DEMONSTRATIO MATHEMATICA
Vol. XXII No 1 1989

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GEOMETRIC CORRELATION LATTICES

In projective geometry the dualities play an essential role. They can often be described by semibilinear forms of vector spaces coordinatizing geometries. These classical results come into a new context being connected to lattice theory. Menger and Birkhoff introduced the concept of geometric lattices which generalizes projective geometries and includes Boolean lattices.

Of course the dualities themselves have also a lattice theoretic counterpart. Some special cases like the orthocomplementation of distributive lattices are well known under the name of Boolean algebras. Also other dualities were introduced to distributive lattices and studied under their geometric name like polarities and correlations [6], [7].

In this paper the main topic are geometric lattices of finite height which admit a correlation as an additional operation. It is an important observation that a geometric lattice of finite heights becomes modular by a duality. An analogue of Birkhoff's theorem on the decomposition of modular geometric lattices as a product of projective geometries can be shown for geometric correlation lattices of finite height. The decisive instrument of the decomposition are σ-central elements which are connected to the correlation σ. The description of the finite simple geometric correlation lattice can further

* This author gratefully acknowledges the support by the DAAD (German Academic Exchange Authorities).

be simplified by the knowledge of the center of these lattices. We present three types of simple correlation lattices using the results in [6].

Furthermore we have included several examples of correlation lattices some of which illustrate the theory and others which demonstrate some of the many open problems.

We like to thank very much M. Szymańska for her helpful suggestions.

1. Some facts of correlation lattices

Definition 1.1. Let $\underset{\sim}{L} = (L;\wedge,\vee)$ be a lattice. An antiordermorphism τ of $\underset{\sim}{L}$ is a bijective map $\tau : L \to L$ which reverses the order, i.e. $a \leqslant b$ iff it follows $\tau(b) \leqslant \tau(a)$. A correlation σ of $\underset{\sim}{L}$ is a bijective map $\sigma : L \to L$ such that $\sigma(x \vee y) = \sigma(x) \wedge \sigma(y)$ and $\sigma(x \wedge y) = \sigma(x) \vee \sigma(y)$ hold. A correlation σ of L is called a polarity if $\sigma^2(x) = x$.

Proposition 1.2. Every antiordermorphism of a lattice L is a correlation.

Proof. If σ is an antiordermorphism of L then from $x,y \leqslant x \vee y$ it follows $\sigma(x \vee y) \leqslant \sigma(x) \wedge \sigma(y)$ and similarly $\sigma(x) \vee \sigma(y) \leqslant \sigma(x \wedge y)$. We have $\sigma^{-1}(\sigma(x \vee y)) \geqslant \sigma^{-1}(\sigma(x) \wedge \sigma(y)) \geqslant$ $\geqslant \sigma^{-1}\sigma(x) \vee \sigma^{-1}\sigma(y)$ and hence $\sigma^{-1}(\sigma(x) \wedge \sigma(y)) = x \vee y$ or $\sigma(x) \wedge \sigma(y) = \sigma(x \vee y)$.

B. Sands gave the first example of a poset which admits an antiordermorphism σ with the property that $\sigma^2 \neq$ id. In the following we give two families of lattices which admit correlations but no polarities.

Theorem 1.3. For every $n \in \mathbb{N}$ with $n+1 = 2^s$, $s \geqslant 1$, there exists a lattice L of height $h(L) \leqslant 5$ such that there is a correlation σ with $\sigma^{2(n+1)} =$ id but no correlation of lower order.

Proof. We consider the following lattice L where the element 0, 1 are not drawn in the figure 1.

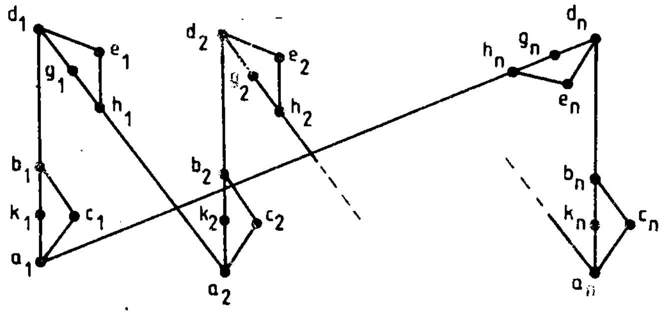


Fig. 1

We define $\sigma : L \to L$ in the following way:

