is connected while $\mathcal{T}_3$ is totally disconnected.

F.D. Tall has proved (see [8], Theorem 3.11) that there is no regular topology $\mathcal{T}$ on the real line such that (a) the algebra of $\mathcal{T}$-regular open sets is isomorphic to the reduced Borel algebra of euclidean Borel sets modulo first category sets, and (b) if G is open and $P \in \mathcal{J}$, then $G - P \in \mathcal{T}$. The three topologies of this paper are not regular (th. 1.6), indeed, the only regular subspaces are the first category subspaces (th. 1.11), nevertheless the three topologies satisfy conditions (a) and (b).

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