$\sigma(a_i) = d_i$, $\sigma(c_i) = e_i$, $\sigma(b_i) = h_i$, $\sigma(k_i) = g_i$, $i=1,\dots,n$

$\sigma(d_i) = a_{i+1}$, $\sigma(e_i) = c_{i+1}$, $\sigma(h_i) = b_{i+1}$, $\sigma(g_i) = k_{i+1}$, $i=1,\dots,n-1$

$\sigma(d_n) = a_1$, $\sigma(e_n) = c_1$, $\sigma(h_n) = b_1$, $\sigma(g_n) = k_1$,

σ reverses the order, and is bijective with $\sigma^{2n+2}(x) = x$. By the above Proposition 1.2 it follows that σ is a correlation.

Now let $\tau : L \to L$ be another correlation of L. We have $\tau(a_1) = d_i$ for some $i = 1,\dots,n$. From $a_1 < c_1 < b_1 < d_1$ and $a_1 < k_1 < b_1 < d_1$ it follows $d_i = \tau(a_1) > \tau(c_1) > \tau(b_1) > \tau(d_1)$ and $\tau(a_1) > \tau(k_1) > \tau(b_1) > \tau(d_1)$. Hence it follows $\tau(c_1) = l_i$, $\tau(b_1) = h_i$, $\tau(k_1) = e_i$, $\tau(d_1) = a_{i+1}$. This again forces $\tau(a_2) = d_{i+1}$ as $d_1 > h_1 > a_2$. Hence we have $\tau(x) = \sigma^{2i-1}(x)$, for some $i = 1,\dots,n$.

In the case $n+1 = 2^s$, $s > 1$ we have from $\sigma^{2(n+1)} = \sigma^{2^{s+1}} = id$ that $id = \tau^{2m} = \sigma^{(2i-1)2m} = \sigma^{2^{s+1}}$. We conclude that 2^{s+1} divides $(2i-1)\cdot 2m$. The order of τ is at least as large as the order of σ.

Problem. Is there a finite modular lattice L which admits a correlation but no polarity?

In most cases there are a lot of polarities and correlations on a modular lattice. For instance a projective line, i.e. the lattice M_n consisting of n atoms, and the element 0 and 1 has p(n) correlations where p(n) is the number of unrestricted partitions on n.

L e m m a 1.4. Let L be a geometric lattice of finite height. If L admits a correlation σ then L is modular.

P r o o f . By definition [4] p.162 a geometric lattice is semimodular and fulfills the upper covering condition: $a \prec b$ implies $a \vee c \prec b \vee c$ or $a \vee c = b \vee c$ for every $c \in L$. For the lower covering condition we consider $d \prec e$, $m = \sigma(m_1)$, and $\sigma(d_1) = d$, $\sigma(e_1) = e$. We have $e_1 \prec d_1$ applying σ^{-1} and by the upper covering condition we have $e_1 \vee m_1 \prec d_1 \vee m_1$ or $e_1 \vee m_1 = d_1 \vee m_1$. Hence we have $\sigma^{-1}(e) \vee \sigma^{-1}(m) \prec \sigma^{-1}(d) \vee \sigma^{-1}(m)$ and by applying σ we get $d \wedge m \prec e \wedge m$ or $d \wedge m = e \wedge m$. Hence L fulfills the lower covering condition and by [4] Corollary 3 p.174, L is modular.

D e f i n i t i o n 1.5. A correlation lattice $\underset{\sim}{L} = (L;\wedge,\vee,\sigma,\sigma^{-1},0,1)$ is a bounded lattice endowed with two unary operations such that

1) $\sigma^{-1}(\sigma(x)) = \sigma(\sigma^{-1}(x)) = x$
2) $\sigma(x \wedge y) = \sigma(x) \vee \sigma(y)$
3) $\sigma(x \vee y) = \sigma(x) \wedge \sigma(y)$
4) $\sigma(0) = 1$, $\sigma(1) = 0$.

For a fixed natural number m a m-correlation lattice $\underset{\sim}{L} = (L;\wedge,\vee,\sigma,0,1)$ is a bounded lattice with a unary operation σ such that

1) $\sigma^{2m}(x) = x$
2) $\sigma(x \wedge y) = \sigma(x) \vee \sigma(y)$
3) $\sigma(x \vee y) = \sigma(x) \wedge \sigma(y)$
4) $\sigma(0) = 1$, $\sigma(1) = 0$.

N o t a t i o n 1.6 [4] p.139. The element $z \in L$, L a lattice, is called neutral if $(z \wedge x) \vee (x \wedge y) \vee (y \wedge z) = (z \vee x) \wedge (x \vee y) \wedge (y \vee z)$ for every $x,y \in L$. The center $C(\underset{\sim}{L})$ of a bounded lattice consists of the neutral elements. [4] p.156.

D e f i n i t i o n 1.7. Let $\underset{\sim}{L}$ be a correlation lattice or a m-correlation lattice, respectively. $z \in \underset{\sim}{L}$ is called a σ-central element if

1) z is neutral
2) $x = (x \wedge z) \vee (x \wedge \sigma(z))$ for every $x \in L$
3) $\sigma^2(z) = z$

$\Sigma(L) = \{z | z \text{ is } \sigma\text{-central}\}$ is called the σ-center of L. From 2) we have $z \vee \sigma(z) = 1$, and also $z \wedge \sigma(z) = 0$ by 3).

P r o p o s i t i o n 1.8. The σ-center of a correlation lattice $\underset{\sim}{L}$ or a m-correlation lattice $\underset{\sim}{L}$ respectively is a Boolean algebra.

P r o o f . a) We show that from $z \in \Sigma(L)$ it follows $\sigma(z) \in \Sigma(L)$.

1) $\sigma(z)$ is neutral because the equation in 1.6 holds, applying σ to it.

2) We have $(x \wedge \sigma(z)) \vee (x \wedge \sigma^2(z)) = (x \wedge \sigma(z)) \vee (x \wedge \sigma(z)) = x$ by 2) and 3) of 1.7.

3) $\sigma^2(\sigma(z)) = \sigma(z)$.

b) We show for $z_1, z_2 \in \Sigma(L)$ we have $z_1 \vee z_2 \in \Sigma(L)$.

1) $z_1 \vee z_2$ is neutral because of [4] theorem 9 (iii) p.143.

2) $[x \wedge (z_1 \vee z_2)] \vee [x \wedge \sigma(z_1 \vee z_2)] =$
$x \wedge [(z_1 \vee z_2) \vee \sigma(z_1 \vee z_2)] =$ by [4] theorem 5(i) p.142
$x \wedge [(z_1 \vee z_2) \vee (\sigma(z_1) \wedge \sigma(z_2))] =$
$x \wedge [(z_1 \vee z_2 \vee \sigma(z_1)) \wedge (z_1 \vee z_2 \vee \sigma(z_1)] =$ by [4] theorem 5(ii) p.142
$x \wedge 1 = x$.

3) $\sigma^2(z_1 \vee z_2) = \sigma^2(z_1) \vee \sigma^2(z_2) = z_1 \vee z_2$.

c) It is clear that by a) and b) it follows from $z_1, z_2 \in \Sigma(L)$ that $z_1 \wedge z_2 \in \Sigma(L)$.

2. Decomposition of correlation lattices

T h e o r e m 2.1. Let $\underset{\sim}{L}$ be a correlation lattice and $\Sigma(\underset{\sim}{L})$ the σ-center of $\underset{\sim}{L}$. If $\Sigma(\underset{\sim}{L})$ contains more then two elements 0 and 1 then $\underset{\sim}{L}$ is isomorphic to a direct product of correlation lattices.

P r o o f . a) Let $z \in \sum(\underset{\sim}{L}) \setminus \{0,1\}$ and define $L_1 = \{a \in L \mid a \leq z\}$ and $L_2 = \{b \in L \mid b \leq \sigma(z)\}$. L_1 and L_2 are lattices as for $a_1, a_2 \in L_1$ we have $a_1 \vee a_2 \leq z$ and $a_1 \wedge a_2 \leq z$. We define $\sigma_1(a) = \sigma(a) \wedge z$ for every $a \in L_1$ and $\sigma_1^{-1}(a) = \sigma^{-1}(a) \wedge z$. This is a correlation as $\sigma_1^{-1}(\sigma_1(a)) = \sigma_1^{-1}(\sigma(a) \wedge z) = \sigma^{-1}(\sigma(a) \wedge z) \wedge z = (a \vee \sigma^{-1}(z)) \wedge z = (a \wedge z) \vee (\sigma^{-1}(z) \wedge z) = a$ because from $\sigma(z) \vee z = 1$ it follows $\sigma^{-1}(z) \wedge z = 0$. Hence L_1 and similarly L_2 is a correlation lattice.

b) Now consider the map $\alpha : L \to L_1 \times L_2$ defined by $\alpha(x) = (x \wedge z,\ x \wedge \sigma(z))$ α is injective as for $(d \wedge z,\ d \wedge \sigma(z)) = (h \wedge z,\ h \wedge \sigma(z))$ we have $d = (d \wedge z) \vee (d \wedge \sigma(z)) = (h \wedge z) \vee (h \wedge \sigma(z)) = h$ by the definition of the σ-center. α is surjective as for $(w_1, w_2) \in L_1 \times L_2$ one considers $w = w_1 \vee w_2$ and has $\alpha(w) = ((w_1 \vee w_2) \wedge z,\ (w_1 \vee w_2) \wedge \sigma(z)) = (w_1 \wedge z), w_2 \wedge \sigma(z)) = (w_1, w_2)$ because $w_2 \wedge z \leq \sigma(z) \wedge z = 0$.

c) α is a correlation lattice homomorphism. $\alpha(\sigma(x)) = (\sigma(x) \wedge z, \sigma(x) \wedge \sigma(z)) = \sigma(x \wedge z, x \wedge \sigma(z)) = \sigma(\alpha(x))$ by the definition of the correlation on $L_1 \times L_2$.
$\alpha(x \vee y) = ((x \vee y) \wedge z, (x \vee y) \wedge \sigma(z)) = (x \wedge z, x \wedge \sigma(z)) \vee \vee (y \wedge z, y \wedge \sigma(z)) = \alpha(x) \vee \alpha(y)$ because z and $\sigma(z)$ are σ-central.

T h e o r e m 2.2. Let L be a m-correlation lattice and $\sum(\underset{\sim}{L})$ the σ-center of $\underset{\sim}{L}$. If $\sum(\underset{\sim}{L})$ contains more then two elements 0, 1 then $\underset{\sim}{L}$ is a isomorphic to a direct product of m-correlation lattices.

P r o o f . The proof is the same as above besides the part a). There we define $\sigma_1(a) = \sigma(a) \wedge z$ and have $\sigma_1^{2m}(a) = \sigma_1^{2m-1}(\sigma(a) \wedge z) = \sigma_1^{2m-2}(\sigma(\sigma(a) \wedge z) \wedge z) = \sigma_1^{2m-2}[(\sigma^2(a) \vee \sigma(2)) \wedge z] = \sigma_1^{2m-2}(\sigma^2(a) \wedge z) = \ldots = \sigma^{2m}(a) \wedge z = a \wedge z = a.$

T h e o r e m 2.3. Let $\underset{\sim}{L}$ be a geometric m-correlation lattice of finite height. Then $\underset{\sim}{L}$ is isomorphic to a direct product of simple geometric m-correlation lattices.

P r o o f . Let θ be a non trivial congruence of the correlation lattice $\underset{\sim}{L}$ and consider $z = \sup\{x \mid (x,0) \in \theta\}$. Then θ is also a lattice congruence of the form θ_z where z is neutral according Corollary 11 [4] p.149. Consider $\sigma(z) =$

$= \inf\{\sigma(x) | (\sigma(x),1) \in \Theta\} = \inf\{y | (y,1) \in \Theta\}$. We have $\sigma^2(z) = z$. Now consider $(x,x) \in \Theta$, $(0,z) \in \Theta$ and hence $(0,x \wedge z) \in \Theta$. Furthermore consider $(x,x) \in \Theta$, $(1,\sigma(z)) \in \Theta$ and hence $(x,x \wedge \sigma(z)) \in \Theta$. Together we have $(x,(x \wedge z) \vee (x \wedge \sigma(z))) \in \Theta$ or as z is neutral $(x,x \wedge (z \vee \sigma(z))) \in \Theta$. Now let d be a complement of $z \vee \sigma(z)$ in L. Then we have $(d,d \wedge (z \vee \sigma(z))) \in \Theta$ and hence $(d,0) \in \Theta$. Therefore $d \leqslant z$ and therefore $1 = d \vee (z \vee \sigma(z)) \leqslant z \vee (z \vee \sigma(z)) = z \vee \sigma(z)$. It follows $0 = z \wedge \sigma(z)$ by $\sigma^2(z) = z$. Therefore for every non trivial congruence Θ of the correlation lattice we have a non trivial σ-central element z and vice versa. The theorem follows from Theorem 2.2.

R e m a r k . Simple correlation lattices which stand apart from the class of simple m-correlation lattice are not described in the literature till now. Hence we will confine us to m-correlation lattices in the following. The above theorems generalize some results of MacLaren [7] on orthomodular lattices and ortholattices.

3. On simple, geometric m-correlation lattices

R e m a r k . Let $\underset{\sim}{L}$ be a simple geometric m-correlation lattice of finite height. Then $\underset{\sim}{L}$ has a trivial σ-center $\Sigma(\underset{\sim}{L})$ but need to have trivial center $C(\underset{\sim}{L})$ concerning the lattice. But $C(\underset{\sim}{L})$ is a Boolean m-correlation lattice as for $z \in C(\underset{\sim}{L})$ we have $\sigma(z) \in C(\underset{\sim}{L})$.

L e m m a 3.1. $\underset{\sim}{L}$ is a simple geometric m-correlation lattice if and only if the center $C(\underset{\sim}{L})$ of $\underset{\sim}{L}$ is a simple m-correlation lattice.

P r o o f . a) Let $C(\underset{\sim}{L})$ be a simple m-correlation lattice. Assume that $\Sigma(\underset{\sim}{L})$ is not trivial and let $z \in \Sigma(\underset{\sim}{L}) \setminus \{0,1\}$. Then we put $(u,v) \in \Theta$ if and only if $u \wedge z = v \wedge z$ for $u,v \in C(\underset{\sim}{L})$. Θ is a lattice congruence. Furthermore for $(u,v) \in \Theta$ we have $\sigma(u \wedge z) = \sigma(v \wedge z)$ and hence $\sigma(u) \vee \sigma(z) = \sigma(v) \vee \sigma(z)$ and also $(\sigma(u) \vee \sigma(z)) \wedge z = (\sigma(v) \vee \sigma(z)) \wedge z$. Therefore $(\sigma(u) \wedge z) \vee (\sigma(z) \wedge z) = (\sigma(v) \wedge z) \vee (\sigma(z) \wedge z)$ and hence $\sigma(u) \wedge z = \sigma(v) \wedge z$. As $(\sigma(u),\sigma(v)) \in \Theta$ it follows that Θ is a non trivial congruence of the m-correlation lattice $C(\underset{\sim}{L})$. Contradiction.

b) Let $\Sigma(\underset{\sim}{L})$ be trivial consisting only of 0 and 1. Assume that there is a non trivial congruence θ on $C(\underset{\sim}{L})$ and put $z = \sup\{u \mid (0,u) \in \theta\}$. We have $(0,z) \in \theta$ and hence $(0,\sigma^2(z)) \in \theta$ and hence $\sigma^2(z) \leq z$. As L is of finite length and σ is bijective we have $\sigma^2(z) = z$. As $C(\underset{\sim}{L})$ is Boolean we have from $(0,z) \in \theta$ that $(1,z') \in \theta$ and as $\sigma(z) = \inf\{y \mid (y,1) \in \theta\}$ we have $\sigma(z) \leq z'$. Hence we have $\sigma(z) \wedge z \leq z' \wedge z = 0$ and also $\sigma(z) \vee z = 1$. Now property 2) of Definition 1.7 is implied by $\sigma(z) \wedge z = 0$ and $\sigma(z) \vee z = 1$. Hence $z \in \Sigma(\underset{\sim}{L}) \setminus \{0,1\}$, a contradiction.

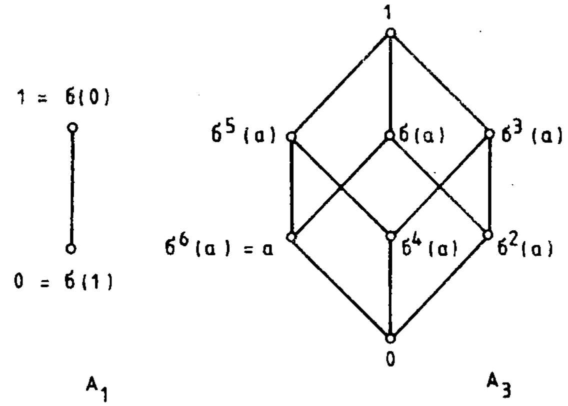
The simple Boolean m-correlation lattices are classified in [6]. From this it follows that there are three types of simple geometric correlation lattices of finite height.

1) $C(\underset{\sim}{L}) = \{0,1\} = \underset{\sim}{A}_1$ (a projective geometry with correlation),

2) $C(\underset{\sim}{L}) \in \{\underset{\sim}{A}_n\}$, $n \in 2N+1$, $n > 1$,

3) $C(\underset{\sim}{L}) \in \{\underset{\sim}{B}_n\}$, $n \in 2N$.

The last two types will be described in the following theorems. The first members of the series $\{\underset{\sim}{A}_n\}$, $n \in 2N+1$, $n \geq 1$ are in figure 2.



The first members of the series $\{A_n\}_{n \in 2N+1}$

Fig.2

If $C(\underset{\sim}{L}) \in \{\underset{\sim}{A}_n\}$, $n \in 2N+1$, then $C(\underset{\sim}{L})$ is a Boolean lattice with an odd number of atoms and a correlation σ without any fixed point. σ^2 permutes the atoms such that $C(\underset{\sim}{L})$ is simple. For details consider [6].

T h e o r e m 3.2. Let $\underset{\sim}{L}$ be a finite, simple, geometric correlation lattice with $C(\underset{\sim}{L}) \in \{A_n\}$, $n \in 2N+1$, $n > 1$. Then $\underset{\sim}{L}$ is as a lattice isomorphic to a direct product of self-dual projective geometries which are isomorphic to each other.

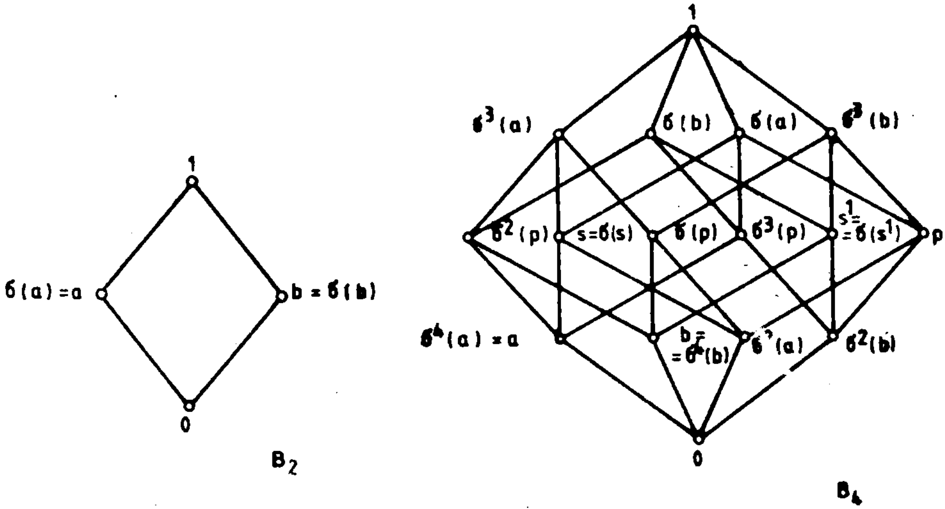
P r o o f . Let $C(\underset{\sim}{L}) \cong A_n$, $n > 1$, n odd and let z be an atom of $C(\underset{\sim}{L})$. Then we have $\underset{\sim}{L} \simeq [0,z] \times [0,\sigma^2(z)] \times \dots \times [0,\sigma^{2n-2}(z)]$ if we consider $\underset{\sim}{L}$ as a lattice. The factors $[0,\sigma^{2i}(z)]$ and $[0,\sigma^{2j}(z)]$, $i,j = 1,\dots,n-1$, are isomorphic as lattices because $\sigma^{2j-2i} = \sigma^{2(j-i)}$ is a lattice isomorphism for $j > i$. The factors are simple and hence projective geometries. It is easy to define a correlation σ on the direct product

$$[0,z] \times \dots \times [0,\sigma^{2n-2}(z)] \quad \text{by} \quad \sigma((a_1,\dots,a_n)) =$$

$$= ((\sigma(d) \wedge z),\dots,(\sigma(d) \wedge \sigma^{2n-2}(z))) \text{ for } d = a_1 \vee \dots \vee a_n.$$

This correlation has the required property that the direct product becomes a correlation lattice isomorphic to the correlation lattice $\underset{\sim}{L}$. Now we consider $f_2 : [0,z] \to [0,\sigma^2(z)]$ defined by $f_2(x) = \sigma(x) \wedge \sigma^2(z)$ which is a dual homomorphism. Consider $f_4 : [0,\sigma^2(z)] \to [0,\sigma^4(z)]$ defined by $f_4(x) = \sigma(x) \wedge \sigma^4(z)$ and so on. Iterating we have $f_{2n} \circ \dots \circ f_4 \circ f_2(x) = x$ and we conclude that $[0,z]$ is dual isomorphic to $[0,\sigma^2(z)]$.

The first members of the series $\{\underset{\sim}{B}_n\}$, $n \in 2N$ are in figure 3. If $C(\underset{\sim}{L}) \in \{B_n\}$, $n \in 2N$, then $C(\underset{\sim}{L})$ is a Boolean lattice with an even number of atoms and a correlation σ with 2 fixed points. There are two different orbits of atoms under σ^2 which are glued together by σ. Hence $C(\underset{\sim}{L})$ is simple by the correlation σ. Furthermore we have $\sigma^n(x) = x$. For further details consider [6].



The first members of the series $\{B_{2n}\}_{n\in\mathbb{N}}$

Fig.3

T h e o r e m 3.3. Let $\underset{\sim}{L}$ be a finite, simple, geometric correlation lattice with $C(\underset{\sim}{L}) \in \{B_n\}$, $n \in 2\mathbb{N}$. Then $\underset{\sim}{L}$ is as a lattice isomorphic to a direct product of $\frac{n}{2}$ projective geometries $G_1,\dots,G_{\frac{n}{2}}$ and $\frac{n}{2}$ projective geometries $H_1,\dots,H_{\frac{n}{2}}$ such that G_i, G_j, $i,j = 1,\dots,\frac{n}{2}$ are isomorphic, H_i, H_j, $k,j = 1,\dots,\frac{n}{2}$ are isomorphic and G_k and H_l are dual isomorphic, k, $l_i = 1,\dots,\frac{n}{2}$.

P r o o f . If $\underset{\sim}{L}$ has the center $\underset{\sim}{C}(L) \simeq \underset{\sim}{B}_n$, $n \in 2\mathbb{N}$, then $L \simeq [0,z]\times[0,6^2(z)]\times\dots\times[0,6^n(z)]\times[0,\bar{z}]\times\dots\times[0,6^n(\bar{z})]$, where z is atom of $C(\underset{\sim}{L})$ and $\bar{z}$ is an atom of $C(\underset{\sim}{L})$ with the properties $\bar{z} \in \{z, 6^2(z),\dots,6^n(z)\}$ and $6(z)\wedge\bar{z} = 0$. We put $G_1 = [0,z],\dots,G_{\frac{n}{2}} = [0,6^n(z)]$ and $H_1 = [0,\bar{z}],\dots,H_{\frac{n}{2}} = [0,6^n(\bar{z})]$. G_i is isomorphic to G_j by $f(x) = 6^{2j-2i}(x)$ for $j \geq i$. Furthermore we consider $g\colon G_1 \to H_1$ defined by $g(x) = 6(x)\wedge\bar{z}$ and $h : H_1 \to G_1$ defined by $h(x) = 6^{n-1}(x)\wedge z$ which both are dual homomorphisms.

Now we have $(h \circ g)(x) = \sigma^{n-1}(\sigma(x) \wedge \bar{z}) \wedge z = (\sigma^{n}(x) \vee \sigma^{n-1}(\bar{z})) \wedge z =$ $= (x \wedge z) \vee (\sigma^{n-1}(\bar{z}) \wedge z) = x$ because $\sigma^{n-1}(\bar{z}) \wedge z = 0$. For this assume $z \leqslant \sigma^{n-1}(\bar{z})$. It follows $\bar{z} \leqslant \sigma(z)$ which is a contradiction to $\bar{z} \wedge \sigma(z) = 0$. Hence H_1 and G_1 are dual isomorphic.

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Received April 7, 1987.